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CHEBYSHEV CENTRES IN NORMED SPACES

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Abstract. The existence of Chebyshev centres and best compact approximants supposing special geometrical properties for the normed space in investigated. The positive results are obtained using a slightly changed of quisi-uniform convexity noted in [**1**]

Let X be a normed space, A and B bounded subset of X and x an element of X. Let us denote

$$d(x,A) = \inf_{Y \in A} \|x-y\|,\ K(A,r) = \{x \in X \mid d(x,A) \leq r\},$$
$$\partial(B,A) = \inf\{r \geq 0 \mid B \subset K(A,r)\}.$$

The number $R(A) = \inf\{\partial(A,x) \mid x \in X\}$ is called *Chebyshev radius* of A and the set $(C(A) = \{x^* \in X \mid \partial(A,x^*)\}$ is called *Chebyshev centre* set of A. We say that X *admits centre* if for every bounded set A of $X, C(A) \neq \varnothing$.

If $\mathcal{K}$ is the family of all compact subset of X then number $R_{\mathcal{K}}(A) = \inf_{K \in \mathcal{K}} \partial(A,K)$ is a *compact radius* of A. If there exists a $K^* \in \mathcal{K}$ such that $\partial(K^*,A) = R_{\mathcal{K}}(A)$ then we say that set A has the *best compact approximant.*

Definition 1. We say that the normed space X is α-approximative iff $\forall \varepsilon (0 < \varepsilon < 1) \exists (\delta)(\varepsilon)$ tending to 0 when ε tends to 0 and $0 \leq \delta(\varepsilon) < 1$ such that $\forall x \in X\ \exists y \in X$ with $\|y\| \leq \delta(\varepsilon)$, and such that if $z \in X$ and

$$\|z\| \leq 1 \text{ and } \|z-x\| \leq 1-\varepsilon$$

then also

$$\|z-y\| \leq 1-\varepsilon(1-\alpha).$$

The definition above has the following geometrical meaning. For every ball $K(x,\ 1-\varepsilon)$ heaving nonempty intersection with the unit ball $K(0,1)$, there exists a ball $K(y, 1-\varepsilon(1-\alpha))$ containg $K(x,1-\varepsilon) \cap K(0,1)$ so that the centre y of

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$K(y, 1-\varepsilon(1-\alpha))$ is not "so far away" from the origin, i. e. y is contained in $K(0, \delta(\varepsilon))$. It is represented by Fig. 1.

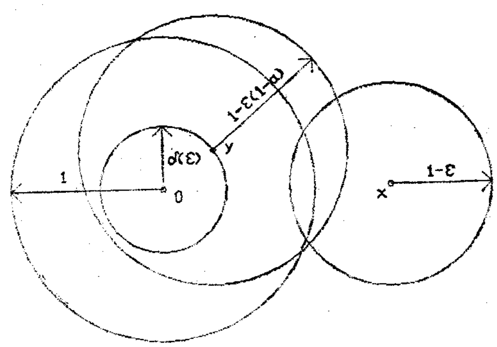

Fig. 1.

Corollary 1.

(i) If $\alpha \geq 1$ then every normed space is α-approximative.

(ii) If $\alpha < 0$ then there is no normed space which would be α-approximative.

(iii) If X is α-approximative and $\delta(\varepsilon)$ is not decreasing, we can replace the function $\delta(\varepsilon)$ with a decreasing function so that X remains α-approximative.

(iv) $\delta(\varepsilon) \geq \varepsilon(1-\alpha)$.

(v) If X is α-approximative and $r > 0$ then $\forall \varepsilon (0 < \varepsilon < r) \exists \delta_1(\varepsilon)$ tending to 0 when ε tends to 0, and $0 < \delta_1(\varepsilon < r$ such that $\forall x \in X$ $\exists y \in X$ with $\|y\| \leq \delta_1(\varepsilon)$, and such that if $z \in X$ and

$$\|z\| \leq r \text{ and } \|z-x\| \leq r-\varepsilon \text{ then also } \|z-y\| \leq r-\varepsilon(1-\alpha)$$

where $\delta_1(\varepsilon) = r\delta(\varepsilon/r)$.

(vi) If X is α-approximative and $0 < R_1 \leq r \leq R_2$ then $\forall \varepsilon (0 < \varepsilon < R_1) \exists \delta_2(\varepsilon)$ tending to 0 when ε tends to 0 and $0 < \delta_2(\varepsilon) < R_2$ such that $\forall x \in X \, \exists y \in X$ with $\|y\| \delta_2(\varepsilon)$, and such that if $z \in X$ and

$$\|z\| \leq r \text{ and } \|z-x\| \leq r-\varepsilon \text{ then also } \|z-y\| \leq r-\varepsilon(1-\alpha)$$

where $\delta_2(\varepsilon) = R_2 \delta(\varepsilon/R_1)$.

