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SOME PROPERTIES OF THE QUASIASYMPTOTIC OF SCHWARTZ DISTRIBUTIONS PART II: QUASIASYMPTOTIC AT 0

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Abstract. We give the definition of the quasiasymptotic behaviour at 0 of Schwartz distributions from $\mathcal{D}'$ and compare this definition with the definition of the quasiasymptotic of tempered distributions at 0 [**2**].

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1. Definitions. The quasiasymptotic behaviour at 0 of tempered distributions was considered in [**8**] and [**2**]. First we reformulate the definition from [**2**].

Definition 1. Let $f \in \mathcal{S}'$ and $c(x),\ x \in (0,a),\ a > 0$, be a mearurable positive function. It is said that f has (in $\mathcal{S}'$) the quasiasymptotic at 0 with recpect to $c(1/k)$ if there is a $g \in \mathcal{S}', g \neq 0$, such that

$$\lim_{k\to\infty}\Big\langle \frac{f(x/k)}{c(1/k)},\ \varphi(x)\Big\rangle = \langle g(x),\ \varphi(x)\rangle, \quad \varphi \in \mathcal{S}. \tag{1}$$

In this case we write $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{S}'$).

We extend this definition.

Definition 2. Let $f \in \mathcal{D}'$ and c be as in Definition 1. It is said that f has (in $\mathcal{D}'$) the quasiasymptotic at 0 with respect to $c(1/k)$ if there is a $g \in \mathcal{D}',\ g \neq 0$, such that

$$\lim_{k\to\infty}\Big\langle \frac{f(x/k)}{c(1/k)},\ \varphi(x)\Big\rangle = \langle g(x),\ \varphi \in \mathcal{D}. \tag{2}$$

In this case we write $f \sim^q g$ at 0 with recpect to $c(1/k)$ (in $\mathcal{D}'$).

Obviously, if $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{S}'$) then $f \sim^q g$ at 0 with respect to $c(1/k)$ in $\mathcal{D}$.

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THEOREM 1. *Let f and c satisfy conditions of Definition 2. Assume further that c is continuous. Then for some real number ν and some slowly varying function L at 0^+*

$$c(x) = x^\nu L(x),\ x \in (0, a).$$

Moreover, g is homogeneous with the order of homogeneity ν.

(Slowly varying functions are studied in [**4**]).

Proof. Let $f \in \mathcal{D}$ be such that $< g, \varphi > \neq 0$. For any $m > 0$ we have

$$\Big\langle \frac{f(mx/k)}{c(m/k)},\ \varphi(x)\Big\rangle \to \langle g(x),\ \varphi(x)\rangle,\ k \to \infty,$$

$$\Big\langle \frac{f(mx/k)}{c(1/k)},\ \varphi(x)\Big\rangle \to \langle g(mx),\ \varphi(x)\rangle,\ k \to \infty;$$

$$\Big(\frac{c(m/k)}{c(1/k)}\Big)\Big\langle \frac{f(mx/k)}{c(m/k)},\ \varphi(x)\Big\rangle \to \langle g(mx),\ \varphi(x)\rangle,\ k \to \infty. \tag{3}$$

This implies that for any $m > 0$

$$\lim_{k\to\infty} \frac{c(m/k)}{c(1/k)} \text{ exists.} \tag{4}$$

From [**4**, 1.4] it follows that the limit function defined by (4) must be equal to $m^\nu, m > 0$, and that for that $\nu \in R$ and for some function L slowly varying at 0^+ one has

$$c(x) = x^\nu L(x),\ x \in (0, a). \tag{5}$$

Also, (3) implies that $g(mx) = m^\nu g(x),\ m > 0,\ x \in R$, which completes the proof.

This theorem directly implies:

THEOREM 2. *Let f and c satisfy conditions of Definition 1 and let c be continuous. Then the assertion of Theorem 1 holds.*

Some obivous properties of the quasiasymptotic at 0 in $\mathcal{S}'$ are given in the next theorem.

