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TENSOR FIELDS AND CONNECTIONS ON CROSS-SECTIONS IN THE FRAME BUNDLE OF SECOND ORDER OF A PARALLELIZABLE MANIFOLD

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Abstract. Let V be a field of global frames on a parallelizable manifold. Then V defines a cross-section in the frame bundle of second order F^2M of M. The behaviour of the lifts of tensor fields and connections on M to F^2M along this cross-section is studied.

Introduction

Let M be an n-dimensional differentiable manifold, TM its tangent bundle and T^2M its tangent bundle of order 2. When a vector field V is gixen on M, then V defines a cross-section in TM and a cross-section in T^2M. The behaviour of the lifts of tensor fields and connections on M to TM and T^2M along the corresponding cross-sections are studied in [**10**] and [**9**], respectively.

When a field of global frames V is given on a parallelizable manifold M, it defines a cross-section in the frame bundle FM of M and cross-section in the frame bundle of second order F^2M of M. The behaviour of the lifts of tensor fields and connections on M to FM along this cross-section is studied in [**1**]. In this paper, we study the behaviour on cross-section in F^2M of lifts of tensor fields and connections on M to F^2M.

In § 1 we first recall some properties of the lifts of tensor fields and connections on M to F^2M.

In § 2 and § 3, we study the lifts of tensor fields on M to F^2M along the cross-section determined by field of global frames on M.

Finally, § 4 will be devoted to the study of the lifts of connections on M to F^2N along this cross-section.

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§ 1. Prolongations of tensor fields and linear connections to the frame bundle of order 2

We shall recall, for later use, some properties of the frame bundle F^2M of order 2 over a differentiable manifold M of dimension n, and those of prolongations of tensor fields and linear connections on M to F^2M (cf. [**2, 3, 4, 5, 8**]).

The frame bundle F^2M of order 2 is the set of all 2-jets of diffeomorphisms of open neighbourhoods of 0 in R^n onto open subsets of M. Let $\pi : F^2 \to M$ be the target projection $\pi(j_0^2\gamma) = \gamma(0)$. Then $\pi : F^2M \to M$ is a prinpal fibre bundle over M with the stuctural group L_n^2 of all 2-jets with the source and with the target at 0 of local diffeomorphisms of R^n.

Let (U, x^h) be a coordinate neighbrohood with the local coordinate system (x^h). A system of local coordinates $(x^h, X_\alpha^h, X_{\alpha\beta}^h)$, $X_{\alpha\beta}^h = X_{\beta\alpha}^h$, $1 \le \alpha,\ \beta \le n$, can be introduced in $\pi^{-1}(U)$ in such a way that a 2-jet $j_0^2\gamma$ with $\gamma(0) \in U$ has coordinates as

$$x^h = x^h \circ \gamma(0),\ X_\alpha^h = \frac{\partial(x^h \circ \gamma)}{\partial t^\alpha}(0),\ X_{\alpha\beta}^h = \frac{\partial^2(x^h \circ \gamma)}{\partial t^\alpha \partial t^\beta}(0), \tag{1.1}$$

where $(t^1, \dots, t^n)$ are the usual coordinates in R^n.

Let (U, x^h) and $\overline{U}, \bar{x}^h)$ be two coordinate neighborhouods of M related by coordinate transformation $\bar{x}^h = \bar{x}^h(x^h)$ in $U \cap \overline{U}$. If we denote by $(x^h, X_\alpha^h, X_{\alpha\beta}^h)$ and $(\bar{x}^h, \overline{X}_\alpha^h, \overline{X}_{\alpha\beta}^h)$ the induced coordinates in $\pi^{-1}(U)$ and $\pi^{-1}(\overline{U})$, respectively, the coordinate transformation in $\pi^{-1}(U) \cup \pi^{-1}(\overline{U})$ is given by

$$\bar{x}^h = \bar{x}^h(x^h),\ \overline{X}_h^\alpha = \frac{\partial \bar{x}^h}{\partial x^k} X_\alpha^k,\ \overline{X}_{\alpha\beta}^h = \frac{\partial \bar{x}^h}{\partial x^r \partial x^s} X_\alpha^r X_\beta^s + \frac{\partial \bar{x}^h}{\partial x^r} X_{\alpha\beta}^r \tag{1.2}$$

We shall denote by $\mathcal{I}_s^r(M)$ (resp., $\mathcal{I}_s^r(F^2M)$) the space of all tensor fields of type (r, s) on M (resp., F^2M).

1.1 Lifts of tensor fields. For any element $f \in \mathcal{I}_0^0(M)$, its lifts $f^0, f^{(\alpha)}$, $f^{(\alpha,\beta)}, f^{(\alpha,\beta)} = f^{(\beta,\alpha)}$, $1 \le \alpha,\ \beta \le n$, to F^2M are elements of $\mathcal{I}_0^0(F^2M)$ given by the following local expressions:

$$f^0 : f(x^h),\ f^{(\alpha)} : X_\alpha^i \partial_i f(x^h),\ f^{(\alpha,\beta)} : X_\alpha^i X_\beta^j \partial_i \partial_j f(x^h) + X_{\alpha\beta}^i \partial_i f(x^h) \tag{1.3}$$

in the induced coordinate system $(x^i, X_\alpha^i, X_{\alpha\beta}^i)$, $f(x^h)$ being the local expression of f in (x^h), where $\partial_i = \partial/\partial x^i$.

