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FIXED POINT THEOREMS FOR PAIRS OF SELFMAPS ON A METRIC SPACE

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Abstract. An attempt is made to find out conditions an the orbits of a pair of selfmaps on a metric space so as to ensure the existence of (common) fixed points when the maps satisfy a variety of generalized contraction conditions governed by a control function.

We obtain fixed points theorems for two selfmaps on a metric space and derive certain results of Ding [1] and Fisher [2] as corollaries.

In Section 2, we provide a number of examples to give insight into the results discussed in Section 1.

Throughout this paper:

(X, d) is a metric space;
f, g are selfmaps on X;
i, j, r, s, m, n are nonnegative integers;
for any selfmap h on X and x in X, $O_h(x) = \{h^n x \mid n = 0, 1, 2, \dots\}$;
for any subset A of X, $\delta(A) = \sup\{d(x, y) \mid x, y \in A\}$;
for x, y in X, $\alpha(x, y) = \delta(O_f(x) \cup O_g(y))$ and
$\beta(x, y) = \sup\{d(f^i x, g^i y) \mid i \geq 0,\ j \geq 0\}$;
R^+ is the set of all nonnegative real numbers; and
$\varphi : [0, \infty] \to [0, \infty]$ as an increasing function.

Definition. A selfmap h on (X, d) is said to be orbitally continuous at $z \in X$ if $hz = z$, when $\{h^n x\}$ converges to z for some x in X.

Section 1. We begin with:

LEMMA 1. *If* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for every* t *in* $(0, \infty)$, *then* $\varphi^2(t+) \leq \varphi(t) < t$ *for every* t *in* $(0, \infty)$ *and* $\varphi(0+) = 0$.

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THEOREM 1. *Suppose that*

$$\inf_{1\le n<\infty} \beta(f^n x, g^n y) \le \varphi(\alpha(x,y)) \tag{I}$$

for all x, y in X, where $\varphi(t+) < t$ for every t in $(0,\infty)$. Suppose also that there is an x_0 in X such that $\{d(f^n x_0, g^n x_0)\}$ converges to zero and one of the sequences $\{f^n x_0\}$ and $\{g^n x_0\}$ is bounded. Then $\{f^n x_0\}$ and $\{g^n x_0\}$ are Cauchy sequences and if one of them converges, then the other also converges to the same limit. Furthermore, if either f or g has a fixed point w, then the two sequences converge to w. Suppose that $\{f^n x_0\}$ converges to some z in X. Then z is a fixed point of $f(g)$ if f or f^2 (g or g^2) is orbitally continuous at x.

Proof. Let $\alpha_n = \alpha(f^n x, g^n x_0)$ $(n = 0,1,2,\ldots)$. Since $\{d(f^n x_0, g^n x_0)\}$ converges to zero and one of the sequences $\{f^n x_0\}$ and $\{g^n x_0\}$ is bounded, it is clear that $\alpha_0 < +\infty$. The sequence $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers. So it converges to some nonnegative real number α. If possible, suppose that $\alpha > 0$. Then $\varphi(\alpha+) < \alpha$. Hence, there exists a real number $\beta > \alpha$ such that $\varphi(\beta) < \alpha$. Choose β^1 such that $\varphi(\beta) < \beta^1 < \alpha$. Since $\{\alpha_n\}$ decreases to α, there exists a positive integer N such that $\alpha_N < \beta$. For $x = f^N x_0$ and $y = g^N x_0$, the right-hand side of inequality (I) is $\varphi(\alpha_N)$ which is less than β^1. Hence, from inequality (I) for $x = f^N x_0$ and $y = g^N x_0$, it follows that there exists an integer $N_1 \ge N$ such that $\beta(f^{N_1} x_0, g^{N_1} x_0) < \beta^1$. Since $\{d(f^n x_0, g^n x_0)\}$ converges to zero, there exists an integer $N_2 \ge N_1$ such that $d(f^n x_0, g^n x_0) < (\alpha - \beta^1)/2$ for every $n \ge N_2$. For $n > N_2$, it is now clear that $\alpha_n \le \beta^1 + (\alpha - \beta^1)/2 = (\alpha + \beta^1)/2$. Since $\{\alpha_n\}$ decreases to α, it now follows that $\alpha \le (\alpha + \beta^1)/2$. This is a contradiction, since $\beta^1 < \alpha$. Hence, $\alpha = 0$. Hence, $\{f^n x_0\}$ and and $\{g^n x_0\}$ are Cauchy sequences and if one of them converges, then the other also converges to the same limit.