Proof. The properties (i) and (ii) suggest that it is not interesting to consider the cases $\alpha \geq 1$ or $\alpha < 0$. The proof of the properties is obvious. The property (iii) suggests that we always may assume that $\delta(\varepsilon)$ is decreasing, without loss of generality. Suppose that $\delta(\varepsilon)$ is not decreasing. Let us consider the function:

$$\delta_1(\varepsilon) = \sum_n x_{J_n} \sup_{I_n} \delta(\varepsilon)$$

where $J_n = [\varepsilon_{n+1}, \varepsilon_k]$, $I_n = [0, \varepsilon_n]$ and x_{J_n} is the characteristic function of J_n, and finally, (ε_n) is a sequence decreasing to 0. As $\delta_1(\varepsilon) \geq \delta(\varepsilon)$, X remains α-approximative when we replace $\delta(\varepsilon)$ with $\delta_1(\varepsilon)$. So the property (iii) is proved.

For proving the property (iv), we shall choose z from the definition 1, such that $\|z\| = 1$. When, further, we apply the triangle rule to the elements z, y and $z - y$, we have $\|z\| \leq \|y\| + \|z - y\|$. So we get $1 \leq \delta(\varepsilon) + 1 - \varepsilon(1 - \alpha)$, whence $\delta(\varepsilon) \geq \varepsilon(1 - \alpha)$ and (iv) is proved.

We have to map the given balls $K(0, r)$ and $K(x, r - \varepsilon)$ homoteticaly with factor $1/r$. So we get balls $K(0, 1)$ and $K(x/r, 1 - \varepsilon/r)$. Applying the definition we get the element $y \in K(0, \delta(\varepsilon/r))$. With the inverse homotetical map we are going back to the starting position. Thus the element $y_1 = ry$ is contained in $K(0, r\delta(\varepsilon/r))$ which proves (v). The property (vi) immediately follows form the (iii) and (v) since for every $r, R_1 \leq r \leq R_2$ we have $r\delta(\varepsilon/r) \leq R_2\delta(\varepsilon/R_1)$.

COROLLARY 2. *If the space X is uniformly convex then X is $0-$approximative.*

Proof. From uniform convexity of the space X it is easy to show that $\forall \varepsilon < 0\, \exists \eta(\varepsilon)$ such that if $x, y \in X$ and $\|x\| \leq 1$ and $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ then also $\|(x - y)/2\| \leq 1 - \eta(\varepsilon)$. When we replace x and y by $x - z$ and $y - z$, respectively, where z is an arbitrary element form X, we obtain

$$\|x - z\| \leq 1 \wedge \|y - z\| \leq 1 \wedge \|x - y\| \geq \varepsilon \Rightarrow \|(x - y)/2 - z\| \leq 1 - \eta(\varepsilon).$$

The relation above has the simple geometrical meaning. If $K(x, 1)$ and $K(y, 1)$ are balls in X, having nonempty intersection, and $\|x - y\| \geq \varepsilon$, then $K(x, 1) \cap K(y, 1)$ is contained in the ball whose centre is the midpoint between x and y and whose radius is equal to $1 - \eta(\varepsilon)$. Let X, be uniformly convex and $K(0, 1)$ and let $K(x, 1 - \varepsilon)$ are given balls in X. Put

$$\delta_1(\varepsilon) = \min\{\sigma \mid (0, 1) \cup K(x, 1 - \varepsilon) \subset K(\sigma x/\|x\|, 1 - \varepsilon)\}.$$

The $\delta_1(\varepsilon)$ is well defined because $\sigma\|x\|$ is contained in the set at the right-hand side. Let us prove that $\delta_1(\varepsilon)$ tends to zero when ε tends to zero. On the contrary, let us suppose that $\delta_1(\varepsilon) \to \delta_0 > 0$. Then we can choose ε such that

$$1 - \varepsilon > 1 - \eta(\delta_0). \tag{1}$$

Thus we have $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|x\|, 1 - \varepsilon)$ and $K(\delta_1(\varepsilon)x/\|x\|, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|x\|, 1) \cup K(0, 1)$. Now we apply the geometrical consequence of uniform convexity, noted before, to balls on the right-hand side of the last relation: $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|2x\|,\ 1 - \delta(\delta_(\varepsilon)))$. Taking into account the inequality (1) we get $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|2x\|, 1 - \varepsilon)$. Using the definition 1. we finally obtain $\delta_1(\varepsilon)/2 \geq \delta_1(\varepsilon)$ which is the contradiction.

Examples. We shall mention some examples of different degrees of approximativity.

The space $\mathcal{C}[0, 1]$ of continous functions on $[0, 1]$ is 0-approximative with $\delta(\varepsilon) = \varepsilon$.

Let us consider the space R^3 with the norm $\|(x, y, z)\| = (x^2 + y^2 + z^2)^{1/2}$. This space is 0-approximative with $\delta(\varepsilon) = (2\varepsilon)^{1/2}$.