For any element $X \in \mathcal{I}_0^1(M)$, its prolongations $X^0, X^{(\alpha)}, X^{(\alpha,\beta)}\ X^{(\alpha,\beta)} = X^{(\beta,\alpha)}$, $1 \le \alpha,\ \beta \le n$, are elements of $\mathcal{I}_0^1(F^2M)$ and have the following properties:

$$\begin{aligned} &X^0 f^0 = (Xf)^0,\ X^0 f^{(\alpha)} = (Xf)^{(\alpha)},\ X^0 f^{(\alpha,\beta)} = (Xf)^{(\alpha,\beta)}, \\ &X^{(\alpha)} f^0 = 0, X^{(\alpha)} f^{(\lambda)} = \delta^{\alpha\lambda}(Xf)^0, X^{(\alpha)} f^{(\lambda,\mu)} = \delta^{\alpha\lambda}(Xf)^{(\mu)} + \delta^{\alpha\mu}(Xf)^{(\lambda)} \\ &X^{(\alpha,\beta)} f^0 = 0,\ X^{(\alpha,\beta)} f^{(\lambda)} = 0,\ X^{(\alpha,\beta)} f^{(\lambda,\mu)} = \delta^{\alpha\lambda}\delta^{\beta\mu}(Xf)^0 \end{aligned} \tag{1.4}$$

f being an arbitrary element of $\mathcal{I}_0^0(M)$, $1 \leq \lambda,\ \mu \leq n$.

For any element τ of $\mathcal{I}_1^0(M)$, its prolongations $\tau^0, \tau^{(\alpha)}, \tau^{(\alpha,\beta)}, \tau(\alpha,\beta) = \tau^{(\beta,\alpha)}$, $1 \leq \alpha,\ \beta \leq n$, are elements of $\tau_1^0(F^2M)$ and have the following properties:

$$
\begin{aligned}
&\tau^0 X^0 = (\tau X)^0,\ \tau^0(X^{(\lambda)}) = 0,\ \tau^0(X^{(\lambda,\mu)}) = 0\\
&\tau^{(\alpha)} X^0 = (\tau X)^{(\alpha)},\ \tau^{(\alpha)}(X^{(\lambda)}) = \delta^{\alpha\lambda}(\tau X)^0,\ \tau^{(\alpha,\beta)}(X^{(\lambda,\mu)}) = 0\\
&\tau^{(\alpha,\beta)} X^0 = (\tau X)^{(\alpha,\beta)},\ \tau^{(\alpha,\beta)}(X^{(\lambda)}) = \delta^{\alpha\lambda}(\tau X)^{(\beta)} + \delta^{\beta\lambda}(\tau X)^{(\alpha)},\\
&\tau^{(\alpha,\beta)}(X^{(\lambda,\mu)}) = \delta^{\alpha\lambda}\delta^{\beta\mu}(\tau X)^0,
\end{aligned}
\tag{1.5}
$$

X being an arbitrary element of $\mathcal{I}_0^1(M)$, $1 \leq \alpha,\ \beta \leq n$.

For any element K of $\mathcal{I}_q^0(M)$ (resp., $\mathcal{I}_q^1(M)$), $q \geq 1$, its prolongations $K^0, K^{(\alpha)}, K^{(\alpha,\beta)}, K^{(\alpha,\beta)} = K^{(\beta,\alpha)}$, $1 \leq \alpha,\ \beta \leq n$, are elements of $\mathcal{I}_q^0(F^2(M))$ (resp., $\mathcal{J}_q^1(F^2(M))$ and are characterized by the following identities (cf. [**3**]):

$$
\begin{aligned}
K^0(X_1^0, \dots, X_q^0) &= (K(X_1, \dots, X_q))^0\\
K^{(\alpha)}(X_1^0, \dots, X_q^0) &= (K(X_1, \dots, X_q))^\alpha\\
K^{(\alpha,\beta)}(X_1^0, \dots, X_q^0) &= (K(X_1, \dots, X_q))^{(\alpha,\beta)}
\end{aligned}
\tag{1.6}
$$

for any vector fields $X_1, \dots, X_q$ on M.

1.2. Lifts of linear connections. Let there be given a linear connection ∇ on M. Then there exists a unique linear connection ∇^0 on F^2M characterized by the following identities:

$$
\begin{aligned}
&\nabla^0_{X^0} Y^0 = (\nabla_X Y)^0,\ \nabla^0_{X^0} Y^{(\alpha)} = \nabla^0_{X^{(\alpha)}} Y^0 = (\nabla_X Y)^{(\alpha)},\\
&\nabla^0_{X^0} Y^{(\alpha,\beta)} = \nabla^0_{X^{(\alpha,\beta)}} Y^0 = (\nabla_X Y)^{(\alpha,\beta)}\\
&\nabla^0_{X^{(\alpha)}} Y^{(\beta)} = (\nabla_X Y)^{(\alpha,\beta)} + (\nabla_X Y)^{(\beta,\alpha)},\\
&\nabla^0_{X^{(\alpha)}} Y^{(\beta,\gamma)} = \nabla^0_{X^{(\alpha,\beta)}} Y^{(\gamma)} = \nabla^0_{X^{(\alpha,\beta)}} Y^{(\gamma\mu)} = 0,
\end{aligned}
\tag{1.7}
$$

for any vector fields X, Y, Z on M, $1 \leq \alpha, \beta, \gamma, \mu \leq n$.

