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NEW CHARACTERIZATIONS AND PROPERTIES OF ALMOST-OPEN AND ALMOST-CLOSED FUNCTIONS

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Abstract. In this paper several recent discoveries about regular open sets and regular closed sets are utilized to further investigate and further characterize almost-open and almost--closed functions.

1. Introduction

In 1937 regular open sets were introduced. Let (X, T) be space and let $A \subset X$. Then A is regular open, denoted by $A \in RO(X, T)$, iff $A = \operatorname{Int}(\operatorname{Cl}(A))$ [23]. In the 1937 investigation it was shown that $RO(X, T)$ is base for a topology T_s on X coarser than T and (X, T_s) was called the semiregularization space of (X, T). The subset A is regular closed, denoted by $A \in RC(X, T)$, iff $X - A$ is regular open. In 1943 and in 1961 the closure operator was used to define θ-continuous and weakly--continuous functions, respectively. A function $f : (X, T) \to (Y, S)$ is θ-continuous [10] (resp. weakly-continuous [13]) iff for each $x \in X$ and each $U \in S$ such that $f(x) \in U$, there exists $V \in T$ such that $x \in V$ and $f(\operatorname{Cl}(V) \subset \operatorname{Cl}(U)$ (resp. $f(V) \subset \operatorname{Cl} U$). In 1968 and in 1980 the interior and closure operators were used to define almost-continuous and δ-continuous functions, respectively. A function $f : (X, T) \to (Y, S)$ is almost-continuous [22] (resp. δ-continuous [19]) if for each $x \in X$ and each $U \in S$ such that $f(x)\ U$, there exists $V \in T$ such that $x \in V$ and $f(V) \subset \operatorname{Int}(\operatorname{Cl}(U))$ (resp. $f(\operatorname{Int}(\operatorname{Cl}(V))) \subset \operatorname{Int}(\operatorname{Cl}(U))$. Further investigation of almost-continuous and δ-continuous functions revealed that $f : (X, T) \to (Y, S)$ is almost-continuous (resp. δ-continuous) iff $f : (X, T) \to (Y, S_s)$ (resp. $f : (X, T_s) \to (Y, S_s)$) is continuous [17]. Also in 1968 almost-open and almost-closed function sere introduced. A function $f : (X, T) \to (Y, S)$ is almost-open (resp. almost-closed) if for each $A \in RO(X, T)$ (resp. $A \in RC(X, T)$), $f(A)$ is open (resp. closed) in Y [22]. Further investigation of almost-open functions showed that $f : (X, T) \to (Y, S)$ is almost-open iff $f : (X, T_s) \to (X, S)$ in open [17].

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Since 1973 the interior and closure operators have been used to define additional collections of subsets associated with a topological space. In 1963 semi open sets were introduce. Let (X,T) be a space and let $A \subset X$. Then A is semi open iff $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ [14]. In 1965 [18] semi open sets were further investigated as β-sets and α-sets were introduced. The subset A is an α-set, denoted by $A \in \alpha(X,T)$, iff $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. In 1970 semi open sets were used to define semi closed sets, which were used to define the semi closure of a set. The subset A is semi closed iff $X - A$ is semi open and the semi closure of a subset B, denoted by scl$B$, is the intersection of all semi closed sets containing B In [2]. In 1978 the semi closure operator was used to define feebly open sets. The subset A is feebly open, denoted by $A \in FO(X,T)$, iff $A \subset \mathrm{scl}(\mathrm{Int}(A))$ [15]. Further investigations of feebly open sets have shown that $FO(X,T)$ is a topology on X and $T \subset FO(X,T) = FO(X, FO(X,T))$ [4], $FO(X,T) = \alpha(X,T)$ [5], $RO(X,T) = \{\mathrm{scl}\,0 \mid 0 \in T\}$ [6] $= \{\mathrm{Ext}\,(0) \mid 0 \in T\}$ [7], $RC(X,T) = \{\mathrm{Cl}\,(O) \mid 0 \in T\}$ [7], and $FO(X,T)_s = T_s$ [8]. In this paper the definitions and results above are used to further investigate and further characterize almost-open and almost-closed functions.