THEOREM 3. *Let $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$). Then:* (i) *if $g^{(m)} \neq 0$ then $f^{(m)} \sim^q g^{(m)}$ at 0 with respect to $(1/k)^{\nu-m}L(1/k)$ (in $\mathcal{S}'$), $m \in N$,* (ii) *$x^m f(x) \sim^q x^m g(x)$ at 0 with respect to $(1/k)^{\nu+m}L(1/k)$ (in $\mathcal{S}'$) if $m \in N$ and $\nu \notin -N$;* (iii) *if $\nu \in -N,\ m \in N$ and $m <| \nu |$, then $x^m f(x) \sim^q x^m g(x)$ at 0 with respect to $(1/k)^{\nu+m}L(1/k)$ (in $\mathcal{S}'$).*

The same assertions hold for the quasiasymptotic at 0 in $\mathcal{D}'$

The quasiasymptotic at 0 is a local property of a distribution. Namely,

THEOREM 4. *Let $f \in \mathcal{D}'$ and $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{D}'$), and let $f_1 \in \mathcal{D}'$ be such that $f = f_1$ in some neighbourhood of zero. Then $f_1 \sim^q g$ at 0 with respect to $(1/k)^\nu \cdot L(1/k)$.*

Proof. Follows from the equality $\langle f(x),\ \varphi(kx)\rangle = \langle f_1(x),\ \varphi(kx)\rangle$ which holds for any $\varphi \in \mathcal{D}$ if $k > k_0(\varphi)$.

The some assertion holds for the quasiasymptotic at 0 (in $\mathcal{S}'$). This was proved in [**2**, Lemma 1.6].

THEOREM 5. *Let $f \in \mathcal{S}'$, resp. $f \in \mathcal{D}'$, and $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{S}'$, resp. in $\mathcal{D}'$). Let $\omega \in \mathcal{S}$, resp. $\omega \in \mathcal{E}$ and*

$$\frac{\omega(x/k)}{c_1(1/k)} \to \omega_0(x) \ \text{ in } \mathcal{S}, \ \text{ resp. in } \mathcal{E}, \ \text{ as } k \to \infty,$$

where $c(x)$, $x \in (0,a)$, $a > 0$, is a measurable positive function. Then $f\omega \sim^q \omega_0 g$ at 0 with respect to $c(1/k)c_1(1/k)$ (in $\mathcal{S}'$, resp. in $\mathcal{D}'$).

Proof. Follows from [**7**, Y. I, p. 72., Théorème X].

2. Relations between two definitions. Let $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $c(1/k)$ (in $\mathcal{D}'$). The question is: Does the same hold in $\mathcal{S}'$? We shall prove in this section that for $c(1/k) = (1/k)^\nu L(1/k),\ k > 1/a$, the answer to the question is affirmative if $\nu > 0$ or if $0 \geq \nu > -1$ and L is bounded in some interval $(0,\eta),\ \eta > 0$. Otherwise the problem is still open.

Theorem 3(i) and [**2**, Lemma 1.7] directly imply the following:

THEOREM 6. *Let $f \in \mathcal{S}'$ and $f = F^{(m)}$ in some neighbourhood of 0, where $m \in N_0$ and F is a locally integrable function such that for some $\nu > -1, L$ and $(C_+, C_-) \neq (0,0)$,*

$$\lim_{x \to \pm 0} \frac{F(x)}{|\,x\,|^\nu\, L(|\,x\,|)} = C_\pm.$$

Then $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$), where $g = (C_+ x_+^\nu + C_- x_-^\nu)^{(m)}$, and $x_\pm^\nu = H(\pm x)\,|\,x\,|^\nu$ (H is the Heaviside function).

The following theorem is proved in [P].

THEOREM 7. *Let $f \in \mathcal{S}, f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{D}'$), where $\nu > 0$. Then there is a continuous function $F(x)$, $x \in (-1,1)$ and an $m \in N_0$ such that $f = F^{(m)}$ in $(-1,1)$ and*

$$\lim_{x \to \pm 0} \frac{F(x)}{|\,x\,|^{\nu+m}\, L(|\,x\,|)} = C_\pm. \ \text{for some } \ (C_+, C_-) \neq (0,0)$$

If $0 \geq \nu > -1$ and $L(x)$ is bounded in some interval $(0,\eta)$, the assertion holds as well.

Theorems 6 and 7 directly imlpy the following

THEOREM 8. *Let f satisfy conditions of Theorem 7. Then $\sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$).*

For $\nu < 0$ we have a partial answer to the question above.

Let us denote by $\mathcal{Z}$ the space of Fourer transformations of elements from $\mathcal{D}$ supplied by the convergence structure transported from $\mathcal{D}(\mathcal{Z} = \mathcal{F}(\mathcal{D}))$. Let $f \in \mathcal{S}'$. We write $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{Z}'$) if $g \in \mathcal{Z}'$, $g \neq 0$, and

$$\lim_{k\to\infty}\Big\langle \frac{f(x/k)}{(1/k)^\nu L(1/k)},\ \varphi(x)\Big\rangle = \langle g(x),\ \varphi(x)\rangle,\ \varphi \in \mathcal{Z}.$$

Using the Fourier Transformations and Theorem I (i) (Part I) one can easily obtain that $g \in \mathcal{S}'$.