If T and R denote the torsion and curvature tensors of ∇, then the torsion and curvature tensors of ∇^0 are T^0 and R^0, respectively.

Remark. Observe that F^2M is an open subset of the tangent bundle of n^2-velocities T^2M over M (cf. [**3**]). Then the linear connection ∇^0 is nothing but the resctriction to F^2M of the 0-prolongation of ∇ to T_n^2M defined by Morimoto [**8**].

§ 2. Lifts of tensor fields on a cross-section determined by a field of global frames

Let there be given a field of global frames $V = (V_1, \dots, V_n)$ on M, that is, at each point $x \in M$, $(V_1(x), \dots, V_n(x))$ is a linear frame at x. Then each V_α is a vector field globally defined on M. Assume that V_α has local components $V_\alpha^h(x)$ with respect to a coordinate system (U, x^h) in M, that is, $V_\alpha = V_\alpha^h \partial_h$ in U.

If, moreover, ∇ is a torsion-free linear connection on M with local components Γ_{ij}^h, then we can define a cross-section γ_∇ of F^2M locally given by

$$\gamma_\nabla(x^h) = (x^h, V_\alpha^h, \ -\Gamma_{ij}^h V_\alpha^i V_\beta^j). \tag{2.1}$$

Now, let $\overline{\nabla}$ be the flat linear connection associated to the absolute parallelism $V = (V_1, \dots, V_n)$, that is,

$$\overline{\nabla}_X Y = \sum_{\alpha=1}^{n} X(Y^\alpha) V_\alpha, \quad X, Y \in \mathcal{I}_0^1(M), \ Y = Y^\alpha V_\alpha \tag{2.2}$$

As it is well known [7], there exist a unique torsion-free linear connection ∇ with the same geodesics of $\overline{\nabla}$, namely, $\nabla_X Y = \overline{\nabla}_X Y - \overline{T}(X - Y)/2, \overline{T}$ being the torsion of $\bar{\nabla}$. From (2.2), one easily deduces that local components of ∇ are

$$\Gamma_{ij}^h = -1/2 \cdot \{\Lambda_j^\alpha \partial_i V_\alpha^h + \Lambda_i^\alpha \partial_j V_\alpha^h\}, \tag{2.3}$$

(Λ_j^α) being the inverse matrix of (V_α^i).

Then we have a cross-section γ_V of F^2M, which will be said to be associated with V. According to (2.1) and (2.3), γ_V is the n-submanifold of F^2M locally expresed in $\pi^{-1}(U)$ by

$$x^h = x^h, X_\alpha^H = V_\alpha^h(x^s), \ X_{\alpha\beta}^h = 1/2 \cdot \{V_\alpha^i(x^s)\partial_i V_\beta^h(x^s) + V_\beta^i(x^s)\partial_i V_\alpha^h(x^s)\}. \tag{2.4}$$

From (1.3) and (2.4), we have along $\gamma_V(M)$ the equations

$$f^0 - f^0, \ f^{(\alpha)} = \mathcal{L}_{V_\alpha} f, \ f^{(\alpha,\beta)} = 1/2 \cdot \{(\mathcal{L}_{V_\alpha V_\beta} + \mathcal{L}_{V_\beta V_\alpha}) f\}, \tag{2.5}$$

for $f \in \mathcal{I}_0^0(M)$, where $\mathcal{L}_{V_\alpha} f$ denotes the Lie derivative with respect to V and $\mathcal{L}_{V_\alpha V_\beta} = \mathcal{L}_{V_\alpha} \mathcal{L}_{V_\beta}$.

From (2.4) one easily deduces that the n vector fields given with respect to the induced coorinates in F^2M by

$$\begin{aligned} B_i = \partial_i + (\partial_i V_\alpha^h)\partial_{h\alpha} + \\ + 1/2 \cdot (\partial_i V_\alpha^s \partial_s V_\beta^h + V_\alpha^s \partial_s \partial_i V_\beta^h + \partial_i V_\beta^s \partial_s V_\alpha^h + V_\beta^s \partial_s \partial_i V_\alpha^h) \partial h_{\alpha\beta} \end{aligned} \tag{2.6}$$

are tangent to $\gamma_V(M)$, where $\partial_{h_\alpha} = \partial/\partial X_\alpha^h$ and $\partial h_{\alpha\beta} = \partial/\partial X_{\alpha\beta}^h$. For any element X of $\mathcal{I}_0^1(M)$ with local components X^i we denote by BX the vector field on F^2M given in $\pi^{-1}(U)$ by

$$BX = X^i B_i. \tag{2.7}$$

Obviously, BX is tangent to $\gamma_V(M)$ and the correspondence $X \to BX$ determines a mapping $B : \mathcal{J}_0^1(M) \to \mathcal{I}_0^1(\gamma_V(M))$ which is in fact the differential of $\gamma_V : M \to F^2M$ and so an isomorphism of $\mathcal{I}_0^1(M)$ onto $\mathcal{I}_0^1(\gamma_V(M))$.

