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ON THE ASYMPTOTIC BEHAVIOUR OF THE G_θ^κ-MEANS OF EIGENFUNCTION EXPANSION RELATED TO THE SLOWLY OSCILLATING FUNCTIONS WITH REMAINDER TERM

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Abstract. Let $f(Q) = f(x_1, \dots, x_n) \in L^2(D)$ where D is a bounded open domain with the sufficiently regular boundary in the space E^n. Two theorems are proved in this paper. The mains result is expressed by Theorem 2 which connects the asymptotic behaviour of the G_θ^κ means of eigenfunction expansion (2.1) with the behaviour of the spherical mean of function f when this is related to behaviour of a slowly oscillating function with remainder term.

1. (i) The G_θ^κ-method of summation is defined [**3**] by

$$G_\theta^\kappa(\lambda, w) = \sum_{\lambda_\nu \le w} \left\{1 - \exp[(\lambda_\nu - w)w^{-\theta}]\right\}^\kappa a_\nu$$

$$(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\nu \to \infty \quad \text{as} \quad \nu \to \infty),$$

where $0 < \theta < 1$ and $\kappa > 0$, or by

$$G_\theta^\kappa(w) = \int_0^w \left\{1 - \exp[(t - w)w^{-\theta}]\right\}^\kappa d[A(t)],$$

where $A(t)$ is of bounded variation in any finite interval. Without loss of generality we can assume $A(0) = 0$ and in this case we have

$$G_\theta^\kappa(w) = \kappa w^{-\theta} \int_0^w \left\{1 - \exp[(t - w)w^{-\theta}]\right\}^{\kappa-1} \exp[(t - w)w^{-\theta}] A(t)\, dt. \tag{1.1}$$

(ii) It is quite natural to introduce the class of slowly oscillating functions with remainder term everywhere in analisys whenever results about convergence are extended to more general asymptotic results. They appear naturally in problems related to the asymptotic evaluations of certain integrals and sums.

Definition. Let r be a positive increasing function on $[0, \infty)$ such that

$$r(x) \to \infty, \ x \to \infty \tag{1.2}$$

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$$x^{-\delta}r(x) \quad \text{is eventually decreasing}^* \text{ for some } \delta > 0. \tag{1.3}$$

A positive measurable function L on $[0,\infty)$ is called slowly oscilating function with remainder term r if

$$L(tx)[L(x)]^{-1} = 1 + O\left\{[r(x)]^{-1}\right\}, \quad x \to \infty$$

for every $t > 0$, [**1**].

We denote the class of these functions by $K_\delta(r)$.We will use the following properties of functions of class $K_\delta(r)$ [**1**]:

(a) *If* $\lambda > 0$, *then*

$$x^{\lambda}L(x) \to \infty, \quad x \to \infty, \tag{1.4}$$

$$x^{-\lambda}L(x) \to 0, \quad x \to \infty. \tag{1.5}$$

(b) *If* $\lambda > 0$ *and*

$$L_1(x) = x^{-\lambda} \sup_{0 \le t \le x} [t^{\lambda}L(t)], \tag{1.6}$$

$$L_2(x) = x^{\lambda} \sup_{x \le t < \infty} [t^{-\lambda}L(t)], \tag{1.7}$$

then

$$L_1(x) \cong L(x) \quad \text{and} \quad L_2(x) \cong L(x), \quad x \to \infty, \tag{1.8}$$

i.e. both $L_1(x)$ *and* $L_2(x)$ *are slowly oscillating functions.*

(c) From (1.4) we get

$$[L(x)]^{-1} \le x^{\sigma-\delta}, \ \sigma > \delta > 0, \quad \text{for } x \ge M. \tag{1.9}$$

(d) Since the function $x^{-\delta}r(x)$ is decreasing, it follows that

$$w^{-\delta} \le b^{-\delta}r(b)[r(w)]^{-1} \quad \text{for } w \ge b. \tag{1.10}$$

(e) There are the asymptotic relations

$$[L(w)]^{-1}\int_0^1 g(t)L(tw)\,dt = \int_0^1 g(t)\,dt + O\left\{[r(w)]^{-1}\right\}, \quad w \to \infty \tag{1.11}$$

and

$$[L(w)]^{-1}\int_1^\infty g(t)L(tw)\,dt = \int_1^\infty g(t)\,dt + O\left\{[r(w)]^{-1}\right\}, \quad w \to \infty \tag{1.12}$$