2. Almost-Open Functions

THEOREM 2.1. *Let (X,T) be a space, Then $RO(X,T) = \{O \in T \mid O = \mathrm{scl}\,O\}$.*

Proof. If $O \in RO(X,T)$, then $O = \mathrm{scl}\,U$ for some $U \in T$ and $\mathrm{scl}\,O = \mathrm{scl}\,(\mathrm{scl}\,U) = \mathrm{scl}\,U = O$. If $O \in T$ such that $O = \mathrm{scl}\,O$, then $O \in \{\mathrm{scl}\,U \mid U \in T\} = RO(X,T)$. Thus $RO(X,T) = \{O \in T \mid O = \mathrm{scl}\,O\}$.

THEOREM 2.2. *Let (X,T) and (Y,S) be spaces and let $f : X \to Y$. Then the following are equivalent:* (a) *$f : (X,T) \to (Y,S)$ is almost-open,* (b) *$f : (X, FO(X,T)) \to (S,S)$ is almost-open,* (c) *$f : (X,T_s) \to (Y,S)$ is almost-open,* (d) *for each $O \in T$, $f(\mathrm{scl}\,O) \in S$.* (e) *for each $O \in T$, $f(\mathrm{Ext}\,(O)) \in S$, and* (f) *for each $A \subset Y$, $f^{-1}(\mathrm{Cl}\,(A)) \subset \mathrm{Cl}_{T_s}(f^{-1}(A))$.*

The proof is straightforward using the results above and the facts that $(T_s)_s = T_s$ [3] and that $f : (X,T) \to (Y,S)$ is open iff for each $A \subset Y$, $f^{-1}(\mathrm{Cl}\,(A)) \subset \mathrm{Cl}\,(f^{-1}(A))$ [21] and is omitted.

In [1] it was shown that the restriction of an almost-open function to an open set or to a closed set may fail to be almost-open. In [20] restrictions of almost-open functions were further investigated and motivated the results below.

In 1981 the interior and closure operators were used to define pre-open sets. If (X,T) is a space and $A \subset X$, then A is pre-open, denoted by $A \in PO(X,T)$, iff $A \subset \mathrm{Int}\,(\mathrm{Cl}\,(A))$ [16].

THEOREM 2.3. *Let (X,T) and (Y,S) be spaces and let $F : X \to Y$. Then $f : (X,T) \to (Y,S)$ is almost-open iff for each $A \in PO(X,T)$ and each $B \subset Y$ such that $A = f^{-1}(B)$, $f\backslash_A : (A, T_A) \to (B, S_B)$ is almost-open.*

Proof. Suppose $f : (X,T) \to (Y,S)$ is almost-open. Let $U \in RO(A,T_A)$. Then $U = A \cap O$ for some $O \in RO(X,T)$ [11], $f(O) \in S$, and $f\backslash_A(U) = f(A \cap O) = B \cap f(O) \in S_B$. Clearly, since $X \in PO(X,T)$ and $X = f^{-1}(Y)$, the converse statement is true.

The results above raised questions about almost-open images of pre-open sets, which led to the discoveries below.

THEOREM 2.4. *Let (X,T) be a space. Then $RO(X,T) = \{\text{Int}\,(\text{Cl}\,(A)) \mid A \in PO(X,T)\} = \{\text{Int}\,(\text{Cl}\,(A)) \mid A \subset X\}$.*

Proof. Since $T \subset PO(X,T)$ and for each $U \in T$, $\text{scl}\, U = \text{Int}\,(\text{Cl}\,(U))$ [9], then $RO(X,T) = \{\text{Int}\,(\text{Cl}\,(U)) \mid U \in T\} \subset \{\text{Int}\,(\text{Cl}\,(A)) \mid A \in PO(X,T)\} \subset \{\text{Int}\,(\text{Cl}\,(A)) \mid A \subset X\}$. If $A \subset X$, then $\text{Int}\,(\text{Cl}\,(A)) \subset \text{Int}\,(\text{Cl}\,(\text{Int}\,(\text{Cl}\,(A))))$ and since $\text{Int}\,(\text{Cl}\,(A)) \subset \text{Cl}\,(A)$, then $\text{Cl}\,(\text{Int}\,(\text{Cl}\,(A))) \subset \text{Cl}\,(A)$, which implies $\text{Int}\,(\text{Cl}\,(A)) = \text{Int}\,(\text{Cl}\,(\text{Int}\,(\text{Cl}\,(A)))) = \text{scl}\,(\text{Int}\,(\text{Cl}\,(A))) \in RO(X,T)$ and $\{\text{Int}\,(\text{Cl}\,(A)) \mid A \subset X\} \subset RO(X,T)$. Thus $RO(X,T) = \{\text{Int}\,(\text{Cl}\,(A)) \mid A \in PO(X,T)\} = \{\text{Int}\,(\text{Cl}\,(A)) \mid A \subset X\}$.