THEOREM 9. *Let $f \in \mathcal{S}'$ and $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{Z}'$) where $\nu < 0$, and $\nu \notin -N$. Then $f \sim^q q$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$).*

Proof. Since for some $m \in N_0$ and some continuous function F of slow growth $f = F^{(m)}$, the Fourier transformation implies

$$(-i)^m x^m \hat{F}(x) \sim^q \hat{g}(x) \ \text{ at } \ \pm\infty \ \text{ with respect to } \ k^{-\nu-1}L_1(k)$$

in the sence of convergence in $\mathcal{D}'$ and thus, in the sense of convergence in $\mathcal{S}'$ (see Theorem I, Part I). $L_1(\cdot) = L(1/\cdot)$ is slowly varying at ∞. Let us put

$$\hat{F}_+(x) = \begin{cases} \hat{F}(x), & x > 0 \\ 0, & x \le 0 \end{cases} \quad \hat{F}_-(x) = \begin{cases} 0, & x \ge 0 \\ \hat{F}(x), & x < 0 \end{cases}$$

[**2**, Lemma 2.2] implies that for some $N \in \mathbf{N}$ and some $(C_+, C_-) \in \mathbf{C}^2, (C_+, C_-) \neq (0,0)$

$$x^{m+N}\hat{F}_\pm(x) \sim^q C_\pm f_{-\nu+N}(\pm x) \ \text{ at } \ \pm\infty \ \text{ with respect to } \ k^{-\nu-1+N}L_1(k).$$

Now [2, Lemma 2.3] implies, for $-\nu - m > 0$

(6) $$\hat{F}_\pm(x) \sim^q C_\pm f_{-\nu-m}(\pm x) \ \text{ at } \ \pm\infty \ \text{ with respect to } \ k^{-\nu-1-m}L_1(k).$$

and for $-\nu - m < 0$.

(7)

$$\hat{F}_\pm(x) \sim^q C_\pm f_{-\nu+N}(\pm x) + \sum_{j=0}^{p} a_{j\pm}\delta^{(j)}(x) \ \text{ at } \ \pm\infty \text{ with respect to } k^{-\nu-1-m}L_1(k).$$

Using the Inverse Fourier transformation we obtain: for $-\nu - m > 0$

$$F(t) \sim^q \mathcal{F}^{-1}(C_+ f_{-\nu-m}(t) + C_- f_{-\nu-m}(-t))$$
$$\text{at } 0 \text{ with respect to } (1/k)^{\nu+m}L(1/k) \ \text{ (in } \mathcal{S}');$$

for $-\nu - m < 0$

$$F(t) \sim^q \mathcal{F}^{-1}(\tilde{C}_+ f_{-\nu-m}(t) + \tilde{C}_- f_{-\nu-m}(-t) + \sum_{j=0}^{p} a_{j+}\delta^{(j)}(t) +$$
$$+ \sum_{j-0}^{p} a_j - \delta^{(j)}(t)) \ \text{ at 0 with respect to}$$
$$(1/k)^{\nu+m}L(1/k) \ \text{ (in } \mathcal{S}'),\ ((\tilde{C}+, \tilde{C}-) \neq (0,0).)$$

Now Theorem 3 (i) completes the proof.

The proof of Theorem 9 shows that if ν and m satisfy the condition $-\nu - m > 0$, then the assertion of Theorem 9 holds without the asypmtotion $\nu \notin -\mathbf{N}$.

At the end we give a theorem which is a consequence of Theorem 5 from part I.

THEOREM 10. *Let $f \in \mathcal{S}'$ such that $xf \sim^q g$ at 0 with respect to*

$$(1/k)^{\nu+1}L(1/k), \ \nu \in R\backslash(-\mathbf{N}) \ (\text{in } \mathcal{S}').$$

Let $\varphi_0 \in \mathcal{D}$ such that $\hat{\varphi}_0(0) = 1$ and

$$\left\langle \frac{f(x/k)}{(1/k)^\nu L(1/k)}, \ \hat{\varphi}_0(x) \right\rangle \to \langle g_0(x), \hat{\varphi}_0(x) \rangle \ \text{ as } \ k \to \infty$$

such that $g_0 \in \mathcal{S}'$ an $xg_0(x) = g(x)$, $x \in \mathbf{R}$. Then, $f \sim^q g$ at 0 with respect to $(1/k)^\nu L(1/k)$ (in $\mathcal{S}'$) (g and g_0 are homogeneous of order $\nu+1$ and ν, respectively).

REFERENCES

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