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# ON CONVERGENCE OF DERIVATIVES OF LINEAR COMBINATIONS OF MODIFIED LUPAS OPERATORS

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**Abstract.** We study some direct theorems in the simultaneous approximation by certain linear combinations of modified Lupas operators. We also consider a class of unbounded functions with growth of order of $t^{\alpha}$.

## 1. Introduction

Motivated by Derriennic [**1**], Sahai and Prasad [**4**] proposed modified Lupas operators defined, for functions integrable on $[0,\infty)$ by

$$(L_n f)(x) = (n-1)\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\int_0^{\infty} p_{n.\nu}(t)f(t)dt, \tag{1.1}$$

where

$$p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-(n+\nu)}.$$

It turns out that the order of approximation by these operators is at best $O(1/n)$, howsoever smooth the function may be. With the aim of bettering the said rate of approximation, May [**2**] and Rathore [**3**] have described a method for forming linear combination linear of positive operators. The approximation process follows.

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k)L_{djn}(f;x), \tag{1.2}$$

where $d_0, d_1, d_2, \cdots d_k$ are arbitrary but fixed distinct positive integers. We define

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j-d_i}, \quad k\neq 0 \text{ and } C(0,0)=1, \tag{1.3}$$

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The object of the present paper is to study the problem of simultaneous approximation by the above linear combination of medified Lupas operators.

Throughout this paper $\langle a, b\rangle \subset [0,\infty)$ denotes an open interval containing the closed interval $[a, b]$. The superscript $(r)$, $[\lambda]$ and $\|\cdot\|$ stand for the $r$-th derivative of the function, maximum integer not exceeding $\lambda$ and the sup-norm on $[a, b]$ respectively.

## 2. Auxiliary results

We shall need the following results:

LEMMA 2.1. [**4**]. *Let*

$$T_{n,m} = (n-r-1)\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} p_{n-r,\nu+r}(t)(t-x)^m dt.$$

*Then*

$$\begin{aligned} T_{n,0} = 1, \quad T_{n,1} &= \frac{(r+1)(1+2x)}{(n-r-2)}, \quad n > (r+2) \\ (n-m-r-2)T_{n,m+1} &= x(1+x)(T^{(1)}_{n.m} + 2mT_{n,m-1}) \\ &\quad + (m+r+1)(1+2x)T_{n,m}; \quad n > m+r+2. \end{aligned}$$

And hence $T_{n,m} = O(n^{-[(m+1)/2]})$.

LEMMA 2.2. [**4**]. *For* $r = 0, 1, 2, \ldots$ *we have*

$$(L^{(r)}_n f)(x) = \frac{(n-r-1)!(n+r-1)!}{(n-1)!(n-2)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} p_{n-r,\nu+r}(t)f^{(r)}(t)dt.$$

LEMMA 2.3. [**2**]. *If* $C(j,k), j = 0, 1, 2 \ldots k$ *are defined as in* (1.3), *then*

$$\sum_{j=0}^{k} C(j,k)d_j^{-m} = \begin{cases} 1 & m = 0 \\ 0 & m = 1, 2, \ldots k. \end{cases}$$

## 3. Main results

THEOREM 3.1. *Let $f$ be integrable on $[0,\infty)$ admitting $(2k+r+2)$-th derivative at a point $x \in [0,\infty)$ with $f^{(r)}(x) = O(x^{\alpha})$, where $\alpha$ is a positive integer not less that $2k+2$, as $x \to \infty$. Then*

$$\lim_{n\to\infty} n^{k+1}[L^{(r)}_n(f,k,x) - f^{(r)}(x)] = \sum_{i=1}^{2k+2} Q(i,k,r,x)f^{(i+r)}(x), \tag{3.1}$$

$$\lim_{n\to\infty} n^{k+1}[L^{(r)}_n(f,k+1,x) - f^{(r)}(x)] = 0, \tag{3.2}$$

*where $Q(i,k,r,x)$ are certain polynomials in $x$ of degree at most $i$. Furthermore if $f^{(2k+r+2)}$ exists and is continuous on $\langle 1,b\rangle$ then* (3.1) *and* (3.2) *hold uniformly on* $[a,b]$.