Suppose now that $fw = w$. Let $\gamma_n = \sup\{d(w, g^j x_0) \mid j \ge n\}$ $(n = 1,2,\ldots)$ and $\gamma = \inf\{\gamma_n \mid n \ge 1\}$. Taking $x = w$ and $y = g^n x_0$ in equality (I), we obtain:

$$\gamma \le \varphi(\{w\} \cup O_g(g^n x_0)) \le \varphi(\max\{\gamma_n, \delta(O_g(g^n x_0))\}).$$

Since $\{\gamma_n\}$ decreases to γ and $\{g^n x_0\}$ is Cauchy, by taking limits on both sides of the inequality above as $n \to \infty$, we obtain $\gamma \le \varphi(\gamma+)$. Hence $\gamma = 0$. Hence $\{g_n x_0\}$ converges to w. In a similar manner, it can be shown that $\{f^n x_0\}$ converges to w if $gw = w$.

Suppose that f^2 is orbitally continuous at z. Since $\{f^{2n} x_0\}$ converges to z, it follows that $f^2 z = z$. Hence, $O_f(z) = \{z, fz\}$. For any nonnegative integer k, $\inf_{1\le n<\infty} \beta(f^n z, g^n(g^k x_0)) = d(z, fz)$. Hence, from inequality (I), we have $d(z, fz) \le \varphi(\alpha(z, g^k x_0))$ for any nonnegative integer k. Since $\alpha(z, g^k x_0) \to d(z, fz)$ as $k \to \infty$, it now follows that $d(z, fz) \le \varphi(d(z, fz)+)$. Since $\varphi(t+) < t$ for every t in $(0,\infty)$, we must have $d(z, fz) = 0$. Hence $fz = z$. In a similar manner, it can be shown that $gz = z$ if g^2 it orbitally continuous at z.

COROLLARY 1. *Theorem 1 holds with inequality (II) below in the place of inequality (I), where p and q are fixed positive integers:*

$$d(f^p x, g^q y) \le \varphi(\alpha(x,y)). \tag{II}$$

Remark 1. Theorem 1.11 of Sastry and Naidu [4] is a special case of Corollary 1 with $f = g$, $p = q$ and $\varphi(t) = \alpha t$, α being a constant in $[0, 1)$.

COROLLARY 2. *Suppose that*

$$d(f^p x, g^q y) \leq \varphi(\delta(\{f^i x, g^i y \mid 0 \leq i \leq p,\ 0 \leq j \leq q\})) \qquad \text{(III)}$$

for all x, y in X, where p and q are fixed positive integers, $\varphi(t+) < t$ for every t in $(0, \infty)$ and $\lim_{t \to +\infty}[t - \varphi(t)] = +\infty$. *Suppose also that there is an x_0 in X such that $\{d(f^n x_0, g^n x_0)\}$ converges to zero. Then $\{f^n x_0\}$ and $\{g^n x_0\}$ are Cauchy sequences, and if one of them converges, then the other also converges to the same limit. Furthermore, if either f or g has a fixed point w, then both sequences converge to w. Suppose that $\{f^n x_0\}$ converges to some x in X. Then the following statements hold:*

1. *if f or f^2 is orbitally continuous at z or $p = 1$, then $fz = z$.*
2. *if g or g^2 is orbitally continuous at z or $q = 1$, then $gz = z$.*

Proof. We need only prove that z is a fixed point of f or g according as p or q is one; the rest of the Corollary is evident from Corollary 1 and statement 3 of Lemma 3 of [5]. Suppose now that $p = 1$. Then for $n \geq q$, from inequality (III), we have

$$d(fz, g^n x_0) \geq \varphi(\delta(\{z, fz, g^j x_0 \mid n - q \leq j \leq n\})).$$

Since $\{g^n x_0\}$ converges to z, by taking limits on both sides of the inequality above as $n \to \infty$, we obtain $d(fz, z) \leq \varphi(d(fz, z)+)$. Hence, $d(fz, z) = 0$. Hence, $fz = z$. In a similar manner, it can be shown that $gz = z$ when $q = 1$.

Remark 2. Example 1 shows that in Corollary 2, the condition '$\varphi(t+) < t$ for every t in $(0, \infty)$' cannot be replaced by the weaker condition '$\lim_{n \to \infty} \varphi^n(t) = 0$ for every t in $(0, +\infty)$, even if (X, d) is a bounded, complete metric space, f and g are continuous on X and $p = q = 1$.

Remark 3. Examples 5 and 6 of Sastry and Naidu [3] show that in Corollary 2 one cannot drop the condition 'there is an x_0 in X such that $\{d(f^n x_0, g^n x_0)\}$ converges to zero' even if X is finite and $p = q = 1$.