COROLLARY 2.1. *Let (X,T) and (Y,S) be spaces and let $f : X \to Y$. Then the following are equivalent:* (a) *$f : (X,T) \to (Y,S)$ is almost-open,* (b) *for each $A \in PO(X,T)$, $f(\text{Int}\,(\text{Cl}\,(A))) \in S$, and* (c) *for each $A \subset X$, $f(\text{Int}\,(\text{Cl}\,(A))) \in S$.*

THEOREM 2.5. *Let (X,T) and (Y,S) be spaces, let $f : (X,T) \to (Y,S)$ be continuous and almost open, and let $A \in PO(X,T)$. Then $f(A) \in PO(Y,S)$.*

Proof. Since $A \in PO(X,T)$, then $A \subset \text{Int}\,(\text{Cl}\,(A))$, since $f : (X,T) \to (Y,S)$ is almost-open, then $f(\text{Int}\,(\text{Cl}\,(A))) \in S$, and since f is continuous, then $f(\text{Cl}\,(A)) \subset \text{Cl}\,(f(A))$. Then $f(A) \subset f(\text{Int}\,(\text{Cl}\,(A))) \subset \text{Cl}\,(f(A))$, which implies $f(A) \subset \text{Int}\,(\text{Cl}\,(f(A)))$ and $f(A) \in PO(Y,S)$.

The following example shows that continuous in Theorem 2.5 can not be replaced by almost-continuous even if almost-open is replaced by the stronger condition of open.

Example 2.1. Let $X = \{a,b\}$, let $T = \{\emptyset, X\}$, let $S = \{\emptyset, X, \{a\}\}$, and let $f : X \to X$ be the identity function. Then $f : (X,T) \to (Y,S)$ is almost-continuous and open, $\{b\} \in PO(X,T)$, but $f(\{b\}) \notin PO(X,S)$.

THEOREM 2.6. *Let (X,T) and (Y,S) be spaces, let $f : (X,T) \to (Y,S)$ be almost-continuous and almost-open, and let $A \in PO(X,T)$, then $f(A) \in PO(Y,S_s)$.*

Proof. Since $A \in PO(X,T)$, then $A \subset \text{Int}\,(\text{Cl}\,(A))$, since $f : (X,T) \to (Y,S)$ is almost-open, then $f(\text{Int}\,(\text{Cl}\,(A))) \in S$, and since $f(X,T) \to (Y,S_s)$ is continuous, then $f(\text{Cl}\,(A)) \subset \text{Cl}_{S_s}(f(A))$. Then $\text{scl}\, f(\text{Int}\,(\text{Cl}\,(A))) \in S_S$ and since $\text{scll}\, U = \text{scl}_{S_s} U$ for each $U \in S$ [6], then $\text{scl}\, f(\text{Int}\,(\text{Cl}\,(A))) = \text{scl}_{S_s} f(\text{Int}\,(\text{Cl}\,(A))) \subset \text{scl}_{S_s} f(\text{Cl}\,(A)) \subset \text{Cl}_{S_s}(f(A))$, which implies $f(A) \subset \text{Int}_{S_s}(\text{Cl}_{S_s}(f(A)))$ and $f(A) \in PO(Y,S_S)$.

Further investigation of almost-continuous, almost-open functions led to the following discoveries.

In 1968 [22], where almost-continuous functions were introduced, an example was given showing that θ-continuous need not imly almost-continuous, but the question of whether or not almost-continuous implies θ-continuous was unresolved. Below the 1968 question is resolved.