From (2.6) and (2.7), one easily obtains, for any $X, Y \in \mathcal{I}_0^1(M)$,

$$[BX, BY] = B[X, Y]. \tag{2.8}$$

Let U be a coordinate neighbourhood in M; then the local vector fields $B_i, C_{i_\alpha}, D_{i_{\alpha\beta}}, D_{i_{\alpha\beta}} = D_{i_{\beta\alpha}}$ given by

$$B_i = B(\partial_i),\ C_{i_\alpha} = \partial_{i_\alpha} + (\partial_i V_\beta^k)\partial_{h_{\alpha\beta}} + (\partial_i V_\beta^k)\partial_{h_{\beta\alpha}}, D_{i_{\alpha\beta}} = \partial_{i_\alpha\beta} \tag{2.9}$$

form a local family of frames along $\gamma_V(M)$ which will be called the *adapted frame* of $\gamma_V(M)$ in $\pi^{-1}(U)$.

For each vector field X on M with local components X^i in U, we shall denote by $C_\alpha(X), D_{\alpha\beta}(X),\ D_{\alpha\beta}(X) = D_{\beta\alpha}(X),\ 1 \leq \alpha,\ \beta \leq n$, the vector fields

$$C_\alpha(X) = X^i C_{i_\alpha},\ D_{\alpha\beta}(X) = X^i D_{i_{\alpha\beta}}. \tag{2.10}$$

From (1.4), (2.9) and (2.10), we have along $\gamma_V(M)$

$$\begin{aligned} X^0 &= BX + \sum_{\alpha=1}^{n} C_\alpha(\mathcal{L}_{V\alpha}X) + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} D_{\alpha\beta}(\mathcal{L}_{V_\alpha V_\beta}X + \mathcal{L}_{V_\beta V_\alpha}X),\\ X^{(\alpha)} &= \qquad C_\alpha(X) + \sum_{\beta-1}^{n}\{D_{\alpha\beta}(\mathcal{L}_{V\alpha}X + D_{\beta,\alpha}(\mathcal{L}_{V_\beta}X)\},\\ X^{\alpha\beta} &= \qquad D_{\alpha\beta}(X), \end{aligned} \tag{2.11}$$

for $X \in \mathcal{I}_0^1(M)$, and, therefore

$$\begin{aligned} BX &= X^0 - \sum_{\alpha=1}^{n}(\mathcal{L}_{V\alpha}X)^{(\alpha)} - \frac{1}{2}\sum_{\alpha,\beta=1}^{n}(\mathcal{L}_{V_\alpha V_\beta}X + \mathcal{L}_{V\beta V\alpha}X)^{(\alpha,\beta)},\\ C_\alpha(X) &= \qquad X^{(\alpha)} - \sum_{\beta-1}^{n}\Big\{(\mathcal{L}_{V\alpha}X)^{(\alpha,\beta)} + (\mathcal{L}_{V\alpha}X)^{(\beta,\alpha)}\Big\},\\ D_{\alpha\beta}(X) &= \qquad X^{(\alpha,\beta)}. \end{aligned} \tag{2.12}$$

Then we have

PROPOSITION 2.1. *X^0 is tangent to $\gamma_V(M)$ if only if the Lie derivative of X with respect to V_α vanishes, that is, $\mathcal{L}_{V\alpha}X = 0$, for every $\alpha = 1, \ldots, n$.*

The adapted coframe of $\gamma_V(M)$ in F^2M dual to the adapted frame $\{B_i, C_{i\alpha}, D_{i\alpha\beta}\}$ is easily shown to be given along $\gamma_V(M)$ by

$$\begin{aligned} \eta^i &= dx^i,\ \eta^{i_\alpha} = -(\partial_h V_\alpha^i)dx^h + dX_\alpha^i\\ \eta_{\alpha\beta}^i &= 1/2 \cdot \{\partial_h V_\alpha^t \partial_t V_\beta^i + \partial_h V_\beta^t \partial_t V_\alpha^i - V_\alpha^t \partial_t \partial_h V_\beta^t - V_\beta^t \partial_t \partial_h V_\alpha^i\}dx^h\\ &\quad -\{\partial_h V_\beta^i \delta^{\lambda\alpha} + \partial_h V_\alpha^i \delta^{\lambda\beta}\}dX_\lambda^h + dX_{\alpha\beta}^i. \end{aligned} \tag{2.13}$$

Let τ be an element of $\mathcal{I}_1^0(M)$ with local components τ_i. Then its lifts $\tau^0, \tau^{(\alpha)}, \tau^{(\alpha,\beta)}$ have the components of the form

$$(2.14)\quad \begin{aligned} &\tau^0 = (\tau_h, 0, 0), \ \ \tau^{(\alpha)} = ((\mathcal{L}_{V\alpha}\tau)_h, \ \delta^{\lambda\alpha}\tau_h, 0) \\ &\tau^{(\alpha,\beta)} = (1/2 \cdot \{\mathcal{L}_{V_\alpha V_\beta}\tau + \mathcal{L}_{V_\beta V_\alpha}\tau\}_h, \delta^{\lambda\beta}(\mathcal{L}_{V_\alpha}\tau)_h + \delta^{\lambda\alpha}(\mathcal{L}_{V_\beta}\tau)_h, \delta^{\lambda\alpha}\delta^{\lambda\beta}\tau_h) \end{aligned}$$

respectively, in the adapted coframe.