Remark 4. Example 2 shows that the initial hypothesis of Corollary 2 (Corollary 1) cannot guarantee the existence of a fixed point for either f or g, even if (X, d) is compact, f^3 and g^3 are continuous on X, $p = q = 2$ $(p = q = 1)$ and $\varphi(t) = \alpha t$, α being a constant in $[0, 1)$.

Remark 5. Example 3 shows that the initial hypothesis of Corollary 2 (Corollary 1) cannot guarantee the existence of a fixed point for f, even if it is strengthened by assuming that (X, d) is compact, f^3 is continuous on X, $p = 2$ $(p = 1)$, g is continuous on X (consequently $gz = z$), $q = 1$ and $\varphi(t) = t/2$.

Remark 6. In Corollary 1, the condition '$\{d(f^n x_0, g^n x_0)\}$ converges to zero' can be replaced by the commutativity of f and g provided $\{f^i g^j x_0 \mid i \geq 0,\ j \geq 0\}$ is bounded [5]. Example 4 shows that this is not so in the case of Theorem 1 even if (X, d) is a bounded complete metric space, f and g are continuous on X and $\varphi(t) = t/2$.

It is possible to drop the condition '$\{d(f^n x_0, g^n x_0)\}$ converges to zero' from Theorem 1 by suitably strengthening inequality (I).

THEOREM 2. *Suppose that*

$$\inf_{1 \leq n < \infty} \alpha(f^n x, g^n y) \leq \varphi(\alpha(x, y)) \tag{IV}$$

for all x, y in X, where $\varphi(t+) < t$ for every t, in $(0, \infty)$. Suppose also that there is an x_0 in X such that $\alpha(x_0, x_0) < +\infty$. Then $\{f^n x_0\}$ and $\{g^n x_0\}$ are Cauchy sequences and $\{d(f^n x_0, g^n x_0)\}$ converges to zero. In fact, $\{f^n x\}$ and $\{g^n y\}$ are Cauchy sequences and $\{d(f^n x, g^n y)\}$ converges to zero, whenever $\alpha(x, y) < +\infty$. In particular, each of f and g has at most one fixed point and if either f or g has a fixed point w, then both $\{f^n x_0\}$ and $\{g^n x_0\}$ converges to w. Suppose that $\{f^n x_0\}$ converges to some z in X. Then z is fixed point of $f(g)$ if $f^k(g^k)$ is orbitally continuous at z for some positive integer k.

Proof. Let x, y be elements of X such that $\alpha(x, y) < +\infty$. Let $\alpha_n = \alpha(f^n x, g^n y)\,(n = 0, 1, 2, \ldots)$ and $\alpha = \inf\{\alpha_n \mid n \geq 1\}$. Then $\{\alpha_n\}$ is a decreasing sequence of nonnegative real numbers decreasing to the nonnegative real number α. Taking $f^m x$ in the place of x and $g^m y$ in the place of y in inequality (IV), we obtain $\alpha \leq \varphi(\alpha_m)\,(m = 0, 1, 2, \ldots)$. Hence, $\alpha \leq \varphi(\alpha+)$. Hence, $\alpha = 0$. Hence $\{f^n x\}$ and $\{g^n y\}$ are Cauchy sequences and $\{d(f^n x, g^n y)\}$ converges to zero. The theorem is now evident.

COROLLARY 3. *Suppose that*

$$\alpha(f^{p(x)} x, g^{q(y)} y) \leq \varphi(\alpha(x, y)) \tag{V}$$

for all x, y in X where $\lim_{n \to \infty} \varphi^n(t) = 0$ for every t in $(0, \infty)$ and p and q are functions from X into the set of all positive integers. Suppose also that there is an x_0 in X such that $\alpha(x_0, x_0) < +\infty$. Then $\{f^n x_0\}$ and $\{g^n x_0\}$ are Cauchy sequences and $\{d(f^n x_0, g^n x_0)\}$ converges to zero. In fact, $\{f^n x\}$ and $\{g^n y\}$ are Cauchy sequences and $\{d(f^n x, g^n y)\}$ converges to zero, whenever $\alpha(x, y) < +\infty$. In particular, aech of f and g has at most one fixed point and if either f or g has a fixed point w then both $\{f^n x_0\}$ and $\{g^n x_0\}$ converge to w. Suppose that $\{f^n x_0\}$ converges to z for some z in X. Then the following statements hold:

1. *if $f^k(g^k)$ is orbitally continuous at z for some positive integer k, then $fz = z$ $(gz = z)$;*
2. *if $p(z) = 1$ and $\{f^n z\}$ is bounded, then $fz = z$;*
3. *if $q(z) = 1$ and $\{g^n z\}$ is bounded, then $gz = z$.*

