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ON THE CONVEXITY OF HIGH ORDER OF SEQUENCES

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Abstract. We improve some results of Lacković and Simić [**2**] concerning the weighted arithmetic means that preserve the convexity of high order of sequences.

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In [**1**] and [**3**] a characterization is given for triangular matrices which define transformations in the set of sequences preserving covexity of order r. In the particular case of weighted arithmetic means, explicit expressions were given before in Lacković and Simić [**2**]. In this paper we improve the results from [**2**] generalizing some of the properties that we proved in [**7**] for the convexity of order two.

At the beginning, let us specify some notation and definitions which will be used throughout the paper.

Let $a = (a_n)(n = 0, 1, \dots)$ be a real sequence. The r-th order difference of the sequence a is defined by:

$$(1) \qquad \Delta^0 a_n = a_n \quad \Delta^r a_n = \Delta^{r-1} a_{n+1} - \Delta^{r-1} a_n \quad (r = 1, 2, \dots;\ n + 0, 1, \dots)$$

Definition 1. A sequence $a = (a_n)$ is said to be convex of order r if $\Delta^r a_n \geq 0$ for all $n \in N$.

Let $p = (p_n)$ be a sequence of pozitive numbers. It defines a transformation P in the set of sequences: any sequence $a = (a_n)$ is transformed into the sequence $P(a) = A + (A_n)$ given by:

$$(2) \qquad A_n = \frac{p_0 a_0 + \dots + p_n a_n}{p_0 + \dots + p_n} \quad (a = 0, 1, \dots)$$

Definition 2. The Transformation P is said to be r-convex if the sequence $A = P(a)$ is convex of order r for any sequence a convex of order r.

In [**2**] the following theorem is given:

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THEOREM 0. *The transformation P is r-convex if and only if the sequence $p = (p_n)$ is given by:*

$$p_n \frac{(r-1)! \cdot p_{r-1}}{n! \cdot (p_0 + \cdots + p_{r-2}} \prod_{k=r-2}^{n-1} (k+1)(p_0 + \cdots + p_{r-2}) + (r-1)p_{r-1}$$

for $n \geq r$, with $p_0, \dots, p_{r-1}$ arbitraty positive numbers.

Remark. For $a_0 = 0$ and $a_n = (3 + 6n - 2n^2)/3$ if $n \geq 1$, we have $\Delta^3 a_0 = 1$ and $\Delta^3 a_n = 0$ if $n \geq 1$, so that the sequence (a_n) is convex of order 3. Let us choose for $r = 3 : p_0 = 6,\ p_1 = 1$ and $p_2 = 7/2$. From (3) we get $p_3 = 7/2$ and so from (2), we have $A_0 = 0,\ A_1 = 1/3, A_2 = 1$ and $A_3 = 1$, that is $\Delta^3 A_0 = -1$. Hence the result from Theorem 0 is not valid in this form. To amend it, we begin by puting (3) in a simpler shape. For this we use the following notation:

$$\binom{u}{o} = 1,\ \binom{u}{n} = \frac{u(u-1)\dots(u-n+1)}{n!},\ \text{for } n \geq 1 \tag{4}$$

where u is an arbitraty real number.

LEMMA 1. *If the transformation P is r-convex, then the sequence (p_n) must be given by:*

$$p_n = p_{r-1} \binom{u+n-1}{n-r+1} : \binom{n}{r-1},\ \textit{for } n \geq r \tag{5}$$

where

$$u = \frac{(r-1) \cdot p_{r-1}}{p_0 + \cdots + p_{r-2}},\ p_k > 0 \text{ for } k = 0, \dots, r-1, \tag{6}$$

Proof. Because (5) is only a transcription of (3) using (4) and (6), the rusult was proved in [**2**]. However we sketch here another proof by mathematical induction. As in [**2**] we use the sequence $a_n = c \cdot n \cdot (n-1) \cdots (n-r+2)$ for which we have $\Delta^r a_n = 0$ for any n. Hence it is convex of order r for any real c, and so must be (A_n) too. But this happens if and only if for $c = 1$ we have $\Delta^r A_n = 0$ for any n. For $n = 0$ we get $p_r = p_{r-1}(u + r - 1)/r$ whic is (5) for $n = r$. Suppose (5) is valid for $n \leq m$. To obtain A_n for $r \leq n \leq m$, we must calculate:

$$\sum_{k=0}^{n} p_k = \sum_{k=o}^{r-2} p_k + p_{r-1} + \sum_{k=r}^{n} p_k = p_{r-1} \left[\frac{r-1}{u} + 1 + \sum_{i=0}^{n-r} \binom{u+r+i-1}{i+1} : \binom{r+i}{i+1} \right].$$

From this it can be shown, by mathematical induction, that:

$$\sum_{k=0}^{n} p_k = p_{r-1} \frac{n-r+2}{u} \cdot \binom{u+n}{n-r+2} : \binom{n}{n-r+1}. \tag{7}$$