Then we have

PROPOSITION 2.2. *(i) A necessary and sufficient condition for the (α)-lift $\tau^{(\alpha)}$ of a 1-form τ on M to $F^2(M)$ to be zero for all vector fields tangent to $\gamma_V(M)$ is that the Lie derivative of τ with respect to the vector field V_α vanishes, that is, $\mathcal{L}_{V\alpha}\tau = 0$*

(ii) A necessary and sufficient condition for the (α,β)-lift of a 1-form τ on M to F^2M to be zero for all vector fields tangent to $\gamma_V(M)$ is that $\mathcal{L}_{V_\alpha V_\beta}\tau = -\mathcal{L}_{V\beta V\alpha}\tau$. A sufficient condition is that the Lie derivatives of τ with respect to V_α and V_β vanish, that is, $\mathcal{L}_{V\alpha}\tau = \mathcal{L}_{V_\beta}\tau = 0$.

Using (1.6), (2.9), (2.11), (2.12) and (2.13), we can find components of 0-lift, (α)-lift and (α,β)-lift of any tensor field on M of type $(0,q)$ or $(1,q)$, $q \geq 1$, with respect to the adapted frame. For instance, for an element $G \in \mathcal{I}_2^0(M)$ we have

$$G^0 = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G^{(\alpha)} = \begin{pmatrix} (\mathcal{L}_{V_\alpha}G)_{ij} & \delta^{\eta\alpha}G_{ij} & 0 \\ \delta^{\lambda\alpha}G_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(2.15)
$$G^{(\alpha,\beta)} = \begin{pmatrix} 1/2 \cdot (\mathcal{L}_{V_\alpha V_\beta}G + \mathcal{L}_{V_\beta V_\alpha}G)_{ij} & \delta^{\alpha\eta}(\mathcal{L}_{V_\beta}G)_{ij} + \delta^{\beta\eta}(\mathcal{L}_{V_\alpha}G)_{ij} & \delta^{\alpha\eta}\delta^{\beta\gamma}G_{ij} \\ \delta^{\alpha\lambda}(\mathcal{L}_{V_\beta}G)_{ij} + \delta^{\beta\lambda}(\mathcal{L}_{V_\alpha}G)_{ij} & \delta^{\alpha\lambda}\delta^{\beta\eta}G_{ij} + \delta^{\alpha\eta}\delta^{\beta\lambda}G_{ij} & 0 \\ \delta^{\alpha\lambda}\delta^{\beta\mu}G_{ij} & 0 & 0 \end{pmatrix}$$

G_{ij} being the local components of G.

For an element F of $\mathcal{J}_1^1(M)$ we obtain

(2.16)
$$F^0 = \begin{pmatrix} F_{ij} & 0 & 0 \\ \delta^{\alpha\lambda}(\mathcal{L}_{V_\alpha}F)^i_j & \delta^{\lambda\eta}F^i_j & 0 \\ 1/2 \cdot \delta^{\lambda\alpha}\delta^{\mu\beta}(\mathcal{L}_{V_\alpha V_\beta}F + \mathcal{L}_{V_\beta V_\alpha}F)^i_j & \delta^{\mu\eta}(\mathcal{L}_{V\lambda}F)^i_j + \delta^{\lambda\eta}(\mathcal{L}_{V_\mu}F)^i_j & \delta^{\lambda\eta}\delta^{\mu\gamma}F^i_j \end{pmatrix}$$

$$F^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 \\ \delta^{\lambda\alpha}F^i_j & 0 & 0 \\ \delta^{\lambda\alpha}(\mathcal{L}_{V_\mu}F)^i_j + \delta^{\mu\alpha}(\mathcal{L}_{V_\lambda}F)^i_j & \delta^{\alpha\lambda}\delta^{\mu\eta}F^i_j + \delta^{\alpha\mu}\delta^{\lambda\eta}F^i_j & 0 \end{pmatrix}$$

$$F^{(\alpha,\beta)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta^{\lambda\alpha}\delta^{\mu\beta}F^i_j & 0 & 0 \end{pmatrix}$$

F^i_j being the local components of F.

For an element of S of $\mathcal{I}^1_2(M)$, we have

$$
(2.17)\quad
\begin{aligned}
&(S^0)^i_{jk} = S^i_{jk}, (S^0)^{i_\lambda}_{jk} = (\mathcal{L}_{V_\lambda} S)^i_{jk}, (S^0)^{i_{\lambda\mu}}_{jk} = 1/2 \cdot (\mathcal{L}_{V_\lambda V_\mu} S + \mathcal{L}_{V_\mu V_\lambda} S)^i_{jk}\\
&(S^0)^{i_\lambda}_{j_\mu k} = (S^0)^{i_\lambda}_{jk_\mu} = \delta^{\lambda\mu} S^i_{jk}\\
&(S^0)^{i_{\lambda\mu}}_{j_\eta k} = (S^0)^{i_{\lambda\mu}}_{jk_\eta} = \delta^{\lambda\eta}(\mathcal{L}_{V_\mu} S)^i_{jk} + \delta^{\mu\eta}(\mathcal{L}_{V_\lambda} S)^i_{jk}\\
&(S^0)^{i_{\lambda\mu}}_{j_\eta k_\gamma} = \delta^{\lambda\eta}\delta^{\mu\gamma} S^i_{jk} + \delta^{\lambda\gamma}\delta^{\mu\eta} S^i_{jk},\ \ (S^0)^{i_{\lambda\mu}}_{j_{\eta\gamma} k} = (S^0)^{i_{\lambda\mu}}_{jk_{\eta\gamma}} = \delta^{\lambda\eta}\delta^{\mu\gamma} S^i_{jk}
\end{aligned}
$$

and the rest of the components are equal to zero, S^i_{jk} being the local componenets of S.