Proof. For x, y in X, let $x_1 = f^{p(x)} x$, $y_1 = g^{q(y)} y$, $\tilde{p}(x) = p(x) + p(x_1)$ and $\tilde{q}(y) = q(y) + q(y_1)$. Then, from inequality (V), we have

$$\alpha(f^{\tilde{p}(x)} x, g^{\tilde{q}(x)} y) = \alpha(f^{p(x_1)} x_1, g^{q(y_1)} y_1) \leq \varphi(\alpha(x_1, y_1)) \leq \varphi^2(\alpha(x, y))$$

for all x, y in X. Hence, inequality (IV) holds for all x, y in X with φ^2 in the place of φ. From Lemma 1 we have $\varphi^2(t+) < t$ for every t in $(0,\infty)$. Now we need only prove statements 2 and 3, since the rest of the Corollary is evident from Theorem 2.

2. Suppose that $p(z) = 1$ and $\{f^n(z)\}$ is bounded. Let $\alpha(z) = \delta(O_f(z))$. Then $0 \leq \alpha(z) < +\infty$. Let $x_k = g^{q(x_{k-1})}x_{k-1}$ and $\gamma_k = \sup\{d(z, g^n x_k) \mid u = 0, 1, 2, \ldots\}$ $(k = 1, 2, 3, \ldots)$. Since $\{x_k\}$ is a subsequence of $\{g^n x_0\}$ and the latter converges to z, it is clear that $\{\alpha(fz, x_k)\}$ and $\{\alpha(z, x_k)\}$ converge to $\alpha(z)$ and $\{\gamma_k\}$ converges to zero. Since $p(z) = 1$, for $k \geq 2$, from inequality (V), we have

$$\alpha(fz, x_k) \leq \varphi(\alpha(z, x_{k-1})) \leq \varphi(\max\{\alpha(fz, x_{k-1}),\, \alpha(z),\, \gamma_{k-1}\})$$
$$\leq \varphi(\max\{\varphi(\alpha(z, x_{k-2})),\, \alpha(z),\, \gamma_{k-1}\}).$$

Hence, $\alpha(fz, x_k) \leq \max\{\varphi^2(\alpha(z, x_{k-2})),\ \varphi(\alpha(z)),\ \gamma_{k-1}\}$ for $k \geq 2$. Taking limits on both sides of the inequality above as $k \to \infty$, we obtain $\alpha(z) \leq \max\{\varphi^2(\alpha(z)+),\ \varphi(\alpha(z))\}$. Now from Lemma 1 it follows that $\alpha(z) = 0$. Hence, $fz = z$.

3. The proof of statement 3 is analogous to the proof of statement 2.

Remark 7. Example 5 shows that none of the conclusions of the first part of Theorem 6 (or Corollary 3) of Ding [1] is true even if (X, d) is a bounded (complete) metric space, $\varphi(t) = t/2$ and the functions m and n (mentioned in the results of Ding) take the constant value one troughout X. Corollary 3 is an improvement over the second part of Theorem 6 of Ding [1].

THEOREM 3. *Theorem* 2 *holds with the following inequality in the place of inequality* (IV)*:*

$$\inf_{1 \leq n < \infty} \beta(f^n x, g^n y) \leq \varphi(\beta(x, y)).$$

Proof. Let x, y be elements of X such that $\beta(x, y) < +\infty$. Proceeding as in the proof of Theorem 2, it can be shown that $\inf\{\beta(f^n x, g^n y) \mid n = 1, 2, 3, \ldots\}$ is zero. Hence, $\{\beta(f^n x, g^n y)\}$ converges to zero. In particular, $\{d(f^n x, g^n y)\}$ converges to zero. Since $d(f^n x, f^m x) \leq d(f^n x, g^n y) + d(f^m x, g^n y) \leq 2\beta(f^n x, g^n y)$ for all $m \geq n$, it is clear that $\{f^n x\}$ is Cauchy. Similarly it can be shown that $\{g^n y\}$ is Cauchy. The theorem is now evident.

Remark 8. Example 6 shows that either in Theorem 1 or in Theorem 2 or in Theorem 3, the condition '$\varphi(t+) < t$ for every t in $(0, \infty)$' cannot be replaced by the weaker condition '$\lim_{n\to\infty} \varphi^n(t) = 0$ for every t in $(0, \infty)$', even if (X, d) is a bounded, complete metric space, $f = g$ and f is continuous on X.