*Proof*. By Lemma 2.2. and Taylor's expansion of $f$, we are led to

$$\sum_{j=0}^{k} C(j,k)\frac{(d_jn-1)!(d_jn-2)!}{(d_jn+r-1)!(d_jn-r-2)!}L_{djn}^{(r)}(f;x)-f^{(r)}(x)$$

$$=\sum_{j=0}^{k} C(j,k)\left[(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{djn+r,\nu}(x)\int_0^\infty p_{djn-r\nu+r}(t)\{f^{(r)}(t)-f^{(r)}(x)\}dt\right]$$

$$=\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r\nu+r}(t)$$
$$\left\{\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}(t-x)^i+\varepsilon(t-x)(t-x)^{2k+2}\right\}dt$$

$$=\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)$$
$$\int_0^\infty p_{d_jn-r,\nu+r}(t)(t-x)^i dt$$

$$+\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt$$

$$=\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k} C(j,k)T_{d_jn,i}(x)+E_{n,r,k}(x),$$

where

$$\varepsilon(t-x)=(t-x)^{2k-2}\left(f^{(r)}(t)-\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}(t-x)^i\right)\ \text{for}\ t\neq x$$
$$=0,\quad \text{otherwise.}$$

Using Lemma 2.1 and 2.3,

$$\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k} C(j,k)T_{d_jn,i}(x)$$
$$=\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k} C(j,k)O\left(\frac{1}{(d_jn)^{[(i+1)/2]}}\right)$$
$$=n^{-(k+1)}\sum_{i=1}^{2k+2}Q(i,k,r,x)f^{(i+r)}(x),$$

where $Q(i,k,r,x)$ are certain polynomials in $x$ of degree at most $i$.

To prove (3.1) it suffices to show that $n^{k+1}E_{n,r,k}(x) \to 0$ for sufficiently large $n$. For arbitrary $\varepsilon > 0, A > 0$, there exists a $\delta > 0$ such that $\mid \varepsilon(T-X) \mid < \varepsilon$ for $x \le A$ and $\mid t-x \mid < \delta$. Now

$$\begin{aligned} E_{n,r,k}(x) =& \sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x) \\ &\left(\int_{|t-x|<\delta} p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt\right. \\ &+\left.\int_{|t-x|>\delta} p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt\right) \\ =& I_1 + I_2 \quad \text{(say)}. \end{aligned}$$

To estimate $I_1$, using Lemma 2.1 we get

$$\begin{aligned} \mid I_1 \mid \le& \sum_{j=0}^{k} \mid C(j,k) \mid (d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_{|t-x|<\delta} p_{d_j-r,\nu+r}(t) \\ &\qquad \mid \varepsilon(t-x) \mid (t-x)^{2k+2}dt \\ <& \varepsilon\sum_{j=0}^{k}|C(j,k)|(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r,\nu+r}(t)(t-x)^{2k+2}dt \\ =& \varepsilon\sum_{j=0}^{k} \mid C(j,k) \mid T_{d_jn,2k+2}(x) \\ =& \varepsilon\sum_{j=0}^{k} \mid C(j,k) \mid O((d_jn)^{-k-1}) \\ =& \varepsilon O(n^{-k-1}). \end{aligned}$$

Finally,

$$\begin{aligned} I_2 =& \sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_{|t-x|\ge\delta} p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt \\ =& \sum_{j=0}^{k} C(j,k)\left((d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_{|t-x|\ge\delta} p_{d_jn-r,\nu+r}(t)t^{\alpha}dt\right) \\ =& \sum_{j=0}^{k} C(j,k)\left((d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_{|t-x|\ge\delta} p_{d_jn-r,\nu+r}(t)\cdot\right. \\ &\qquad \left.\cdot\left(\sum_{i=0}^{\alpha}\tbinom{\alpha}{i}(t-x)^i x^{\alpha-i}\right)dt\right) \end{aligned}$$

$$= \sum_{j=0}^{k} C(j,k)\bigg((d_j n - r - 1)\sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x)$$