§ 3. Lifts of tensor fields of type (1, 1) and of type (0, 2) on a cross-section

3.1. Lifts of tensor fields of type (1, 1). Let $F \in \mathcal{I}^1_1$ with local components F^i_j. Then, from (2.11) and (2.16), we have along $\gamma_V(M)$ that

(3.1)

$$
\begin{aligned}
F^0(BX) &= B(FX) + \sum_{\alpha=1}^{n} C_\alpha\Big((\mathcal{L}_{V_\alpha} F)X\Big) + 1/2 \sum_{\alpha=1}^{n} D_{\alpha\beta}((\mathcal{L}_{V\alpha v\beta} F + \mathcal{L}_{V_\beta V_\alpha} F)X)\\
F^{(\alpha)}(BX) &= C_\alpha(FX) + \sum_{\lambda,\mu=1}^{n} D_{\lambda\mu}(\delta^{\lambda\alpha}(\mathcal{L}_{V\mu} F)X + \delta^{\mu\alpha}(\mathcal{L}_{V_\lambda} F)X)\\
F^{(\alpha,\beta)}(BX) &= D_{\alpha\beta}(FX)
\end{aligned}
$$

for any vector field X on M.

When $F^0(BX)$ is tangent to $\gamma_V(M)$ for any vector field X on M, F^0 is said to leave $\gamma_V(M)$ invariant. Thus we have from (3.1).

PROPOSITION 3.1. *F^0 leaves $\gamma_V(M)$ invariant if and only if $\mathcal{L}_{V\alpha} F = 0$ for every $\alpha = 1, \dots, n$. The lifts F^α and $F^{(\alpha,\beta)}$, $1 \leq \alpha, \beta \leq n$, do not have $\gamma_V(M)$ invariants unless $F = 0$.*

Now, assume F^0 leaves $\gamma_V(M)$ invariantr. Then we can define an element $(F^0)^\# \in \mathcal{I}^1_1(\gamma_V(M))$ by

$$
(3.2)\qquad (F^0)^\#(BX) = F^0(BX) = B(FX)
$$

for arbitrary $X \in \mathcal{I}^1_0(M)$; $(F^0)^\#$ is called the tensor *field induced* on $\gamma_V(M)$ from F^0.

Let us now recall from [**3**] that if F is a polynomial structure of rank r and structural polynomial $P(t)$ (i. e., rank $F = r$ and $P(F) = 0$) then its 0-lift F^0 to F^2M defines on F^2M a polynomial structure with the same structural polynomial and with rank $F^0 = r(1 + n + n(n+1)/2)$. Moreover, if N_F and N_{F^0} denote the Nijenhuis tensor of F and F^0, respectively, then $(N_F)^0 = N_{F^0}$

So, if F defines on M a polynomial structure of rank r and $P(F) = 0$, and if F^0 leaves $\gamma_V(M)$ invariant, then $(F^0)^\#$ satisfies $P((F^0)^\#) = 0$ and the rank of

$(F^0)^\# = r$, and hence, $(F^0)^\#$ defines on $\gamma_V(M)$ a polynomial structure of the same type.

Taking into account (2.11) and (2.17), one obtains

$$(3.3)\quad \begin{aligned}(N_F)^0(BX,BY) =&B(N_F(X,Y)) + \sum_{\alpha=n}^{n} C_\alpha((\mathcal{L}_{V_\alpha}N_F)(X,Y))+ \\ &+\frac{1}{2}\sum_{\alpha,\beta=1}^{n} D_{\alpha\beta}((\mathcal{L}_{V_\alpha V_\beta}N_F + \mathcal{L}_{V_\beta V_\alpha}N_F)(X,Y))\end{aligned}$$

along $\gamma_V(M)$, for any $X, Y \in \mathcal{I}_0^1(M)$. Thus

PROPOSITIONS 3.2. *$N_{F^0}(BX,BY)$ is tangent to $\gamma_V(M)$ for arbitrary elements $X, Y \in \mathcal{I}_0^1(M)$ if and only if $\mathcal{L}_{V_\alpha}N_F = 0$ for every $\alpha = 1,\ldots,n$.*

Now, we assume that F^0 leaves $\gamma_V(M)$ invariant. Then from (2.8) and (3.2) we obtain

$$N_{F^0}(BX,BY) = N_{(F^0)^\#}(BX,BY)$$

for arbitrary $X, Y \in \mathcal{I}_0^1(M)$. Then, since $\mathcal{L}_{V_\alpha}F = 0$ implies $\mathcal{L}_{V_\alpha}N_F = 0$, from (3.3) we have

PROPOSITION 3.3. *Suppose that the 0-lift of F^0 of F to F^2M leaves $\gamma_V(M)$ invariant. Then $N_{(F^0)^\#} = 0$ if and only if $N_F = 0$.*