So:

$$A_n = \frac{u \cdot (r-1)!}{u+r-1} \binom{n}{r-1},\ n \leq m$$

and

$$A_{m+1} = \left[p_{r-1}(r-1)!\binom{u+m}{m-r+1} + p_{m+1}(r-1)!\cdot\binom{m+1}{r-1}\right] :$$
$$:\left[p_{r-1}\frac{m-r+2}{u}\binom{u+m}{m-r+2} : \binom{m}{m-r+1} + p_{m+1}\right].$$

From $\Delta^r A_{m-r+1} = 0$, we obtain (5) for $m+1$, and so for every n

LEMMA 2. *If the sequence* (a_n) *is given by:*

(8) $$a_n = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} \cdot b_k,$$

then

(9) $$\Delta^r a_n = b_{n+r} \quad (n = 0, 1, \dots)$$

Remark 2. This result is connected with some relations from [**1**] and [**6**]. Because any sequence may be put in the form (8), we obtain a representation theorem simpler than that given in [**6**]:

COROLLARY 1. *The sequence* (a_n) *is convex of order* r *if and only if in its representation* (8), *it has* $b_n \geq 0$ *for* $n \geq r$.

LEMMA 3. *If the transformation* P *is* r*-convex, then for every* $n \leq r$*:*

(10) $$\sum_{k=0}^{n-1} p_k = n \cdot p_n / u.$$

Proof. Let (A_n) be represented by:

(11) $$A_n = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} \cdot c_k.$$

Then:

$$a_n = \left(A_n \sum_{i=0}^{n} p_i - A_{n-1} \sum_{i=0}^{n-1} p_i\right) : p_n.$$

If

$$q_n = \frac{1}{p_n} \sum_{k=0}^{n-1} p_k$$

then

$$\sigma_n = A_n + q_n \cdot (A_n - A_{n-1}) = \sum_{k=0}^{n} \left[\binom{n+r-i-1}{r-1} + q_n \cdot \binom{n+r-i-2}{r-2}\right] c_i$$

for $n \geq 1$ and $a_0 = A_0 = c_0$. So:

$$\Delta^r a_0 = \sum_{j=0}^{r} (-1)^j \binom{r}{j} a_{r-j} =$$

$$= \sum_{j=0}^{r-1} \left\{ \sum_{i=0}^{r-1} \left[\binom{2r-j-i-1}{r-1} + q_{r-j} \cdot \binom{2r-j-i-2}{r-2} \right] \cdot c_i \right\} (-1)^j \binom{r}{j} + (-1)^r c_0$$

$$= \sum_{i=0}^{r} \left\{ \sum_{j=0}^{r-i} \left[\binom{2r-j-i-1}{r-1} + q_{r-j} \cdot \binom{2r-j-i-2}{r-2} \right] (-1)^j \binom{r}{j} \right\} c_i +$$

$$+ \left\{ \sum_{j=0}^{r-1} \left[\binom{2r-j-1}{r-1} + q_{r-j} \cdot \binom{2r-j-2}{r-2} \right] \right\} (-1)^j \binom{r}{j} + (-1)^r \Big\} c_0.$$

But, as it is proved in [**5**]:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0 \text{ for } p < n$$

and hence:

$$\sum_{j=0}^{n} (-1)^j \binom{r}{j} \cdot Q(j) = 0$$

for any polinomial Q of degree less than n. So:

$$\sum_{j=0}^{m} (-1)^j \binom{r}{j} \cdot \binom{m+r-j-1}{r-1} = 0 \text{ for } m = 1, \ldots, r \tag{12}$$

because:

$$\sum_{j=0}^{m} (-1)^j \frac{r!}{j! \cdot (r-j)!} \cdot \frac{(m+r-j-1j)!}{(r-1)! \cdot (m-j)!} = \frac{r}{m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \cdot \binom{m+r-j-1}{m-1}$$

and $\binom{m+r-j-1}{m-1}$ is a polynomial of degree $m-1$ in j. Hence:

$$\Delta^r a_0 = c_r + \sum_{i=0}^{r} \left[\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{2r-j-i-2}{r-2} \cdot q_{r-j} \right] c_i +$$

$$+ \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{2r-j-2}{r-2} \cdot q_{r-j} \cdot c_o.$$

As the coefficient of c_r is $1 + q_r > 0$, $\Delta^r a_0 \geq 0$ implies $\Delta^r A_0 = c_r \geq 0$ if and only f the coefficients of c_i are zero for $i = 0, \ldots, r-1$. For $i = r-1$ we have: $(r-1) \cdot q_r - r \cdot q_{r-1} = 0$ and as (6) means $q_{r-1} = (r-1)/u$, we also have $q_r = r/u$. Assuming (10) valid for $r - j(j = 0, \ldots, m-1; m < r-1)$ it may be deduced for $r - m$, because we have:

$$\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{m+r-j-2}{r-2} \cdot (r-j) = 0, \text{ for } m < r-1$$

and

$$\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\cdot\binom{2r-j-2}{r-2}\cdot(r-j)=0$$

which may be verified as in (12).

