

**APPROXIMATION OF CONTINUOUS FUNCTIONS
BY MONOTONE SEQUENCES OF POLYNOMIALS
WITH RESTRICTED COEFFICIENTS**

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Abstract. The problem of approximation by polynomials with restricted coefficients, is considered in several papers (e.g. [7-9]). In [1-6], I have proved among other things, that every $f \in C_{[0,1]}$ can be approximated uniformly by a polynomial sequence $(P_n)_n$ such that $(P_n)_n$ is monotonically decreasing on $[0, 1]$. The aim of this paper is to extend the ideas of [1-6] to the case of approximation by polynomials with restricted coefficients.

1. Introduction

Let $C_0[0, 1] = \{f \in C_{[0,1]}; f(0) = 0\}$ and let $A = \{A_k\}_{k \geq 1}$ be a sequence of positive real numbers and

$$P_A = \left\{ p = \sum_{k=1}^n a^k x^k; n \in N, |a_k| \leq A_k^k, \text{ for all } k = 1, 2, \dots \right\}.$$

The problem of approximation in the space $C_0[0, 1]$ by polynomials $P_n \in P_A$, is treated in several papers, (see e.g. [7-9]), where it is proved for example [9] that P_A is dense in $C_0[0, 1]$ iff there is a subsequence of natural numbers $\{k_i\}_i$ satisfying the properties

$$\sum_{i=1}^{\infty} 1/k_i = +\infty \text{ and } \lim_{i \rightarrow \infty} A_{k_i} = +\infty.$$

In several papers ([1-6]) I have proved, among other things, that every $f \in C_{[0,1]}$ can be approximated uniformly by a polynomial sequence $(P_n)_n$, monotonically decreasing on $[0, 1]$.

In the present paper we will extend the ideas of [1-6] to the case of approximation in $C_0[0, 1]$ by polynomials from P_A .

2. Basic Result

Let us consider $\{A_k\}_{k \geq 1}$ which satisfies:

$$1 \leq A_1 \leq A_2 \leq \dots \leq A_k \leq A_{k+1} \leq \dots, \lim A_i = +\infty. \quad (1)$$

Then, by [9], P_A is dense in $C_0[0, 1]$. Now, let us denote by

$$C_0^1[0, 1] = \{f \in C_0[0, 1] : f'(0) = 0\}.$$

THEOREM 2.1. *For every $f \in C_0^1[0, 1]$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \dots$, such that $P_n \rightarrow f$ uniformly on $[0, 1]$ and $f(0) = P_n(0)$, $f(x) < P_{n+1}(x) < P_n(x)$, for all $x \in (0, 1]$ and all $n = 1, 2, \dots$.*

Proof. Take $F(x) = f(x)/x$, $x \in (0, 1]$, $F(0) = 0$. Since $f \in C_0^1[0, 1]$ it is evident that $F \in C_0[0, 1]$. Then, P_A is dense in $C_0[0, 1]$ and therefore there exists a polynomial sequence $(R_n)_n$, $R_n \in P_A$, $n = 1, 2, \dots$, such that $|F(x) - R_n(x)| < 1/[n(n+1)]$, for all $x \in [0, 1]$, and $n = 1, 2, \dots$. Hence:

$$|f(x) - xR_n(x)| \leq x/[n(n+1)], \text{ for all } x \in [0, 1], \text{ and } n = 1, 2, \dots \quad (2)$$

Take $Q_n(x) = xR_n(x)$ and $S_n(x) = Q_n(x) + 2x/n$, $x \in [0, 1]$, $n = 1, 2, \dots$. From (2) it is evident that $Q_n \rightarrow f$ uniformly and therefore $S_n \rightarrow f$ uniformly on $[0, 1]$. Then by (2) we obtain:

$$\begin{aligned} |Q_n(x) - Q_{n+1}(x)| &\leq |Q_n(x) - f(x)| + |f(x) - Q_{n+1}(x)| \leq \\ &\leq x/[n(n+1)] + x/[(n+1)(n+2)] \leq 2x/[n(n+1)], \end{aligned}$$

for all $x \in [0, 1]$ and all $n = 1, 2, \dots$, and therefore

$$S_n(x) - S_{n+1}(x) = Q_n(x) - Q_{n+1}(x) + 2x/[n(n+1)] > 0$$

for all $x \in (0, 1]$ and $S_n(0) = S_{n+1}(0) = f(0)$ for all $n = 1, 2, \dots$.