Next, let us suppose that $F \in \mathcal{I}_i^i(M)$ defines an almost complex structure on M, i.e. $F^2 = -I$. Then, F^0 defines an almost complex structure on F^2M. Recall that a submanifold in an almost complex manifold with structure F is said to be invariant or almost analytic when F leaves the submanifold invariant. Thus, from the previous propositions, we deduce

PROPOSITION 3.4. *$\gamma_V(M)$ is almost analytic in the almost complex manifold F^2M with structure F^0 if and only if each vector field V_α is almost analytic, that is, $\mathcal{L}_{V_\alpha}F = 0$. In this case, $\gamma_V(M)$ is an almost complex manifold with structure tensor $F^0)^\#$; moreover $N_{(F^0)^\#} = 0$, that is, $(F^0)^\#$ is complex analytic, if and only if F is complex analytic, that is, $N_F = 0$.*

Let $X \in \mathcal{I}_0^1(M)$ and $F \in \mathcal{I}_1^1(M)$ such that F^0 leaves $\gamma_V(M)$ invariant. Then, $(\mathcal{L}_{BX}(F^0)^\#)(BY) = B((\mathcal{L}_XF)Y)$ for any $Y \in \mathcal{I}_0^1(M)$. Therefore,

PROPOSITION 3.5. *Let F be an almost complex structure on M such that F^0 leaves $\gamma_V(M)$ invariant. Then, for any $X \in \mathcal{I}_0^1(M), BX$ is almost analytic in $\gamma_V(M)$ if and only if X is almost analytic in M.*

3.2. Lifts of tensor fields of type (0, 2).. Let G be a tensor field of type (0, 2) on M. Then, from (2.15) we have along $\gamma_V(M)$.

$$(3.4)\quad \begin{aligned}G^0(BX,BY) &= (G(X,Y))^0 \\ G^{(\alpha)}(BX,BY) &= \{(\mathcal{L}_{V_\alpha}G)(X,Y)\}^0 \\ G^{(\alpha,\beta)}(BX,BY) &= \{1/2(\mathcal{L}_{V_\alpha V_\beta}G + \mathcal{L}_{V_\beta V_\alpha}G)(X,Y)\}^0\end{aligned}$$

for all vector fields X, Y on M, $1 \leq \alpha,\ \beta \leq n$. Then, putting

$$(G^0)^\#(BX, BY) = G^0(BX, BY), (G^{(\alpha)})^\#(BX, BY) = G^{(\alpha)}(BX, BY)$$
$$(G^{(\alpha,\beta)})^\#(BX, BY) = G^{(\alpha,\beta)}(BX, BY)$$

we have elements $(G^0)^\#, (G^{(\alpha)})^\#, (G^{(\alpha,\beta)})^\# \in \mathcal{I}_2^0(\gamma_V(M))$.

If G is a Riemann metric on M, then from (3.4) we deduce

PROPOSITION 3.6. *$\gamma_V(M)$ is a Riemann manifold with metric $(G^0)^\#$ and the projection $\pi : F^2M \to M$ is an isometry.*

Next, assume that $G \in \mathcal{I}_0^2(M)$ is a 2-form; then, $(G^0)^\#$ is a 2-form on $\gamma_V(M)$, and a straightforward computation shows the identity

$$d(G^0)^\#(BX, BY \cdot BZ) = (dG(X, Y, Z))^0$$

along $\gamma_V(M)$, for every $X, Y, Z \in \mathcal{I}_0^1(M)$. Therefore,

PROPOSITION 3.7. *$(G^0)^\#$ is closed along $\gamma_V(M)$ if and only if G is closed.*
Since rank $(G^0)^\#$ along $\gamma_V(M)$ is equal to rank G on M, we easily deduce.

COROLLARY 3.8. *$\gamma_V(M)$ is a symplectic manifold with respect to $(G^0)^\#$ if and only if M is a symplectic manifold with respect to G.*

For an arbitrary $G \in \mathcal{I}_2^0(M)$, we have along $\gamma_V(M)(\mathcal{L}_{BX}(G^0)^\#)(BY, BZ) = ((\mathcal{L}_X G)(Y, Z))^0$ for any $X, Y, X \in \mathcal{I}_0^1(M)$. Therefore

COROLLARY 3.9. *i) Under the hypothesis of Proposition* 3.6*, a vector field X on M is Killing for the metric G on M if and only if BX is Killing for the metric $(G^0)^\#$ on $\gamma_V(M)$.*

ii) Under the hypothesis of Corollary 3.8*, a vector field X on M is an infinitesimal symplectic authomorphism with respect to G on M if and only if BX is such an automorphism with respect to $(G^0)^\#$ on M.*

§ 4. Linear connections induced on γ_V(M)

Let M be a manifold with a linear connection ∇. Then the frame bundle of second order $F^2(M)$ of M is a manifold with linear connection ∇^0. We now study the linear connection ∇', induced from ∇^0 on $\gamma_V(M)$.