The conditions which insure the validity of these results are usually of the form

$$\int_0^1 t^{-\lambda} \mid g(t) \mid \, dt < \infty \quad \text{or} \quad \int_1^\infty t^{\lambda} \mid g(t) \mid \, dt < \infty, \quad \lambda > 0, \tag{1.13}$$

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[*] A function f on $(0,\infty)$ is *eventually decreasing* if there exists $x \ge 0$ such that $x_2 \ge x_1 \ge x$ implies $f(x_1) \ge f(x_2)$.

assuming that

$$t^{\sigma}L(t) \textit{ is bounded on } [0,d] \textit{ for some } \sigma > \delta, \\ \text{where } [0,d] \text{ is any finite interval.} \tag{1.14}$$

(iii) Let D denote a bounded open domain in the Euclidean space E^n $(n \geq 2)$. Let $P(x_1^0, \dots, x_n^0)$ and $Q(x_1, \dots, x_n)$, be two points in D. We suppose that the boundary B of the domain D is sufficiently regular, so that the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u = 0 \qquad & \text{in } D, \\ u = 0 \qquad & \text{on } B, \\ \Delta u = \sum_{m=1}^{n} \frac{\partial^2 u}{\partial x_m^2} & \end{aligned}$$

possesses an infinite number of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_m \to \infty$ as $m \to \infty$ with the corresponding eigenfunctions $\varphi_1(Q), \varphi_2(Q), \dots, \varphi_m(Q), \dots$. We assume that these eigenfunctions form a complete orthonormal set in the space L^2.

Let $f(Q) \in L^2(D)$. We form its eigenfunction expansion

$$f(Q) \sim \sum a_m \varphi_m(Q), \tag{1.15}$$

where

$$a_m = \int_D f(Q)\varphi_m(Q) dV_Q$$

and dV_Q denotes the element of volume in E^n.

$f(P;t)$ is the spherical mean of the function $f(Q)$ over a sphere of radius t with centre at the point P, i.e.,

$$f(P;t) = 2^{-1}\pi^{-s-1}\Gamma(s+1) \int_S f(x_1^0 + t\xi_1, \dots, x_n^0 + t\xi_n)\, dS_\xi, \tag{1.16}$$

where S is the unit sphere $\xi_1^2 + \cdots + \xi_n^2 = 1$, dS_ξ its $(n-1)$-dimensional volume element and

$$s = (n-2)/2. \tag{1.17}$$

ρ_P is the shortest distance between the point P and the boundary B, and ρ is a number such that

$$0 < \rho < \rho_P. \tag{1.18}$$

(iv) $J_m(x)$ is the Bessel function of the first kind of order m. The following results hold [**6**]

$$J_m(x) = O(x^m), \quad x \to 0 \tag{1.19}$$

$$J_m(x) = O(x^{-1/2}), \quad x \to \infty \tag{1.20}$$

$$\int_0^\infty t^{\mu-\nu-1} J_\nu(t)\, dt = 2^{\mu-\nu-1}\Gamma(\mu/2)[\Gamma(1+\nu-\mu/2)]^{-1} \tag{1.21}$$

for $0 < \mu < \nu + 3/2$.

We will make use of the known formula [**6**, p. 46]

$$\frac{d^k}{(xdx)^k}[x^m J_m(x)] = x^{m-k} J_{m-k}(x). \tag{1.22}$$

(v) Throughout this paper all $M, M_0, M_1, \dots$ are the positive constants.

2. In this paper we consider a problem concerning the asymptotic behaviour of the G_θ^κ means of eigenfunction expansion (1.15), i.e.,

$$G_\theta^\kappa(P;w) = \sum_{\lambda_m \le w} \{1 - \exp[(\lambda_m - w)w^{-\theta}]\}^\kappa a_m \varphi_m(P). \tag{2.1}$$

We will formulate and prove a theorem (Theorem 2) which connects the asymptotic behaviour of G_θ^κ means (2.1) with the asymptotic behaviour of the spherical mean $f(P;t)$ of the function $f(Q)$ at the point P, defined by (1.16), when this is related to the behaviour of a function $L \in K_\delta(r)$.