Now, let $n \in \mathbb{N}$ be fixed and $R_n(x) = \sum_{k=1}^{j_n} a_k x^k \in P_A$; therefore $|a_k| \leq A_k^k$, for all $k = \overline{1, j_n}$. Then $xR_n(x) = \sum_{k=1}^{j_n} a_k x^{k+1} = \sum_{i=2}^{j_n+1} a_{i-1} x^i$. We have:

$$|a_{i-1}| \leq A_{i-1}^{i-1} \leq A_i^{i-1} \leq A_i^i, \quad i = \overline{2, j_n+1};$$

therefore, it is evident that $xR_n(x) = Q_n(x) \in P_A$, $n = 1, 2, \dots$. But $S_n(x) = \frac{2}{n}x + \sum_{i=2}^{j_n+1} a_{i-1} x^i$ and it is evident $2/n \leq A_1$ for all $n \geq 2$. Hence $S_n \in P_A$ for all $n \geq 2$ and therefore it is evident that $P_n(x) = S_{n+1}(x)$, $n = 1, 2, \dots$, satisfies the conclusion of Theorem 2.1.

Remark. If in the previous proof we consider $S_n(x) = Q_n(x) - 2x/n$, then it is easily seen that $(S_n)_n$ is monotonically increasing on $(0, 1]$, $n = 2, 3, \dots$

3. Some Extensions

Let us consider $m, r_k \in \mathbb{N}$, $k = \overline{1, m}$ and the numbers $D_i^{(k)} \in \mathbb{R}$, $i = \overline{0, r_k}$, $k = \overline{1, m}$ satisfying

$$\sum_{i=0}^{r_k} D_i^{(k)} = 0, \quad \sum_{i=0}^{r_k} |D_i^{(k)}| > 0 \text{ for all } k = \overline{1, m}. \quad (3)$$

Also, let $A = \{A_k\}_{k \geq 1}$ satisfy (1). We will prove result analogous to Theorem 1.2 of [4]:

THEOREM 3.1. *If there is a sequence $(\alpha_n)_n$, $\alpha_n \in R$, $\alpha_n \searrow 0$ satisfying*

$$\sum_{i=0}^{r_k} D_i^{(k)} \alpha_{n+r_k-i} > 0 \text{ for all } k = \overline{1, m} \text{ and } n \in N \quad (4)$$

then for every $f \in C_0^1[0, 1]$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \dots$, such that $P_n \rightarrow f$ uniformly on $[0, 1]$ and

$$\begin{aligned} \sum_{i=0}^{r_k} D_i^{(k)} P_{n+r_k-i}(x) &> 0, \text{ for all } x \in (0, 1], k = \overline{1, m}, n \in N \\ P_n(0) &= f(0), \text{ for all } n = 1, 2, \dots \end{aligned} \quad (5)$$

Proof. By Lemma 1.1 of [4] we can write:

$$\begin{aligned} \sum_{i=0}^{r_k} D_i^{(k)} \alpha_{n+r_k-i} &= \sum_{i=0}^{r_k-1} C_i^{(k)} (\alpha_{n+r_k-i} - \alpha_{n+r_k-i-1}) \\ \text{with } C_i^{(k)} &= \sum_{j=0}^i D_j^{(k)}, i = \overline{1, r_k-1}, k = \overline{1, m}. \end{aligned}$$