COROLLARY 4. *Corollary* 3 *holds with inequality* (VI) *below in the place of inequality* (V)*:*

$$\beta(f^{p(x)}x, g^{q(y)}y) \leq \varphi(\beta(x, y)). \qquad \text{(VI)}$$

Proof. Corollary 4 can be proved along the lines of the proof of Corollary 3 with obvious modifications, such as the replacement of α with β except in the definition of $\alpha(z)$ which is to be defined here as $\sup\{d(z, f^n z) \mid n = 1, 2, 3, \ldots\}$.

Remark 9. Example 7 shows that the initial hypothesis of Corollary 3 or 4 does not guarantee the existence of a fixed point for f even if (X,d) is compact, g is continuous on X (consequently, g has a fixed point), p takes the value one troughout X, except at a single point where it takes the value 2, $q(y)=1$ for all y in X and $\varphi(t)=t/2$. It also shows that the condition '$p(z)=1$' cannot be dropped from statement 2 of Corollary 3 or 4 even if $p(x)=1$ for all x in $X\setminus\{z\}$.

COROLLARY 5. *Corollary 4 holds with inequality* (VII) *below in the place of inequality* (VI)*, the statement 'p and q are fixed positive integers' in the place of the statement 'p and q are functions from X into the set of all positive integers', p in the place of $p(z)$ and q in the place of $q(z)$:*

$$d(f^p x, g^q y) \le \varphi(\beta(x,y)). \tag{VII}$$

Proof. The validity of inequality (VII) for all x, y in X implies that of inequality (VI) with $p(x)=p$ and $q(y)=q$ for all x, y in X.

Remark 10. Example 8 shows that in statement 2 of Corollary 3, or 4, or 5, one cannot drop the condition '$\{f^n z\}$ is bounded' even if (X,d) is complete, $p=q=1$ and $\varphi(t)=t/2$. In fact, the example shows that the remark is true whether g is continuous on X (and therefore g has a fixed point) or $f=g$.

COROLLARY 6. *Suppose that*

$$d(f^p x, g^q y) \le \varphi \max\{d(f^i x, g^j y) \mid 0 \le i \le p,\ 0 \le j \le q\}) \tag{VIII}$$

for all x, y in X, where p and q are fixed positive integers, $\lim_{n\to\infty}\varphi^n(t)=0$ for every t in $(0,\infty)$ and $\lim_{t\to+\infty}[t-\varphi(t)]=+\infty$. Then for all x, y in X, $\{f^n x\}$ and $\{g^n y\}$ are Cauchy sequences and $\{d(f^n x, g^n y)\}$ converges to zero. In particular, each of f and g has at most one fixed point. Suppose that there is an x_0 in X such that $\{f^n x_0\}$ converges to z for some z in X. Then for all x in X, $\{f^n x\}$ and $\{g^n x\}$ converge to z. Furthermore, the following statements hold:

1. *if either $p=1$ or f^k is orbitally continuous at z for some positive integer k, then $fz=z$;*
2. *if either $q=1$ or f^k is orbitally continuous at z for some positive integer k, then $gz=z$.*

Proof. Let x in X. Let $M=\max\{d(g^s x, g^q x) \mid 0\le s\le q\}$ and $\gamma_n = \max\{d(f^r x, g^q x) \mid 0 \le r \le n\}\,(n=0,1,2,\ldots)$. For $p\le i\le n$, from inequality (VIII), we have

$$d(f^i x, g^q x) \le \varphi(\max\{d(f^r x, g^s x) \mid i-p \le r \le i,\ 0 \le s \le q\}) \le \varphi(\gamma_n + M).$$

Hence, $\gamma_n \le \varphi(\gamma_n+M)+\gamma_p\,(n=1,2,3,\ldots)$. Hence $(\gamma_n+M)-\varphi(\gamma_n+M)\le M+\gamma_p\,(n=1,2,3,\ldots)$. Since $\lim_{t\to+\infty}[t-\varphi(t)]=+\infty$, it now follows that $\{\gamma_n\}$ is bounded. Hence $\{f^n x\}$ is bounded for each x in X. Similarly it can be shown that $\{g^n x\}$ is bounded for each x in X. Hence $\beta(x,y)<+\infty$ for all x, y in X. Now the corollary is evident from Corollary 5.

Remark 11. Theorem 2 of Fisher [2] is a special case of Corollary 6 with $\varphi(t) = \alpha t$, α being a constant in $[0,1)$. Example 9 shows that Corollary 6 is a proper generalization of Fisher's theorem.