T. V. Avadhani has proved for the Riesz means of eigenfunction expansion (1.15) ((3.12) in [**2**]) that

$$\begin{aligned} R_k(P;w) &= \sum_{\lambda \le w} (1 - w^{-1}\lambda_m)^k a_m \varphi_m(P) \\ &= F_k(P;w) + o[w^{-(k-s-1/2)/2}], \quad w \to \infty,\ k > s + 1/2, \end{aligned} \tag{2.2}$$

where

$$F_k(P;w) = c_k w^{(s-k+1)/2} \int_0^\rho t^{s-k} J_{k+s+1}(t\sqrt{w}) f(P;t)\, dt, \tag{2.3}$$

with

$$c_k = 2^{k-s}\Gamma(k+1)[\Gamma(s+1)]^{-1}$$

and s defined by (1.17) and ρ by (1.18). The Riesz method of summation is defined by the integral

$$R_k(w) = kw^{-k} \int_0^w (w-t)^{k-1} A(t)\, dt. \tag{2.4}$$

If k is a positive integer, then it follows from (2.4) that

$$A(w) = \frac{1}{k!} \cdot \frac{d^k}{dw^k}[w^k R_k(w)]. \tag{2.5}$$

To prepare for the proof of Theorem 2 we first prove the following

THEOREM 1. *If*

$$f(P;t) \cong t^{-a} L(t^{-1}), \quad t \to 0, \quad L \in K_\delta(r), \tag{2.6}$$

where

$$s + \sigma + 1/2 < a < 2s + 2 - \sigma, \quad s = (n-2)/2, \quad \sigma > \delta > 0, \tag{2.7}$$

then

$$\begin{aligned}F_0(P;w) &= c_0(\sqrt{w})^{s+1}\int_0^\rho t^s J_{s+1}(t\sqrt{w})f(P;t)\,dt\\ &= b_0(\sqrt{w})^a L(\sqrt{w}) + O\left\{(\sqrt{w})^{2s+2+\sigma}L(\sqrt{w})[r(\sqrt{w})]^{-1}\right\}, \quad w\to\infty,\end{aligned} \tag{2.8}$$

with $c_0 = 2^{-s}[\Gamma(s+1)]^{-1}$ *and*

$$b_0 = 2^{-a}\Gamma(s+1-a/2)[\Gamma(s+1)\Gamma(1+a/2)]^{-1}. \tag{2.9}$$

Proof. We write integral (2.8) in the form

$$\begin{aligned}F_0(P;t) &= c_0(\sqrt{w})^{s+1}\int_0^\rho t^{s-a}J_{s+1}(t\sqrt{w})L(t^{-1})\,dt\\ &\quad + c_0(\sqrt{w})^{s+1}\int_0^\rho t^s J_{s+1}(t\sqrt{w})[f(P;t)-t^{-a}L(t^{-1})]\,dt\\ &= H_1 + H_2.\end{aligned} \tag{2.10}$$

Furthermore,

$$H_1 = c_0(\sqrt{w})^a\left(\int_0^\infty - \int_0^{1/\rho\sqrt{w}}\right)t^{a-s-2}J_{s+1}(t^{-1})L(t\sqrt{w})\,dt = H_{11} - H_{12}. \tag{2.11}$$

Now we estimate the integral

$$H_{11} = c_0u^a\int_0^\infty t^{a-s-2}J_{s+1}(t^{-1})L(tu)\,dt,$$

where $u=\sqrt{w}$.

According to (2.7), (1.19) and (1.20) the function

$$g(t) = t^{a-s-2}J_{s+1}(t^{-1})$$

satisfies the conditions (1.13). Therefore we can apply the relations (1.11) and (1.12) and we get

$$H_{11} = c_0u^aL(u)\left(\int_o^\infty t^{a-s-2}J_{s+1}(t^{-1})\,dt + O\left\{[r(u)]^{-1}\right\}\right), \quad u\to\infty.$$

In virtue of (1.21) we have

$$\begin{aligned}\int_0^\infty t^{a-s-2}J_{s+1}(t^{-1})\,dt &= \int_0^\infty t^{s-a}J_{s+1}(t)\,dt\\ &= 2^{s-a}\Gamma(1+s-a/2)[\Gamma(1+a/2)]^{-1},\end{aligned}$$

i.e.,

$$H_{11} = b_0u^aL(u) + O\left\{u^aL(u)[r(u)]^{-1}\right\}, \quad u\to\infty, \tag{2.12}$$

where b_0 is given by (2.9).