THEOREM 2.7. *Let (X,T) and (Y,S) be spaces and let $f:(X,T)\to(Y,S)$ be almost-continuous. Then $f:(X,T)\to(Y,S)$ is θ-continuous.*

Proof. Let $x\in X$ and let $V\in S$ such that $f(x)\in V$. Since $f:(X,T)\to(Y,S_s)$ is continuous, then for each $A\subset Y$, $\mathrm{Cl}\,(f^{-1}(A))\subset f^{-1}(\mathrm{Cl}_{S_s}(A))$. Since $\mathrm{Cl}_S(U)=\mathrm{Cl}_{S_s}(U)$ for each $U\in S$ [2] and $\mathrm{scl}\,V\in S_s\subset S$, then $\mathrm{Cl}_{S_s}(\mathrm{scl}\,V)=\mathrm{Cl}_S(\mathrm{scl}\,V)=\mathrm{Cl}_S(V)$. Then $x\in f^{-1}(\mathrm{scl}\,V)\in T$ and $f(\mathrm{Cl}\,(f^{-1}(\mathrm{scl}\,V)))\subset f(f^{-1}(\mathrm{Cl}_S(V)))\subset \mathrm{Cl}_S(V)$.

THEOREM 2.8. *Let (X,T) and (Y,S) be spaces and let $f:(X,T)\to(Y,S)$ be almost-open. Then the following are equivalent* (a) *$f:(X,T)\to(Y,S)$ is almost--continuous,* (b) *$f:(X,T)\to(Y,S)$ is θ-continuous, and* (c) *$f:(X,T)\to(Y,S)$ is δ-continuous.*

Proof. Clearly, from earlier results, (c) implies (a) and (a) implies (b). (b) implies (c): Let $V\in S_S$. Let $x\in f^{-1}(V)$. Then $f(x)\in V$ and there exists $W\in RO(Y,S)$ such that $f(x)\in W\subset V$. Let $U\in T$ such that $x\in U$ and $f(\mathrm{Cl}\,(U))\subset \mathrm{Cl}\,(W)$. Then $x\in \mathrm{scl}\,U\in RO(X,T)$, $f(\mathrm{scl}\,U)\in S$, and $f(\mathrm{scl}\,U)\subset f(\mathrm{Cl}\,(U))\subset \mathrm{Cl}\,(W)$, which implies $f(\mathrm{scl}\,U)\subset \mathrm{Int}\,(\mathrm{Cl}\,(W))=\mathrm{scl}\,W=W\subset V$. Thus $f^{-1}(V)\in T_S$.

In [22], it was indicated that almost-continuous implies weakly-continuous. The following example shows that weakly-continuous is not an equivalent condition in Theorem 2.8.

Example 2.2. Let $X=\{a,b,c\}$, let $T=\{\emptyset,X,\{c\},\{a,c\},\{b,c\}\}$, let $S=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$, and let $f:X\to X$ be the identity function. Then $f:(X,T)\to(X,S)$ is weakly-continuous and almost-open, but not almost-continuous.

Combining results above with the fact that weakly-continuous and open implies almost-continuous [22] gives the following corollary.

COROLLARY 2.2. *Let (X,T) and (Y,S) be spaces and let $f:(X,T)\to(Y,S)$ be open. Then the following are equivalent:* (a) *$f:(X,T)\to(Y,S)$ is almost--continuous,* (b) *$f:(X,T)\to(Y,S)$ is θ-continuous,* (c) *$f:(X,T)\to(Y,S)$ is δ-continuous, and* (d) *$f:(X,T)\to(Y,S)$ is weakly-continuous.*

Example 2.1 shows that continuity is not an equivalent condition in Corollary 2.2.

3. Almost-Closed Functions

THEOREM 3.1. *Let (X,T) be a space and let $A\in T$. Then* $\mathrm{scl}\,A=\mathrm{Ext}\,(\mathrm{Ext}\,(A))$ *and* $RC(X,T)=\{\mathrm{Cl}\,(\mathrm{Ext}\,(O))\mid O\in T\}$.