THEOREM 1. *The transformation P is r-convex if and only if the sequence (p_n) is given by:*

$$p_n = p_0\cdot\binom{u+n-1}{n},\quad \textit{for } n\geq 1,\quad \textit{with } u=p_1/p_0. \tag{13}$$

Proof. Necessity: Lemma 1 and Lemma 3 give the necessary conditions (5) and (10). From (10) we have: $u=p_1/p_0$ for $n=1$, and $p_2=u(p_0+p_1)/2=p_0\binom{u+1}{2}$: supposing (13) valid for $n\leq m<r-1$, (10) gives:

$$p_{m+1}=\frac{u\cdot p_0}{m+1}\sum_{k=0}^{m}\binom{u+k-1}{k}=p_0\frac{u}{m+1}\binom{u+m}{m}=p_0\binom{u+m}{m+1}$$

that, is, (13) holds for $n\leq r-1$. Hence, from (5) we also get:

$$p_n=p_0\cdot\binom{u+r-2}{r-1}\cdot\binom{u+n-1}{n-r+1}:\binom{n}{r-1}=p_0\cdot\binom{u+n-1}{n}$$

for $n\geq r$.

Sufficiency: with (13), the sequence (2) becomes:

$$A_n\left[\sum_{k=0}^{n}\binom{u+k-1}{k}a_k\right]:\binom{u+n}{n} \tag{14}$$

and so we have the relation:

$$a_n=A_n+n\cdot(A_n-A_{n+1}):u,\ \text{ for } n>0. \tag{15}$$

Taking A_n of the form (11), from (15) we obtain:

$$a_n=\sum_{k=0}^{n}\binom{n+r-k-2}{r-2}\cdot\left(\frac{n+r-k-1}{r-1}+\frac{n}{u}\right)\cdot c_k. \tag{16}$$

Besause $\Delta^r A_n=c_{n+r}$, applying to (15) the know relation (see [**4**]):

$$\Delta^r(a_n\cdot b_n)=\sum_{i=0}^{r}\binom{r}{i}\Delta^i a_n\cdot\Delta^{r-i}b_{n+i}$$

we obtain:

$$\Delta^r a_n=(n+r+u)u^{-1}c_{n+r}-nu^{-1}c_{n+r-1},\ \ n\geq 1. \tag{17}$$

From the proof of Lemma 3 we have: $\Delta^r a_0 = c_r \cdot (r+u) : u$, that is (17) is valid for $n = 0$ too. Asuming (a_n) given by (8), (9) is valid; thus:

(18) $$b_r = (r+u)/u, \quad b_{n+r} = (n+r+u)/u c_{n+r} - n/u\ c_{n+r-1}.$$

Hence, if $b_n \geq 0$ for $n \geq r$, then also $c_n \geq 0$ for $n \geq r$; that is, if (a_n) is convex of order r, so is (A_n) too.

Remark 3. The sufficiency part of Theorem 1 was also proved in [**1**]. In what follows we improve also this result. Let us denote by K_r, the set of all sequences convex of order r and by K_r^u the set of all sequences (a_n) with the property that (14) gives a sequence (A_n) in K_r.

THEOREM 2. *If* $0 < v < u$ *then the following strict inclusions hold:*

$$K_r \subset K_r^u \subset K_r^v.$$

Proof. The first inclusion was proved in Theorem 1. Its strictness follows from (18): the positivity of $c_n (n \geq r)$ does not imply that of b_n. Now supose (a_n) given by (16) and also by:

$$a_n = \sum_{k=0}^{n} \binom{n+r-k-2}{r-2} \cdot \left(\frac{n+r-k-1}{r-1} + \frac{n}{v} \right) \cdot d_k.$$

So (17) holds and $\Delta^r a_n = (n+r+v)v^{-1} d_{n+r} - nv^{-1} d_{n+r-1}$ that is:

$$(n+r+v)/v d_{n+r} - nv^{-1} d_{n+r-1} = (n+r+u)u^{-1} c_{n+r} - nu^{-1} c_{n+r-1}$$

Hence $d_r = \frac{v \cdot (r+u)}{u \cdot (r+v)} c_r$ and generally, by mathematical induction:

(19) $$d_{r+n} = \frac{u+r+n}{v+r+n} \cdot \frac{c_{r+n}}{uv} + \frac{u-v}{uv} \sum_{i=0}^{n-1} \frac{c_{r+i}}{n-i+1} \binom{n}{i} : \binom{v+r+n}{n-i+1};$$

that is, $c_n \geq o$ for $b \geq r$ implies $d_n \geq 0$ for $b \geq r$ and so, if (a_n) is in K_r^u, it is also in K_r^v. That the inclusion $K_r^u \subset K_r^v$ is strict follows also from (19) as above.

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