With respect to (1.20) we obtain

$$|H_{12}| \leq M u^a L(u) \int_0^{1/\rho u} t^{a-s-3/2} L(tu)[L(u)]^{-1}\, dt$$

and further by (1.9) it follows that

$$\begin{aligned} |H_{12}| \leq & M u^a L(u) u^{-\delta} \int_0^{1/\rho u} t^{a-s-\sigma-3/2}[(tu)^\sigma L(tu)]\, dt \\ \leq & M u^a L(u) u^{-\delta} \sup_{0\leq v\leq 1/\rho} [v^\sigma L(v)] \int_0^{1/u\rho} t^{a-s-\sigma-3/2} dt. \end{aligned}$$

In virtue of (1.10) we have

$$|H_{12}| \leq M u^a L(u)[r(u)]^{-1} \rho^\delta r(1/\rho) \sup_{0\leq v\leq 1/\rho} [v^\sigma L(v)] \int_0^{1/u\rho} t^{a-s-\sigma-3/2} dt$$

and by (2.7) and (1.14) we finally get

$$H_{12} = O\left\{u^a L(u)[r(u)]^{-1}\right\}. \tag{2.13}$$

With respect to (2.11)–(2.13) we get

$$H_1 = b_0 u^a L(u) + O\left\{u^a L(u)[r(u)]^{-1}\right\}, \quad u \to \infty, \tag{2.14}$$

where b_0 is given by (2.9).

Now we estimate the integral H_2. By assumption (2.6), ρ can be chosen so that

$$|f(P;t) - t^{-a}L(t^{-1})| \leq \varepsilon t^{-a} L(t^{-1}), \quad \text{for } 0 \leq t \leq \rho,$$

whence

$$\begin{aligned} |H_2| &\leq \varepsilon c_0 u^{s+1} \int_0^\rho t^{s-a} \mid J_{s+1}(tu) \mid L(t^{-1}) dt \\ &= \varepsilon c_0 u^a \int_{1/u\rho}^\infty t^{a-s-2} \mid J_{s+1}(t^{-1}) \mid L(tu) dt. \end{aligned}$$

Since $\mathrm{Im}(x) = O(x^m)$ on $[0,\infty)$ it follows that

$$\begin{aligned} |H_2| &\leq M\varepsilon u^{a+\sigma} \int_{1/u\rho}^\infty t^{a-2s-3+\sigma}[(tu)^{-\sigma} L(tu)]\, dt \\ &\leq M\varepsilon u^{a+\sigma} \left\{ \sup_{1/\rho\leq v<\infty} [v^{-\sigma} L(v)] \right\} \int_{1/u\rho}^\infty t^{a-2s-3+\sigma} dt. \end{aligned}$$

Since by (2.7) $a - 2s - 2 + \sigma < 0$, we have

$$|H_2| \leq M_1 \varepsilon \rho^{2s+2-a-\sigma} \left\{ \sup_{1/\rho\leq v<\infty} [v^{-\sigma} L(v)] \right\} [L(u)]^{-1} [u^{2s+2} L(u)],$$

and further by (1.9)

$$|H_2| \leq M_1 \varepsilon \rho^{2s+2-a-\sigma} \left\{ \sup_{1/\rho \leq v < \infty} [v^{-\sigma} L(v)] \right\} u^{-\delta} [u^{2s+2+\sigma} L(u)].$$

With respect to (1.10) we have

$$|H_2| \leq M_1 \varepsilon \rho^{2s+2-a+\delta-\sigma} r(1/\rho) \left\{ \sup_{1/\rho \leq v < \infty} [v^{-\sigma} L(v)] \right\} u^{2s+2+\sigma} L(u) [r(u)]^{-1}.$$

According to the property (1.5), ρ can be chosen so that

$$H_2 = O\left\{u^{2s+2+\sigma} L(u)[r(u)]^{-1}\right\}, \quad u \to \infty. \tag{2.15}$$

In virtue of (2.10), (2.14), (2.15), (2.7) and substituting $u = \sqrt{w}$ we finally get

$$\begin{aligned} F_0(P; w) &= b_0 (\sqrt{w})^a L(\sqrt{w}) \\ &+ O\left\{(\sqrt{w})^{2s+2+\sigma} L(\sqrt{w}) [r(\sqrt{w})]^{-1}\right\}, \quad w \to \infty, \end{aligned} \tag{2.16}$$

where b_0 is given by (2.9). This concludes the proof of Theorem 1.

