

A NOTE ON LENGTH DISTORTION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Some interesting results on length distortion for certain classes of analytic functions have been proved recently by Y. Komatu. We derive some results involving analytic functions satisfying $\operatorname{Re}\{f(z)/z\} > \alpha$, starlike functions of order α , and convex functions of order α .

Introduction. Let A denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Komatu [1] introduced a linear operator L of the form

$$Lf(z) = \int_0^1 t^{-1} f(zt) d\sigma(t)$$

defined on A where $\sigma(t)$ denotes a probability measure supported by the interval $I = [0, 1]$. He has shown in [1] that there exists a family of operators $\{L^\lambda\}$ depending on a continuous parameter λ such that

$$L^\lambda L^\mu = L^{\lambda+\mu} \quad \text{where } L^0 = \text{id}$$

and, under certain conditions on σ ,

$$(1.1) \quad f_\lambda(z) \equiv L^\lambda f(z) = \int_I t^{-1} f(zt) d\sigma_\lambda(t)$$

where σ_λ is a probability measure supported by I .

Let $R(\alpha) \subset A$ be the class of all functions satisfying the condition

$$\operatorname{Re}\{f(z)/z\} > \alpha \quad (f(z) \in A)$$

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$.

The main tool in deriving our results for length distortion is based on the following lemma [2].

LEMMA 1. Let $L_\lambda(r)(\lambda \geq 0)$ be the length of the image-curve of $\{z : |z| = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in R(0)$. Then we have

$$(1.2) \quad L_\lambda(r) \leq \int_I L(rt) d\sigma_\lambda(t)$$

where $L = L_0$ satisfies $L(r) \leq 4\pi r/(1 - r^2)$.

2. The class $R(\alpha)$. In view of Lemma 1, we state and prove

THEOREM 1. Let $L_\lambda(r)(\lambda \geq 0)$ be the length of the image-curve of $\{z : |z| = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in R(\alpha)$. Then we have the estimation (1.2) where $L = L_0$ satisfies

$$(2.1) \quad L(r) \leq 4\pi(1 - \alpha)r/(1 - r^2).$$

Proof. The estimation (1.2) is obtained by the method of proof in [2]. For a function $f(z)$ belonging to the class $R(\alpha)$, the Herglotz representation implies

$$(2.2) \quad \frac{f(z)}{z} - \alpha = (1 - \alpha) \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\tau(\varphi)$$

with a probability measure τ supported by $(-\pi, \pi]$. It follows from (2.2) that

$$g(z) \equiv \frac{f(z)}{z} = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + (1 - 2\alpha)z}{e^{i\varphi} - z} d\tau(\varphi).$$

Therefore, we obtain

$$\begin{aligned} L(r) &= r \int_{-\pi}^{\pi} |g'(re^{i\theta})| d\theta \\ &\leq 2(1 - \alpha)r \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} |e^{i\varphi} - re^{i\theta}|^{-2} d\tau(\varphi) \\ &= 2(1 - \alpha)r \int_{-\pi}^{\pi} d\tau(\varphi) \int_{-\pi}^{\pi} |e^{i\varphi} - re^{i\theta}|^{-2} d\theta \\ &= 2(1 - \alpha)r \int_{-\pi}^{\pi} 2\pi(1 - r^2)^{-1} d\tau(\varphi) \\ &= 4\pi(1 - \alpha)r/(1 - r^2) \end{aligned}$$

whish gives (2.1). Furthermore, the bound in (2.1) is attained by the extremal function given by $f(z) = (z + (1 - 2\alpha)\varepsilon z)/(1 - \varepsilon z)$ ($|\varepsilon| = 1$).

Remark 1. By letting $\alpha = 1/2$ in Theorem 1, we have the corresponding result due to Komtu [2, Theorem 2].

3. The class $S^*(\alpha)$. A function $f(z)$ belonging to the class A is said to be starlike of order α ($0 \leq \alpha < 1$) if it satisfies $Re\{zf'(z)/f(z)\} > \alpha$ for some α ($0 \leq \alpha < 1$) and for all $z \in U$. We denote by $S^*(\alpha)$ the class of all functions in A which are starlike of order α . We note that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$ for $0 \leq \alpha < 1$.