If we define norm $\|(x,y,z)\| =| x | + | y | + | z |$ then for some ε and

$$\|(x,y,z)\| = 1, \quad \text{,no} \quad \|(x_1,y_1,z_1)\| < 1 \text{ satisfy}$$
$$K(0,1) \cap K((x,y,z),1-\varepsilon) \subset K((x_1,y_1,z_1),1-\varepsilon).$$

Consequently R^3 cannot be 0-approximative. It is easy to see that in this case R^3 is 0.5-approximative with $\delta(\varepsilon) \leq 3\varepsilon$.

The space l_1 of all absolutely convergent series is not α-approximative for any $\alpha < 1$.

THEOREM. *Let X be an α-approximative Banach space with $0 \leq \alpha < 1$. If the series $\sum_{n=1}^{\infty} \delta(\alpha^n)$ converges, then*

(a) every bounded set M in X has a Chebyshev centre,

(b) every bounded set M in X has a best compact approximant.

Proof. Let $R_1(M) = r_0$ and $K(x_n, r_n)$ be the sequence of balls containing M so that r_n decreasing and tends to r_0. The case $r_0 = 0$ is not of interest. On the other hand we can suppose that r_n is less than the diameter of M. We construct the sequence $K(y_n, \rho_n)$ inductively. $K(y_1, \rho_1) = K(x_1, r_1)$.

Suppose that we already have the ball $K(y_{n-1}, \rho_{n-1})$, where $n \geq 2$. Applying corollary 2. (vi) to $K(y_{n-1}, \rho_{n-1})$ and $K(x_n, r_n)$ we get $K(y_n, \rho_n)$ so that

$$\rho_n = r_n + \alpha(\rho_{n-1} - r_n) = \alpha\rho_{n-1} + (1-\alpha)r_n, \; d(y_n, y_{n-1}) \leq \delta(\rho_{n-1} - r_n).$$

After solving the system of the difference equalities we get

$$\rho_n = r_n + \alpha\varepsilon_{n-1} + \alpha^2\varepsilon_{n-2} + \cdots + \alpha^{n-1}\varepsilon_1,$$
$$d(y_n, y_{n-1}) \leq \delta(\varepsilon_n + \alpha\varepsilon_{n-1} + \alpha^2\varepsilon_{n-1} + \cdots + \alpha^{n-1}\varepsilon_1).$$

where $\varepsilon_n = r_{n-1} - r_n$. If $\alpha = 0$ then $\rho_n = r_n$ and $d(y_n, y_{n-1}) = \delta(\varepsilon_n)$. When we choose ε_n so that $\delta(\varepsilon_k) \leq 1/2^n$, the sequence (y_n) converges. If $0 < \alpha < 1$, then we choose the starting sequence $K(x_n, r_n)$ so that $\varepsilon_n < \alpha^{2n}$. Then ρ_n obviously converges to 0, and moreover, $d(y_n, y_{n-1} \leq \delta(\alpha^{n+1}(\alpha^n - 1)/(\alpha - 1))$. Therefore, because of $0 < \alpha < 1$ there exists a integer k so that $\alpha^k/(1-\alpha) < 1$. Consequently $d(y_n, y_{n-1}) \leq \delta(\alpha^{n+1-k})$, and so we conclude that (y_n) converges to the Chebyshev centre of M.

In order to prove the second part of the theorem, we suppose that $R_{\mathcal{K}}(M) = r_0$. Then there exists a real sequence (r_n) tending to r_0, and sequence of nets $(N_n \mid N_n \subset X \wedge \partial(M, N_n) = r_n \wedge \mathit{card}(N_n) < \infty)$. If $r_0 = 0$ then $\mathit{cl}(M)$ is the best compact approximant of M. If r_0 is different from 0, then we will repeat a procedure similar to the proof of the first part of the theorem. Naimelly, we construct the sequences (ρ_n) and (K_n) as follows.

Let $\rho_1 = r_1$ and $K_1 = N_1$. Suppose that the members of sequences of indices less than n are already done. Let us consider the pairs $(x, y) \in N_n \times M_{n-1}$ such that $K(x,r_n)\cap K(y,\rho_{n-1})\cap M\neq\varnothing$. Applying the corollary 2 (vi) to balls noted above, we get the set K_n such that.

$$\partial(M,K_n)=\rho_n,\ \ \rho_n=r_n+\alpha(\rho_{n-1}-r_n)=\alpha\rho_{n-1}+(1-\alpha)r_n$$
$$\partial(K_{n-1},K_n)\le\delta(\rho_{n-1}-r_n),\ \ card(K_n)\le card(N_n)card(K_{n-1}).$$

Finally, when we chose the starting sequences so that $\varepsilon_n<\alpha^{2n}$ we get $\partial(K_{n-1},K_n)\le\delta(\alpha^{n+1-k})$. Then the set $K=\cup_n K_n$ is totally bounded, hence $cl(K)$ is a compact set. As ρ_n tends to r_0 we have $\partial(M,cl(K))=r_0$ and the second part of theorem is proved.

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