3. Now we formulate and prove the mentioned theorem on the asymptotic behaviour of G_θ^κ-means (2.1) of eigenfunction expansion (1.15)

THEOREM 2. *If*

$$f(P; t) \cong t^{-a} L(t^{-1}), \quad t \to 0, \quad L \in K_\delta(r) \tag{3.1}$$

with

$$s - k + \sigma + 1/2 < a < 2s + 2 - \sigma, \quad k > s + 1/2, \quad s = (n-2)/2, \tag{3.2}$$

where σ *is any number such that* $\sigma > \delta > 0$, *then*

$$\begin{aligned} G_\theta^\kappa(P; w) &= O[w^{1-\theta+a/2} L(\sqrt{w})] \\ &+ O\left\{w^{2+\sigma-\theta+\sigma/2} L(\sqrt{w}) [r(\sqrt{w})]^{-1}\right\} + o(1), \quad w \to \infty \end{aligned} \tag{3.3}$$

for $\kappa > k$, *where* k *is the smallest positive integer greater than* $s + 1/2$ *and* θ *is such that* $1/2 < \theta < 1$ *and* $2^{-1}(2s+1)(2\theta-1)^{-1}$ *is an integer.*

Proof. If k is a positive integer, then according to (2.5), (1.1) and (2.2) we can write expression (2.1) in the form

$$\begin{aligned} G_\theta^\kappa(P; w) &= \kappa w^{-\theta} \int_0^w \left\{1 - \exp[(t-w)w^{-\theta}]\right\}^{\kappa-1} \exp[(t-w)w^{-\theta}] \\ &\times \frac{1}{k!} \frac{d^k}{dt^k} [t^k R_k(P; t)] dt \\ &= \kappa w^{-\theta} \int_0^w \left\{1 - \exp[(t-w)w^{-\theta}]\right\}^{\kappa-1} \exp[(t-w)w^{-\theta}] \\ &\times \frac{1}{k!} \frac{d^k}{dt^k} [t^k F_k(P; t) + t^{k-\beta} \varepsilon(t)] dt \\ &= I_1 + I_2, \end{aligned} \tag{3.4}$$

where $\beta = (k - s - 1/2)/2$ and $\varepsilon(t) \to 0$ as $t \to 0$.

In [**4**, Chapter II] it is proved that

$$\begin{aligned} I_2 = \frac{(-1)^k}{k!}\frac{\kappa}{w^\theta}\int_0^w \frac{d^k}{dt^k}\Big(\big\{1-\exp[(t-w)w^{-\theta}]\big\}^{\kappa-1}\exp[(t-w)w^{-\theta}]\Big) \\ \times\, t^{k-\beta}\varepsilon(t)\,dt = o(1), \qquad w \to \infty, \end{aligned} \tag{3.5}$$

if $\beta = k(1-\theta)$ i.e., $(k-s-1/2)/2 = k(1-\theta)$, whence

$$k = \frac{2s+1}{2(2\theta-1)}, \quad \theta = \frac{1}{2}\left(1 + \frac{s+1/2}{k}\right),$$

i.e., $1/2 < \theta < 1$, but such θ that $2^{-1}(2s+1)(2\theta-1)^{-1}$ is an integer, because k is a positive integer.

Now we estimate the integral I_1. With respect to (2.3) we have

$$\frac{1}{k!}\frac{d^k}{dt^k}[t^kF_k(P;t)] = \frac{c_k}{k!}\int_0^\rho u^{-2k-1}\frac{d^k}{dt^k}\left[(u\sqrt{t})^{k+s+1}J_{k+s+1}(u\sqrt{t})\right]f(P;u)\,du.$$

By the successive differentiation exploring (1.22) we get

$$\frac{d^k}{dt^k}[(u\sqrt{t})^{k+s+1}J_{k+s+1}(u\sqrt{t})] = 2^{-k}u^{2k}(u\sqrt{t})^{s+1}J_{s+1}(u\sqrt{t})$$

i.e.,

$$\frac{1}{k!}\frac{d^k}{dt^k}[t^kF_k(P;t)] = \frac{(\sqrt{t})^{s+1}}{2^s\Gamma(s+1)}\int_0^\rho u^sJ_{s+1}(u\sqrt{t})f(P;u)du = F_0(P;t),$$

and according to (2.8),

$$\begin{aligned} I_1 &= \kappa w^{-\theta}\int_0^w \big\{1-\exp[(t-w)w^{\kappa-1}]\big\}^{\kappa-1}\exp[(t-w)w^{-\theta}] \\ &\quad\times \Big(b_0(\sqrt{t})^aL(\sqrt{t}) + O\Big\{(\sqrt{t})^{2s+2+\sigma}L(\sqrt{t})[r(\sqrt{t})]^{-1}\Big\}\Big)\,dt \\ &= I_{11}(w) + O[I_{12}(w)], \quad w \to \infty. \end{aligned} \tag{3.6}$$