Remark 12. Example 10 shows that in Corollary 6 one cannot conclude that the sequence $\{f^n x\}$ is bounded (and therefore Cauchy) even if (X,d) is complete, $f = g$ and $p = q = 1$ if the condition '$\lim_{t\to+\infty}[t - \varphi(t)] = +\infty$' is dropped. From the example it is also evident that a similar remark holds in the case of Corollary 2.

Remark 13. Example 11 shows that, when $p = q = 2$, the initaial hypothesis of Corollary 6 (and therefore that of Corollary 5) does not guarantee the existence of a fixed point for f even if (X,d) is compact, $f = g$ and $\varphi(t) = t/2$.

Remark 14. Example 12 shows that when $p = 2$, the initial hypothesis of Corollary 6 cannot ensure the existence of a fixed point for f even if (X,d) is compact, g has a fixed point and $\varphi(t) = t/2$.

Remark 15. Example 13 shows that in Corollary 6 it is not possible to take p and q even as bounded functions from X into the set of all positive integers and replace p with $p(x)$ and q with $q(y)$ in inequality (VIII) even if (X,d) is a bounded, complete metric space, $f = g$ and $\varphi(t) = t/2$. In fact, the example shows that it is not possible to take even p alone as a bounded function and $q = 1$. It is evident that similar remarks hold also in the case of Corollaries 1, 2 and 5.

2. Examples. 1. Let X be the set of all integers with a metric d defined on it by $d(m,-m) = 1/m$ if $m > 0$, $d(0,m) = 1$ if $m > 0$, $d(0,-m) = 1 + 1/m$ if $m > 0$, $d(m,n) = d(-m,-n) = 1 + 1/n$ if $0 < m < n$ and $d(m,-n) = 1$ if $m > 0$, $n > 0$ and $m \neq n$. Define $f, g : X \to X$ by:

$$fx = \begin{cases} -x & \text{if } x < 0, \\ 1 + x & \text{if } x \geq 0, \end{cases} \qquad gy = \begin{cases} 0 & \text{if } y > 1, \\ -2 & \text{if } y = 1, \\ y - 1 & \text{if } y \leq 0. \end{cases}$$

Define $\varphi : R^+ \to R^+$ by $\varphi(t) = \begin{cases} 1 & \text{if } t > 1, \\ 0 & \text{if } t \leq 1. \end{cases}$ Then (X,d) is bounded, complete metric space; f and g are continuous on X; φ is increasing on R^+, $\varphi^2(t) = 0$ for every t in R^+, and

$$d(fx, gy) \leq \varphi(\delta\{x, fx, y, gy\})$$

for all x, y in X. The sequence $\{d(f^n 0, g^n 0)\}$ converges to zero. But there is no x in X for which either $\{f^n x\}$ or $\{g^n x\}$ is Cauchy. In particular, neither f nor g has a fixed point.

2. Let $X = \{0\} \cup \{2^{-n},\ -2^{-n} \mid n = 0, 1, 2, \dots\}$ with the usual metric. Define $f, g : X \to X$ by $f0 = 1$, $f1 = -1$, $f(-1) = 0$, $fx = x/2$ if $x = 2^{-n}$ and n is odd or $x = -2^{-n}$, n is even and $n \neq 0$, $fx = -x/2$ if $x = 2^{-n}$ n is even and $n \neq 0$ or $x = -2^{-n}$ and n is odd, $g0 = 2^{-1}$, $g1 = g(-1) = g(-2^{-1}) = 0$, $g(2^{-1}) = -2^{-1}$ and $gx = fx$ for all x in $X \setminus \{0, 1, -1, -2^{-1}\}$. Then f^3 and g^3 are continuous on X; $f^3 0 = g^3 0 = 0 = \lim_{n\to\infty} g^n x = \lim_{n\to\infty} f^n x$ for all x in

$\{2^{-k}, -2^{-k} \mid k = 2, 3, \dots\}$ and $|fx - gy| \le (4/5) \max\{\delta\{x, fx, f^2x\},\ \delta\{y, gy, g^2y\}\}$, $|f^2x - g^2y| \le (3/4) \max\{\delta\{x, fx, f^2x\},\ \delta\{y, gy, g^2y\}\}$ for all x, y in X. In fact, for all integers $p, q \ge 2$, we have

$$|f^p x - g^q y| \le (3/4) \max\{\delta\{x, fx, f^2x\},\ \delta\{y, gy, g^2y\}\}$$

for all x, y in X. But neither f nor g has a fixed point.