From (1.7) and (2.11) trough a direct computation we get along $\gamma_V(M)$

$$\nabla^0_{B_i} B_j = \Gamma^h_{ij} B_h + \sum_{\alpha=1}^{n} (\mathcal{L}_{v_\alpha} \nabla)^h_{ij} C_{h_\alpha} + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} (\mathcal{L}_{V_\alpha V_\beta} \nabla + \mathcal{L}_{V_\beta V_\alpha} \nabla)^h_{ij} D_{h_{\alpha\beta}}$$

$$\nabla^0_{B_i} C_{j\alpha} = \Gamma^h_{ij} C_{h_\alpha} + \sum_{\beta=1}^{n} \{ (\mathcal{L}_{V_\beta} \nabla)^h_{ij} D_{h_{\alpha\beta}} + (\mathcal{L}_{V_\beta} \nabla)^h_{ij} D_{h_{\beta\alpha}} \} \tag{4.1}$$

$$\nabla^0_{B_i} D_{j_{\alpha\beta}} = \Gamma^h_{ij} D_{h_{\alpha\beta}}$$

where Γ_{ij}^h are the components of ∇. Therefore

$$\nabla'_{B_i} B_j = \Gamma_{ij}^h B_h$$

defines the induced linear connection ∇' on $\gamma_V(M)$, and

$$\nabla^0_{B_i} B_j = \nabla'_{B_i} B_j + \sum_{\alpha=1}^{n} (\mathcal{L}_{V_\alpha} \nabla)_{ij}^h C_{h_\alpha} + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} (\mathcal{L}_{V_\alpha V_\beta} \nabla + \mathcal{L}_{V_\beta V_\alpha} \nabla)_{ij}^h D_{h_{\alpha\beta}}$$

is the Gauss formula for $\gamma_V(M)$.

PROPOSITION 4.1. *$\gamma_V(M)$ is autoparallel with respect to ∇^0 if and only if each V_α, $1 \leq \alpha \leq n$, is an infinitesimal affine transformation on M, i.e. $\mathcal{L}_{V_\alpha} \nabla = 0$, for any $\alpha = 1, \dots, n$.*

Now we recall that if R is the curvature tensor of ∇, then the cutvature tensor of ∇^0 is R^0. Using (1.7), (2.11) and (2.12) we obtain along $\gamma_V(M)$.

$$R^0(BX, BY)BZ = B(R(X,Y)Z) + \sum_{\alpha=1}^{n} C_\alpha((\mathcal{L}_{V_\alpha} R)(X,Y,Z)) + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} D_{\alpha\beta}((\mathcal{L}_{V_\alpha V_\beta} R + \mathcal{L}_{V_\beta V_\alpha} R)(X,Y,Z))$$

for all vector fields X, Y, Z on M.

Then we have

PROPOSITION 4.2. *Let R be the curvature tensor of a linear connection ∇ on M. Then, for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ tangent to $\gamma_V(M)$, $R^0(\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ is tangent to $\gamma_V(M)$ if and only if $\mathcal{L}_{V_\alpha} R = 0$, for $\alpha = 1, \dots, n$.*

Let $F \in \mathcal{I}_0^1(M)$ be such that F^0 leaves $\gamma_V(M)$ invariant. Then, along $\gamma_V(M)$ we obtain $\nabla'_{BX}(F^0)^\#(BY) = B((\nabla_X F)Y)$, for any $X, Y \in \mathcal{I}_0^1(M)$. Therefore

PROPOSITION 4.3. *Let $F \in \mathcal{I}_1^1(M)$ be such that F^0 leaves $\gamma_V(M)$ invariant. Then $\nabla'(F^0)^\# = 0$ if and only if $\nabla'(F^0)^\# = 0$.*

Let $G \in \mathcal{I}_2^0(M)$. Then we obtain along $\gamma_V(M)$.

$$(\nabla'_{BX}(G^0)^\#)(BY, BZ) = \{(\nabla_X G)(Y,Z)\}^0 \text{ for any } X, Y, Z \in \mathcal{I}_0^1(M).$$

Therefore, using Propositions 3.6. and 3.7 and Corollary 3.9, we deduce

PROPOSITION 4.4. *i) Let G be a Riemann metric on M and ∇ its Riemann connection. Then, the connection ∇', induced on $\gamma_V(M)$ from ∇^0, is the Riemann connection constructed from the metric $(G^0)^\#$ induced on $\gamma_V(M)$ from G^0.*

ii) Let G be an almost symplectic (resp., symplectic) 2-form on M and ∇ an adapted connection, i.e. $\nabla G = 0$. Then, the linear connection ∇', induced on $\gamma_V(M)$ from ∇^0, is adapted with respect to the almost symplectic (resp., symplectic) from $(G^0)^\#$ induced from G^0 on $\gamma_V(M)$.

Now, let $F \in \mathcal{I}_1^1(M)$ and $G \in \mathcal{I}_2^0(M)$ such that F^0 leaves $\gamma_V(M)$ invariant. Then, along $\gamma_V(M)$.

$$(G^0)^\#((F^0)^\#(BX),\ (F^0)^\#(BY)) = (G^0)^\#(B(FX),\ B(FY)) = \{G(FX,FY)\}^0,$$

for all vector fields Y, Y on M.

If a Riemann metric G and a complex structure F on M satisfy the conditions $G(FX,FY) = G(X,Y)$, $\nabla_X F = 0$, for all vector fields X, Y, ∇ being the Riemann connection determined by G, then (F,G) is a Kahlerian structure. Thus, taking into accound the previous results, we have

PROPOSITION 4.5. *Let* (F,G) *be a Kahlerian structure on* M *such that* F^0 *leaves* $\gamma_V(M)$ *invariant. Then* $((F^0)^\#),\ (G^0)^\#)$ *is a Kahlerian structure on* $\gamma_V(M)$.

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