$$\int_{|t-x|\geq\delta} p_{d_j n-r,\nu+r}(t)\frac{(t-x)^{2k+3}}{\delta^{2k+3}}\bigg(\sum_{i=0}^{\alpha}\tbinom{\alpha}{i}(t-x)^i x^{\alpha-i}\bigg)dt\bigg)$$

$$= \sum_{j=0}^{k}\frac{C(j,k)}{\delta^{2k+3}}\sum_{i=}^{\alpha}\tbinom{\alpha}{i}x^{\alpha-i}\cdot O(T_{d_j n,2k+i+3}(x)) = O\bigg(\frac{1}{n^{k+2}}\bigg)$$

and (3.1) follows

The assertion (3.2) can be proved along similar lines using $L_n((t-x)^i,\ k+1,\ x) = O(n^{-(k+2)}),\ i = 1,2,\ldots$ which follows from Lemma 2.3.

The last assertion follows due to the uniform continuity of $f^{(2k+r+2)}$ on $[a,b]$ (enabling $\delta$ to become independent of $x \in [a,b]$).

This completes the proof.

*Remark*. We may note here that $\frac{(d_j n-1)!(d_j n-2)!}{(d_j n+r-1)!(d_j n-r-2)!} \to 1$ as $n \to \infty$.

THEOREM 3.2. *Let $1 \leq p \leq 2k+2$ and $f$ be integrable on $[0,\infty)$. If $f^{(p+r)}$ exists and is continuous on $\langle a,b\rangle$ having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $\langle a,b\rangle$ and $f^{(r)}(x) = O(x^\alpha)$($\alpha$ is a positive integer $\geq p$) then for $n$ sufficiently large*

$$\|L_n^{(r)}(f,k,x) - f^{(r)}\| \leq \mathrm{Max}\{C_1 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}),\ C_2 n^{-(k+1)}\}$$

*where $C_1 = C_1(k,p,r)$ and $C_2 = C_2(k,p,r,f)$.*

*Proof*. For every $t \in [0,\infty)$ and $x \in [a,b]$ we have

$$f^{(r)}(t) = \sum_{i=0}^{p}\frac{f^{i+r}(x)}{i!}(t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{p!}(t-x)^p + h(t,x)\chi(t), \tag{3.3}$$

where $\xi$ lies between $t$ and $x$ and $\chi(t)$ is the characteristic function of the set $[0,\infty)/\langle a,b\rangle$. The function $h(t,x)$ for $x \in [a,b]$ is bounded by $M\ t^\alpha \mid t-x\mid^p$, for some constant $M$. Using (3.3) we get

$$\sum_{j=0}^{k} C(j,k)\frac{(d_j n-1)!(d_j n-2)!}{(d_j n+r-1)1(d_j n-r-2)!}L_{d_j n}^{(r)}(f;x)$$

$$= \sum_{j=0}^{k} C(j,k)(d_j n - r - 1)\sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x)\int_0^\infty p_{d_j n-r,\nu+r}(t) f^{(r)}(t)dt$$

$$= \sum_{j=0}^{k} C(j,k)(d_j n-r-1)\sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x)\int_0^\infty p_{d_j n-r,\nu+r}(t)\bigg\{\sum_{i=0}^{p}\frac{f^{(i+r)}(x)}{i!}(t-x)^i$$

$$+ \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{p!}(t-x)^p + h(t,x)\chi(t)\bigg\}dt$$

$$=\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r,\nu+r}(t)\cdot\sum_{i=0}^{p}\frac{f^{(i+r)}(x)}{i!}(t-x)^i dt+$$
$$+\sum_{j=0}^{k} C(j,k)(d_jn-r-1).$$

$$\cdot\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r,\nu+r}(t)\left(\frac{f^{(p+r)}(\xi)-f^{(p+r)}(x)}{p!}\right)(t-x)^p dt$$
$$+\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r,\nu+r}(t)h(t,x)\chi(t)dt$$
$$=I_1+I_2+I_3 \quad \text{(say)}$$

An application of Lemma 2.1 gives us $I_1 = f^{(r)}(x) + O(n^{-(k+1)})$ uniformly in $x \in [a,b]$