3. Let $X = \{-1, 0, 1\} \cup \{2^{-n} \mid n = 1, 2, \dots\}$ with the usual metric. Define $f, g : X \to X$ by $f(-1) = 0$, $f0 = 1$, $f1 = -1$, $f(2^{-n}) = 2^{-n-1} (n = 1, 2, \dots)$ and $gx = 0$ for all x in X. Then f^3 and g are continuous on X, $\lim_{n\to\infty} f^n x = 0 = g(0) = \lim_{n\to\infty} g^n x$ for all x in $\{2^{-k} \mid k = 1, 2, \dots\}$ and

$$|fx - gy| \le \delta\{x, fx, f^2x, gy\}/2,$$
$$|f^2x - gy| \le \delta\{x, fx, f^2x, gy\}/2$$

for all x, y in X. In fact, for all positive integers p, q, we have

$$|f^p x - g^q y| \le 2^{-1} \max\{\delta\{x, fx, f^2x\},\ d(x, gy)\}$$

for all x, y in X. But f has no fixed point.

4. Let X be the set of all positive integers with a metric d defined on it by $d(x, x+1) = 2$ for all x in X and $d(x, y) = 1$ if x and y are distinct nonconsecutive positive integers. Define $f : X \to X$ as $fx = x + 1$ for all x in X. Let g be the identity map on X. Then (X, d) is a bounded, complete metric space with no accumulation points, and

$$\inf_{1 \le n < \infty} \beta(f^n x, g^n y) = 2^{-1} d(x, fx) = 2^{-1} \alpha(x, y)$$

for all x, y in X. But, for no x in X, $\{f^n x\}$ is Cauchy.

5. Let $X = \{3, 4, 5, \dots\}$ with a metric d defined on it as in Example 4. Define $f, g : X \to X$ by

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even}, \\ 2^n + 3 & \text{if } n \text{ is odd}, \end{cases} \qquad g(n) = \begin{cases} 2^n & \text{if } n \text{ is even}, \\ n+1 & \text{if } n \text{ is odd}. \end{cases}$$

Then (X, d) is a bounded, complete metric space with no accumulation points and

$$\delta(O_f(fx) \cup O_g(gx)) = 2^{-1} \max\{d(x, fx),\ d(x, gx)\} = 2^{-1} \delta(O_f(x) \cup O_g(x))$$

for all x in X. But there is no x in X for which either $\{f^n x\}$ or $\{g^n x\}$ is Cauchy. In particular, neither f nor g has a fixed point in X.

6. Let X be the set of all positive integers with a metric d defined on it by $d(x, y) = 1 + 1/y$ if $x < y$. Define $f : X \to X$ by $fx = x + 1$. Define $\varphi : R^+ \to R^+$ by $\varphi(t) = 1$ if $t > 1$ and $\varphi(t) = 0$ if $t \le 1$. Then (X, d) is a bounded, complete

metric space with no accumulation points, φ is increasing on R^+, $\varphi^2(t) = 0$ for every t in R^+ and

$$\inf_{1\le n<\infty} \delta(O_f(f^n x) \cup O_f(f^n y)) \le \varphi(\min\{\max\{d(x,y),\ d(fx,y)\},$$

$$\max\{d(x,y),\ d(x,fy)\},\ \max\{d(fx,y),\ d(x,fy)\}\})$$

for all x, y in X. But, for no x in X, the sequence $\{f^n x\}$ is Cauchy. Clearly, f has no fixed point.

7. Let $X = \{0\} \cup \{2^{-n} \mid n = 0,1,2,\ldots\}$ with the usual metric. Define $f, g : X \to X$ by $f0 = 1$, $fx = x/2$ for $x \ne 0$ and $gx = x/2$ for all x in X. Then X is compact, g is continuous on X, $g0 = 0$, $\{f^n x\}$ and $\{g^n x\}$ converge to zero for each x in X,

$$\alpha(fx, gy) \le 2^{-1}\beta(x,y)$$

for all x in $X \setminus \{0\}$ and for all y in X, and

$$\alpha(f^2 0, gy) \le 2^{-1}\beta(0,y)$$

for all y in X. But f has no fixed point.

8. Let $X = \{2^{-n} \mid n = 0,1,2,\ldots\} \cup \{0,-1,-2,-3,\ldots\}$ with the usual metric. Define $f : X \to X$ by $fx = x - 1$ if $x \in \{0,-1,-2,-3,\ldots\}$ and $fx = x/2$ if $x \in \{2^{-n} \mid n = 0,1,2,\ldots\}$. Let g be the constant map zero on X. Then

$$\alpha(fx, gy) \le 2^{-1}\delta(O_f(x)) = 2^{-1}\beta(x,y),$$
$$\delta(O_f(fx) \cup O_f(fy)) \le 2^{-1}\sup\{|f^i x - f^j y| \mid i \ge 0,\ j \ge 0\}$$

for all x, y in X. For $x > 0$, $\{f^n x\}$ and $\{g^n x\}$ converge to zero. But $\{f^n 0\}$ is unbounded. Clearly, f has no fixed point.

