

ON COMMON FIXED POINTS IN LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE

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Abstract. We state a new common fixed point theorem for a continuous mapping and a multivalued upper semicontinuous mapping in locally convex topological vector spaces, under some additional conditions.

1. Introduction. The aim of this short paper is to use results given by Fan [3] and Browder [1], to prove the existence of a common fixed point for a single-valued mapping and a multivalued mapping in locally convex topological vector spaces.

For this sort of result the reader may be referred to [2], [4] and [5], for example.

Throughout, by E we will denote a Hausdorff topological vector space. Furthermore, if K is a nonempty subset of E , then 2^K will be the family of all subsets of K and $C(K)$ the family of all nonempty compact convex subsets of K .

Before proceeding to the statement of our result, we recall the concepts of algebraic boundary and multivalued upper semicontinuous mapping.

Definition 1 [1]. Let C be a closed convex subset of E . Then a point x of C is said to lie in the algebraic boundary $\delta(C)$ of C if there exists a finite dimensional subspace F of E such that x lies in the boundary of $C \cap F$.

Definition 2 [4]. A multivalued mapping $T : K \rightarrow 2^K$, $K \subset E$, is said to be *upper semicontinuous* if for any closed subset F of K , $T^{-1}(F) = \{x \in K : T(x) \cap F \neq \emptyset\}$ is a closed subset of K .

2. Main result. The main result is inspired by an interesting paper by Itoh and Takahashi [4].

Next we combine a theorem of Browder [1] with the classic Fan's fixed point theorem [3], to obtain a common fixed point theorem for a continuous mapping f

of X into E and an upper semicontinuous mapping T of X into 2^X , where X is a nonempty compact convex subset of E .

THEOREM 1. *Let E be a locally convex space and X a nonempty compact convex subset of E . Suppose that $f : X \rightarrow E$ is a continuous mapping and $T : X \rightarrow 2^X$ is an upper semicontinuous mapping, with $T(x) \in C(X)$ for all $x \in X$. Suppose further that the following condition is satisfied:*

(*) *For each $x \in X$ and for every $u \in \delta(T(x))$ there exists an element v of $T(x)$ and a real number $\lambda > 0$ (both depending on x), such that $f(u) - u = \lambda(v - u)$.*

If $A = \{x \in X : f(x) = x\}$ is convex, then there exists an element w of X such that $w = f(x) \in T(w)$.

Before we prove the main result (Theorem 1 above), we recall results given by Fan [4] and Browder [1].

THEOREM 2 [3]. *If $K \in C(E)$, $T : K \rightarrow 2^K$ is an upper semicontinuous mapping and for any $x \in K$, $T(x) \in C(K)$, then $\{x \in K : x \in T(x)\}$ is nonempty compact.*

Remark 1. Theorem 2 is the classic ‘‘Fan’s fixed point theorem’’ [3].

THEOREM 3 [1]. *Let $K \in C(E)$ and f be a continuous mapping of K into E . Suppose that for each u in $\delta(K)$, there exists an element v of K and a real number λ (both depending on u), such that $f(u) - u = \lambda(v - u)$.*

Then f has a fixed point in K .

Proof of Theorem 1. By theorem 3 we observe that, for each $x \in X$, f has a fixed point in $T(x)$. This shows that $x \in X$ implies $T(x) \cap A$ is nonempty. Now, we define a multivalued mapping S of A into 2^A , by $S(x) = A \cap T(x)$, $x \in A$. Since X is compact and f is continuous in X , it follows that A is compact; moreover by hypothesis A is convex. Hence, $S(x)$ is nonempty compact convex, for each $x \in A$.

Furthermore, since $F \subset A$ implies that $S^{-1}(F) = A \cap T^{-1}(F)$, and T is upper semicontinuous, we conclude that S is upper semicontinuous. Thus, by Theorem 2 there exists a fixed point w of S in A . For this w we have $f(w) = w \in T(w)$. The proof is now complete.

Remark 2. Theorem 1 remains true when the condition (*) is replaced by the following one:

(**) *If $x \in X$ and $u \in \delta(T(x))$, then $f(u) \in T(x)$.*

Indeed, if $x \in X$ and $u \in \delta(T(x))$, then $v = f(u) \in T(x)$ by the condition (**). For these v and $\lambda = 1$ we have $f(u) - u = \lambda(v - u)$.

Remark 3. Note that this reduces to Fan’s fixed point theorem if f is the identity on X . In this case if $x \in X$ and $u \in \delta(T(x))$, then the condition (*) of the hypothesis of Theorem 1 is satisfied for $v = u$ and $\lambda > 0$, λ arbitrary.

Remark 4. Condition (*) is not necessary for the conclusion of Theorem 1, since one adds the following hypothesis:

- a) f is a single-valued mapping of X into X and
- b) $f(T(x)) \subset T(f(x))$ for all $x \in X$ (see [4]).

Remark 5. In Theorem 1 if f is affine, then A is automatically convex.

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