For the function $f(z)$ in the class $S^*(\alpha)$, we prove

THEOREM 2. Let $L_\lambda(r)$ ($\lambda \geq 0$) be the length of the image-curve of $\{z : |z| = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in S^*(\alpha)$. Then we have the estimation (1.2) where $L = L_0$ satisfies

$$L(r) \leq \frac{8(1-\alpha)r}{(1-r)^{2(1-\alpha)}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right),$$

where K denotes the complete elliptic integral of the first kind with modulus $2\sqrt{r}/(1+r)$.

Proof. Again we use the estimation (1.2). Applying Herglotz representation, we have, for $g(z) = f(z)/z$ and $f(z) \in S^*(\alpha)$,

$$\begin{aligned} (3.1) \quad g'(z) &= f(z)z^{-2}\{(zf'(z)/f(z) - \alpha) - (1-\alpha)\} \\ &= \frac{g(z)}{z} \left\{ (1-\alpha) \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\tau(\varphi) - (1-\alpha) \right\} \\ &= 2(1-\alpha)g(z) \int_{-\pi}^{\pi} (e^{i\varphi} - z)^{-1} d\tau(\varphi) \end{aligned}$$

with a probability measure τ supported by $(-\pi, \pi]$. Thus from (3.1) we derive

$$\begin{aligned} L(r) &= r \int_{-\pi}^{\pi} |g'(re^{i\theta})| d\theta \\ &= 2(1-\alpha)r \int_{-\pi}^{\pi} |g(re^{i\theta})| d\theta \left| \int_{-\pi}^{\pi} (e^{i\varphi} - re^{i\theta})^{-1} d\tau(\varphi) \right| \\ &\leq 2(1-\alpha)r \int_{-\pi}^{\pi} d\tau(\varphi) \int_{-\pi}^{\pi} \left| \frac{g(re^{i\theta})}{e^{i\varphi} - re^{i\theta}} \right| d\theta \end{aligned}$$

We note that a function $f(z)$ belonging to the class $S^*(\alpha)$ satisfies [4]:

$$r(1+r)^{-2(1-\alpha)} \leq |re^{i\theta}| \leq r(1-r)^{-2(1-\alpha)}$$

which implies

$$(3.2) \quad |g(re^{i\theta})| \leq (1-r)^{-2(1-\alpha)},$$

and we have the following explicit expression [2]:

$$(3.3) \quad J(r) = 2 \int_0^\pi (1+r^2 - 2r \cos \theta)^{-1/2} d\theta = \frac{4}{1+r} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

Consequently, by using (3.2) and (3.3), we obtain

$$L(r) \leq \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} \int_{-\pi}^{\pi} \frac{1}{|1-re^{i\theta}|} d\theta = \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

This completes the proof of Theorem 2.

Remark 2. For $\alpha = 0$, Theorem 2 implies the result for $f(z) \in S^*$ proved in [2].

4. The class $K(\alpha)$. A function $f(z) \in A$ is said to be convex of order α ($0 \leq \alpha < 1$) if it satisfies $Re\{1 + zf''(z)/f'(z)\} > \alpha$ for some α ($0 \leq \alpha < 1$) and for all $z \in U$. We denote by $K(\alpha)$ the class of all functions in A which are convex of order α . Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that $K(\alpha) \subseteq K(0) \equiv K$ for $0 \leq \alpha < 1$.

In order to derive our theorem for $f(z) \in K(\alpha)$, we recall here the following lemma by MacGregor [3].

LEMMA 2. *If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where $\beta(\alpha)$ is given by*

$$\beta(\alpha) = \begin{cases} (2\alpha - 1)/2(1 - 2^{1-2\alpha}) & (\alpha \neq 1/2) \\ 1/2 \log 2 & (\alpha = 1/2). \end{cases}$$

Combining Theorem 2 and Lemma 2, we have

THEOREM 3. *Let $L_\lambda(r)$ ($\lambda \geq 0$) be the length of the image-curve of $\{z : |z| = r < 1\}$ by mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in K(\alpha)$. Then we have the estimation (1.2) where $L = L_0$ satisfies*

$$L(r) \leq \frac{8(1-\beta(\alpha))r}{(1-r)^{2(1-\beta(\alpha))}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

where K denotes the complete elliptic integral of the first kind with modulus $2\sqrt{r}/(1+r)$.

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