9. Let $X = \{0, 1, 1/2, 1/3, \ldots\}$ with the usual metric. Define $f : X \to X$ as $f0 = 0$, $f(1/n) = 1/(n+1)$ $(n = 1,2,\ldots)$. Define $\varphi : R^+ \to R^+$ by $\varphi(t) = t/(1+t)$. Then φ is an increasing function on R^+, $\varphi(t) < t$ for every $t > 0$,

$$\lim_{t\to+\infty}[t - \varphi(t)] = +\infty \text{ and } |fx - fy| \le \varphi(\max\{|x - fy|, |fx - y|\})$$

for all x, y in X. But there is no constant α in $[0,1)$ such that

$$|fx - fy| \le \alpha \max\{|x-y|, |fx-y|, |x-fy|\}$$

for all x, y in X.

10. Let $X = [1,\infty)$ with the usual metric. Define $f : X \to X$ by $fx = 2x$ and $\varphi : R^+ \to R^+$ as $\varphi(t) = 2t^2/(1+2t)$. Then φ is an increasing continuous function on R^+, $\varphi(t) < t$ for every $t > 0$, $\lim_{t\to+\infty}[t - \varphi(t)] = 1/2$,

$$|fx - fy| \le \varphi(\max\{|x-y|, |fx-y|, |x-fy|\})$$

for all x, y in X, and, for each x in X, $f^n x \to +\infty$ as $n \to \infty$.

11. Let $X = \{-1, 0, 1\} \cup \{2^{-n} \mid n = 1, 2, 3, \dots\}$ with the usual metric. Define $f : X \to X$ by $f(-1) = 2^{-1}$, $f(0) = -1$, $f(2^{-n}) = 2^{-n-1} (n = 0, 1, 2, \dots)$. Then

$$|f^2x - f^2y| \le 2^{-1} \max\{|x - y|, |x - fy|, |fx - y|\}$$

for all x, y in X. But f has no fixed point.

12. Let $X = \{-1, 0, 1\} \cup \{2^{-n} \mid n = 1, 2, 3, \dots\}$ with the usual metric. Define $f, g : X \to X$ by $f(-1) = g(-1) = 2^{-1}$, $f(0) = -1$, $g(0) = 0$, $f(2^{-n}) = g(2^{-n}) = 2^{-n-1} (n = 0, 1, 2, \dots)$. Then

$$|f^2x - gy| \le 2^{-1} \max\{|x - y|, |fx - y|\}$$

for all x, y in X. Then g has a unique fixed point, namely, zero. But f has no fixed point.

13. Let $X = \{0\} \cup \{2, 3, 4, \dots\} \cup \{-1/n \mid n = 2, 3, 4, \dots\}$. Define a metric d on X by $d(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \le 2, \\ 2 & \text{if } |x - y| > 2. \end{cases}$ Define $f, g : X \to X$ by $g(x) = 0$ for all x in X and

$$f(x) = \begin{cases} -1/2 & \text{if } x = 0, \\ -1/(x + 1) & \text{if } x \in \{2, 3, 4, \dots\}, \\ -1/x & \text{if } x \in \{-1/n \mid n = 2, 3, 4, \dots\}. \end{cases}$$

Define $p : X \to \{1, 2, 3\}$ by $p(0) = 3$, $p(x) = 1$ if $x \in \{2, 3, 4, \dots\}$ and $p(x) = 2$ if $x \in \{-1/n \mid n = 2, 3, 4, \dots\}$. Then (X, d) is a bounded, complete metric space and

$$d(f^{p(x)}x, gy) \le 2^{-1} d(f^{p(x)-1}x, gy),$$
$$d(f^{p(x)}x, f^{p(y)}y) \le 2^{-1} d(f^{p(x)-1}x, f^{p(y)-1}y)$$

for all x, y in X. But, for no x in X, the sequence $\{f^n x\}$ is Cauchy. In particular, f has no fixed point. Furthermore, $(X, |\cdot|)$ is an unbounded complete metric space and

$$|f^{p(x)}x - gy| \le 2^{-1} |f^{p(x)-1}x - gy|,$$
$$|f^{p(x)}x - f^{p(y)}y| \le 2^{-1} |f^{p(x)-1}x - f^{p(y)-1}y|$$

for all x, y in X, where $|\cdot|$ denotes the modulus function. For any x in X, $\{f^n x\}$ is unbounded in $(X, |\cdot|)$.

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