Then, as in the proof of Theorem 1.2 of [4], take:

$$\begin{aligned} a_n &= \min \left\{ \sum_{i=0}^{r_k-1} C_i^{(k)} (\alpha_{n+r_k-i} - \alpha_{n+r_k-i-1}); k = \overline{1, m} \right\}, \\ D &= \max \left\{ \sum_{i=0}^{r_k-1} |C_i^{(k)}|; k = \overline{1, m} \right\} > 0, \gamma_m = a_n / (2D), n \in N \end{aligned}$$

and let us consider $\beta_n < \gamma_n$, $n \in N$, $\beta_n \searrow 0$. Now, for $F(x) = f(x)/x$, $x \in (0, 1]$, $F(0) = 0$, $F \in C_0[0, 1]$, there exists a polynomial $(R_n)_n$, $R_n \in P_A$ such that $|F(x) - R_n(x)| < \beta_n$, for all $x \in [0, 1]$, and all $n = 1, 2, \dots$. Hence $|f(x) - xR_n(x)| \leq x\beta_n$, for all $x \in [0, 1]$ and all $n = 1, 2, \dots$.

Take $Q_n(x) = xR_n(x)$ and $P_n(x) = Q_n(x) + x\alpha_n$. Then, reasoning exactly as in the proof of Theorem 1.2 of [4], (5) follows immediately. Also, as in the proof of Theorem 2.1 (because there is an $n_0 \in N$ such that $\alpha_n \leq A_1$, for all $n \geq n_0$), $P_n \in P$, for all $n \geq n_0$ and therefore the sequence $P_{n+n_0}(x)$, $n = 1, 2, \dots$, satisfies the conclusion of Theorem 3.1.

As in immediate corollary (analogous to Corollary 1.3 of [4]) we obtain the following:

COROLLARY 3.2. *For any $f \in C_0^1[0, 1]$ and any $r_0 \in N$, there exists a polynomial sequence $P_n \in P_A$, $n = 1, 2, \dots$, uniformly convergent to f on $[0, 1]$*

which satisfies

$$(1-)^{r_k} \sum_{i=0}^{r_k} (-1)^i \binom{r_k}{i} P_{n+r_k-i}(x) > 0, \text{ for all } x \in (0, 1], n \in N,$$

$$r_k \in N, r_k \leq r_0, k = \overline{1, m} \text{ and } P_n(0) = f(0) \text{ for all } n = 1, 2, \dots$$

Remark. (1) If $r_0 = 2$, from the previous inequality we obtain that $P_{n+1}(x) < P(x)$ and $P_{n+2}(x) - 2P_{n+1}(x) + P_n(x) > 0$, (namely $(P_n)_n$ is "convex" on $[0, 1]$), and $P_n(0) = f(0)$ for all $n = 1, 2, \dots$

Added in proof. (2) These results do not remain valid for all $f \in C_0[0, 1]$. One such example is the following. Let us denote $D(f; x) = [f(x) - f(0)]/x$, $(D^*f)(0) = \limsup_{x \downarrow 0} D(f; x)$ and let $f : [0, 1] \rightarrow R$ be defined by $f(0) = 0$, $f(x) = -1/\ln x$ if $x \in (0, 1/2]$ and $f(x) = 1/\ln 2$ if $x \in (1/2, 1]$, where $\ln(x)$ represents the hyperbolic logarithm of x . We have: $\lim_{x \downarrow 0} f(x) = -1/-\infty = 0$ and $\lim_{x \uparrow 1/2} f(x) = 1/\ln 2$ which implies $f \in C_0[0, 1]$, but f does not satisfy Theorem 2.1.

Indeed, if Theorem 2.1 held for the function f so defined, it would follow that there exists a sequence $P_n \in P_A$, $n \in N$ such that $P_n \rightarrow f$ uniformly on $[0, 1]$ and $0 = f(0) = P_n(0)$, $f(x) < P_{n+1}(x) < P_n(x)$, $\forall x \in (0, 1]$, $\forall n \in N$. Hence we would obtain $D(f; x) = f(x)/x < P_n(x)/x = D(P_n; x) \forall x \in (0, 1]$, therefore $(D^*f)(0) \leq (D^+P_n)(0) = P'_n(0)$, $n = 1, 2, \dots$. But $D(f; x) = -1/[x \cdot \ln x] \forall x \in (0, 1/2)$ which immediately implies $(D^+f)(0) = +\infty$ choosing for example $x_m \searrow 0$, $x_m = e^{-m}$, $m \in N$) contradicting the inequalities $(D^+f)(0) \leq P'_n(0)$, $n \in N$.

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