Since $\kappa > k$ and $k \ge 1$, k is a positive integer, the function

$$T(t,w) = \big\{1-\exp[(t-w)w^{-\theta}]\big\}^{\kappa-1}\exp[(t-w)w^{-\theta}], \quad 0 \le t \le w$$

has the maximum $\kappa^{-1}(1-\kappa^{-1})^{\kappa-1}$ for $t = w - w^\theta\log\kappa$.

Therefore

$$\begin{aligned} I_{11}(w) &\le b_0(1-\kappa)^{\kappa-1}w^{-\theta}\int_0^w (\sqrt{t})^{a-\sigma}[(\sqrt{t})^\sigma L(\sqrt{t})]\,dt \\ &\le b_0(1-\kappa^{-1})^{\kappa-1}w^{-\theta}\int_0^w (\sqrt{t})^{a-\sigma}\sup_{0\le v\le\sqrt{t}}[v^\sigma L(v)]\,dt. \end{aligned}$$

According to (3.2) it follows that

$$I_{11}(w) \leq M w^{-\theta}(\sqrt{w})^{\sigma}\left\{(\sqrt{w})^{-\sigma} \sup_{0\leq v\leq\sqrt{w}} [v^{\sigma} L(v)]\right\} w^{1+a/2-\sigma/2}$$

and with respect to(1.6) and (1.8) we get

$$I_{11}(w) = O[w^{1-\theta+a/2} L(\sqrt{w})], \quad w \to \infty. \tag{3.7}$$

Finally we estimate the integral $I_{12}(w)$.

$$\begin{aligned} I_{12}(w) &\leq (1-\kappa^{-1})^{\kappa-1} w^{-\theta} \int_0^w (\sqrt{t})^{2s+2+\sigma} L(\sqrt{t})[r(\sqrt{t})]^{-1}\,dt \\ &\leq (1-\kappa^{-1})^{\kappa-1} w^{-\theta} \int_0^w (\sqrt{t})^{2s+2} \sup_{0\leq v\leq\sqrt{t}} [v^{\sigma} L(v)][r(\sqrt{t})]^{-1}\,dt \\ &\leq (1-\kappa^{-1})^{\kappa-1} w^{-\theta}(\sqrt{w})^{\sigma}\left\{(\sqrt{w})^{-\sigma} \sup_{0\leq v\leq\sqrt{w}} [v^{\sigma} L(v)]\right\} \\ &\quad\times \int_0^w (\sqrt{t})^{2s+2-\delta}[(\sqrt{t})^{-\delta} r(\sqrt{t})]^{-1}\,dt. \end{aligned}$$

According to the property (1.3) of the function $x^{-\delta} r(x)$ we see that the function $[x^{-\delta} r(x)]^{-1}$ is increasing on $[0,\infty)$, and further with respect to (1.6) we get

$$I_{12} \leq (1-\kappa^{-1})^{\kappa-1} w^{-\theta}(\sqrt{w})^{\sigma} L_1(\sqrt{w})[(\sqrt{w})^{-\delta} r(\sqrt{w})]^{-1} \int_0^w (\sqrt{t})^{2s+2-\delta}\,dt$$

In virtue of (1.8) and 3.2 we obtain

$$I_{12}(w) = O\left\{w^{2+s-\theta+\sigma/2} L(\sqrt{w})[r(\sqrt{w})]^{-1}\right\}, \quad w \to \infty. \tag{3.8}$$

According to (3.6)–(3.8) we have

$$I_1 = O[w^{1-\theta+a/2} L(\sqrt{w})] + O\left\{w^{2+s-\theta+\sigma/2} L(\sqrt{w})[r(\sqrt{w})]^{-1}\right\}, \quad w \to \infty. \tag{3.9}$$

Finally from (3.4), (3.5) and (3.9) substituting $u = \sqrt{w}$ we get (3.3). This concludes the proof of Theorem 2.

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