To estimate $I_2$, for every $\delta > 0$, we have

$$\begin{aligned}| f^{(p+r)}(\xi) - f^{(p+r)}(x) | &\le \omega_{f^{(p+r)}}(| \xi - x |) \le \omega_{f^{(p+r)}}(| t-x |)\\ &\le (1+ | t-x | /\delta)\omega_{f^{(p+r)}}(\delta).\end{aligned}$$

Hence

$$| I_2 | \le \frac{1}{p!}\sum_{j=0}^{k} | C(j,k) | (d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r\nu+r}(t)$$
$$(1+ | t-x | /\delta) | t-x |^p \omega_{f^{(p+r)}}(\delta)dt$$

$$=\frac{\omega_{f^{(p+r)}}(\delta)}{p!}\sum_{j=0}^{k} | C(j,k) | (d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\cdot$$
$$\int_0^{\infty} p_{d_jn-r,\nu+r}(t)\cdot(| t-x |^p + | t+x |^{p+1} /\delta)dt.$$

Using Schwardz inequality for summation and then for integration we find that

$$\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^{\infty} p_{d_jn-r\nu+r}(t) | t-x |^p dt \le \left\{\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x).\right.$$
$$\left.\left(\int_0^{\infty} p_{d_jn-r\nu+r}(t) | t-x |^p dt\right)^2\right\}^{1/2}$$
$$\le \left\{\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)1/(d_jn-r-1)\left(\int_0^{\infty} p_{d_jn-r\nu+r}(t)(t-x)^{2p}dt\right)\right\}^{1/2}. \quad (3.4)$$

It may be remarked that (3.4) is true when $p$ is replaced by $p+1$ and consequently

$$| I_2 | \leq \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | \left\{ (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \cdot \int_0^{\infty} p_{d_j n - r, \nu + r}(t) ((t-x)^{2p} + (t-x)^{2(p+1)}/\delta) dt \right\}^{1/2}$$

$$= \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | O\{T_{d_j n, 2p} + \delta^{-1} T_{d_j n, 2(p+1)}\}^{1/2}$$

$$= \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | \{O((d_j n)^{-p}) + \delta^{-1} O((d_j n)^{-p-1})\}^{1/2}.$$

Choosing $\delta = n^{-1/2}$ we get $| I_2 | \leq C_1 n^{-p/2} \omega_{f^{(p+r)}}(n^{-1/2})$.

Finally, choosing a positive number $\eta$ such that $| t - x | \geq \eta$ we get

$$| I_3 | \leq \sum_{j=0}^{k} | C(j,k) | O\left( (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{|t-x| \geq \eta} p_{d_j n - r, \nu + r}(t) t^{\alpha} | t - x |^p \, dt \right)$$

$$= \sum_{j=0}^{k} | C(j,k) | O\left( (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{|t-x| \geq \eta} p_{d_j n - r, \nu + r}(t) \left( \sum_{i=0}^{\alpha} \binom{\alpha}{i} | t - x |^i x^{\alpha - i} \right) | t - x |^p \, dt \right)$$

$$= \sum_{j=0}^{k} | C(j,k) | O\left( (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{|t-x| \geq \eta} p_{d_j n - r, \nu + r}(t) \frac{| t - x |^{2m-p}}{\eta^{2m-p}} \left( \sum_{i=0}^{\alpha} \binom{\alpha}{i} | t - x |^i x^{\alpha - i} \right) | t - x |^p \, dt \right), \ m > k+1$$

$$= \sum_{j=0}^{k} \frac{| C(j,k) |}{\eta^{2m-p}} O\left( (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_0^{\infty} p_{d_j n - r, \nu + r}(t) \left( \sum_{i=0}^{\alpha} \binom{\alpha}{i} | t - x |^{i+2m} x^{\alpha - i} \right) dt \right)$$

$$= \sum_{j=0}^{k} \frac{| C(j,k) |}{\eta^{2m-p}} \cdot O((d_j n)^{-m}) = C_3 n^{-m}$$

uniformly in $x \in [a,b]$. Combining the estimates of $I_1 - I_3$, we obtain the required result.

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