Proof. Since $X=\mathrm{Ext}\,(A)\cup \mathrm{Fr}\,(\mathrm{Ext}\,(A))\cup \mathrm{Ext}\,(\mathrm{Ext}\,(A))$, where $\mathrm{Ext}\,(A)$, $\mathrm{Fr}\,(\mathrm{Ext}\,(A))$, and $\mathrm{Ext}\,(\mathrm{Ext}\,(A))$ are mutually disjoint sets, $\mathrm{scl}\,A\in T$, and $\mathrm{scl}\,A\subset$

$\mathrm{Cl}(A) = X - \mathrm{Ext}(A)$, then $\mathrm{scl}\, A \subset \mathrm{Ext}(\mathrm{Ext}(A)) \subset \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{scl}\, A$, which implies $\mathrm{scl}\, A = \mathrm{Ext}(\mathrm{Ext}(A))$.

Let $C \in RC(X,T)$. Let $U \in T$ such that $C = \mathrm{Cl}(U)$. Then $C = \mathrm{Cl}(U) = \mathrm{Cl}(\mathrm{scl}\, U) = \mathrm{Cl}(\mathrm{Ext}(\mathrm{Ext}(U)))$. Thus $RC(X,T) \subset \{\mathrm{Cl}(\mathrm{Ext}(O)) \mid O \in T\}$ and since $\{\mathrm{Cl}(\mathrm{Ext}(O)) \mid O \in T\} \subset \{\mathrm{Cl}(V) \mid V \in T\} = RC(X,T)$, then $RC(X,T) = \{\mathrm{Cl}(\mathrm{Ext}(O)) \mid O \in T\}$.

The results above can be combined with the fact that $RO(X,T) = RO(X,T_S) = RO(X, FO(X,T))$ [8] to obtain the following characterizations of almost-closed functions.

COROLLARY 3.1. *Let (X,T) and (Y,S) be spaces and let $f : X \to Y$. Then the following are equivalent:* (a) *$f : (X,T) \to (Y,S)$ is almost-closed,* (b) *$f : (X,T_S) \to (Y,S)$ is almost-closed,* (c) *$f : (X, FO(X,T))) \to (Y,S)$ is almost-closed,* (d) *for each $O \in T$, $f(\mathrm{Cl}(O)) = \mathrm{Cl}(f(\mathrm{Cl}(O)))$,* (e) *for each $O \in T$, $f(\mathrm{Cl}(\mathrm{Ext}(O))) = \mathrm{Cl}(f(\mathrm{Cl}(\mathrm{Ext}(O))))$, and* (f) *for each $A \in PO(X,T)$ and each $B \subset Y$ such that $A = f^{-1}(B)$, $f\backslash_A : (A, T_A) \to (B, S_B)$ is almost-closed.*

The following example shows that if $f : (X,T) \to (Y,S)$ is continuous and closed, and $A \in PO(X,T)$, then $f(A)$ need not be pre-open in (Y,S) or (Y,S_S).

Example 3.1. Let $X = \{a,b,c\}$, let $T = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, and let $f : X \to X$ defined by $f = \{(a,a),(b,c),(c,c)\}$. Then $f : (X,T) \to (X,S)$ is continuous and closed, $\{b\} \in PO(X,T)$, and $f(\{b\}) \notin PO(X,T) = PO(X,T_S)$.

The example given in [12] shows that a continuous, closed function need not be δ-continuous. The following example shows that a θ-continuous, closed function need not be almost-continuous.

Example 3.2. Let $X = \{a,b,c,d,e,f,g,h,i,j\}$, let T be the topology on X with base $\{X, \{a\}, \{b\}, \{a,b,c,d,e\}, \{f\}, \{g\}, \{f,g,h,i,j\}\}$, let $Y = \{a,b,c,f,g,h\}$, let S be the topology on Y with base $\{Y, \{a\}, \{a,c\}, \{f\}, \{f,h\}\}$ and let $k : X \to Y$ defined by $k = \{(a,a),(b,b),(c,c),(d,b),(e,g),(f,f),(g,g),(h,h),(i,b),(j,g)\}$. Then $k : (X,T) \to (Y,S)$ is θ-continuous and closed, but not almost-continuous.

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