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A COMPLETENESS THEOREM FOR AN INFINITARY INTUITIONISTIC LOGIC WITH BOTH ORDINARY AND PROBABILITY QUANTIFIERS

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Abstract. Our main result is a proof of the completeness theorem for an infinitary intuitionistic logic with both ordinary and probability quantifiers with respect to the Kripke semantics.

1. Sometimes it is not enough to prove that formula $\varphi(x)$ is satisfied by an element $a \in A$ in the interested model $\mathfrak{A}$ or not. It may happen that we are looking for the quantity of those $a \in A$ for which $\varphi[a]$ is true. Of course, the mathematical discipline for this kind of considerations is probability theory.

The logic suitable for this kind of reasoning was introduced by H. J. Keisler in 1976. This logic has formulas similar to those of $L_{\mathcal{A}} \subseteq L_{\omega_1\omega}$ ($\mathcal{A}$ is a countable admissible set), except that the quantifiers $(Px \geq r)$ ($r \in \mathcal{A} \cap [0,1]$ is a real number) are used instead of the usual quantifiers $(\forall x)$ and $(\exists x)$. A model for this logic is a pair $(\mathfrak{A}, \mu)$, where $\mathfrak{A}$ is a classical structure and μ is a probability measure defined in such a way that definable subsets of the universe A are measurable. The quantifiers are interpreted in the natural way, i.e.

$$(\mathfrak{A}, \mu) \models (Px \geq r)\varphi(x) \qquad \text{iff} \qquad \mu\{a \in A : (\mathfrak{A}, \mu) \models \varphi[a]\} \geq r.$$

This logic, denoted by $L_{\mathcal{A}P}$, is essentially infinitary. That means that if we allow only finite formulas we still must have an infinite rule of inference in order to prove even the weak form of the completeness theorem (see [**4**]).

Our aim is to build an intuitionistic logic with probability quantifiers. However we do not want to miss ordinary quantifiers therefore we include them in our logic, too. In our logic, (denoted by L_{API}) which allows the conjunction and disjunction of countable sets of sentences, all conectives and quantifiers are treated intuitionistically. Therefore, all of the following three classically valid equivalences are not valid in our logic

$$(Px < r)\varphi \iff \neg(Px \geq r)\neg\varphi,$$

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$$(Px \leq r)\varphi \iff (Px \geq 1-r)\neg\varphi,$$
$$(Px > r)\varphi \iff \neg(Px \geq 1-r)\neg\varphi$$

We try to find motives for introducing some particular axioms in intuitionistic mathematics and in the intuition of Kripke semantics as well.

Let us say something briefly about Kripke models in the classical case. A Kripke model consists of an ordered tree $\langle T_0, \leq\rangle$ with a single root t_0 and at each node t of the tree we have a classical model $\mathfrak{A}_t$. The nodes satisfy the following condition: $s \leq t$ implies $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ (i.e. for each n-ary relation R, $R^{\mathfrak{A}_s} \subseteq R^{\mathfrak{A}_t} \cap A_s^n$). In essence, a classical model for predicate calculus is attached to each node, with the restriction that true atomic sentences are preserved in extensions. We can interpret this kind of model as representing the growth of knowledge. We want to extend this intuition to $L_{\mathcal{A}PI}$ logic. So in each node of T_0 we have a triple $(\mathfrak{A}, \mu_i^t, \mu_o^t)$, where μ_i^t is a superadditive inner measure and μ_o^t a subadditive outer measure. Also, μ_i^t is increasing and μ_o^t is decreasing, i.e. $\mu_i^s \leq \mu_i^t \leq \mu_o^t \leq \mu_o^s$ for $s \leq t$. (For the precise definition see 2.6 and 2.7).

The logic $L_{\mathcal{A}PI}$ is infinitary for two reasons. First, this logic is much more expressive than the same first-order version, so many important notions of probability theory can be expressed, as for example: convergence almost surely, convergence in probability, random variables X_1 and X_2 have the same distribution, etc.

Secondly, in order to prove the completeness theorem we need infinity in a similar way as in the classical case. The full finiteness (this means finite formulas and finite rules of inference) can be ensured only for some logic with finite number of quantifiers of the type $Px \geq r$, $Px > r$, $Px \leq r$, $Px < r$. The infinitary intuitionistic logic has already been treated in [**6**] and [**7**], of course for ordinary quantifiers.

Finally, we shall make some remarks. We work in an admissible fragment $\mathcal{A}$ and all relevant facts about admissible sets can be found in [**1**].

The present paper written by a model theorist is intended as a work in model theory which may be of interest to both model theorists and intuitionists.

2. We can now introduce $L_{\mathcal{A}PI}$ logic and prove the completeness theorem.

2.1. *Convention*. We shall assume throughout that A is an admissible set (possibly with urelements) such that $\omega, Q \in \mathcal{A}$ (Q is a set of rational numbers) and each $a \in \mathcal{A}$ is countable (that is $\mathcal{A} \subseteq HC$, where HC is the set of hereditarily countable sets).

2.2 *Definition*. We shall assume throughout that L is a countable $\mathcal{A}$-recursive set of finitary relation and constant symbols (no function symbols).

The logic has the following logical symbols:

(a) a countable list of individual variables v_n, $n \in \mathbf{N}$,

(b) the connectives $\wedge, \vee, \Rightarrow, \neg$,

(c) the quantifiers $Px \geq r$, $Px > r$, $Px \leq r$, and $Px < r$, where x is a variable and $r \in \mathcal{A} \cap [0,1] \cap Q$,

(d) quantifiers $(\forall x)$ and $(\exists x)$,

(e) the equality symbol $=$.

2.3. *Definition.* The set of formulas of $L(\mathcal{A}P)$ is the least set such that:

(a) Each atomic formula of first-order logic is a formula of $L_{\mathcal{A}PI}$.

(b) If φ and ψ are formulas of $L_{\mathcal{A}PI}$, then $\neg\varphi$ and $\varphi \Rightarrow \psi$ are formulas of $L_{\mathcal{A}PI}$.

(c) If $\Phi \in \mathcal{A}$ is a set of formulas of $L_{\mathcal{A}PI}$ with only finitely many free variables, then $\bigwedge \Phi$ and $\bigvee \Phi$ are formulas of $L_{\mathcal{A}PI}$.

(d) If φ is a formula of $L_{\mathcal{A}PI}$ and $Px \geq r$, $Px > r$, $Px \leq r$ and $Px < r$ are quantifiers of $L_{\mathcal{A}PI}$, then $(Px \geq r)\varphi$, $(Px > r)\varphi$, $(Px \leq r)\varphi$ and $(Px < r)\varphi$ are formulas of $L_{\mathcal{A}PI}$.

(e) If φ is a formula of $L_{\mathcal{A}PI}$ and $\forall x$ and $\exists x$ are quantifiers of $L_{\mathcal{A}PI}$, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas of $L_{\mathcal{A}PI}$.

It is understood that the formulas are constructed set theoretically so that $L_{\mathcal{A}PI} \subseteq \mathcal{A}$. We denote $L_{\mathcal{A}PI}$, where $\mathcal{A} = HC$, by $L_{\omega_1 PI}$. Thus, $L_{\mathcal{A}PI} = \mathcal{A} \cap L_{\omega_1 PI}$.

If $\Box$ is a contradiction, $\varphi \Rightarrow \Box$ is shortened by $\neg\varphi$.

$L_{\mathcal{A}PI}$ has the following set of axioms, where $\varphi \in L_{\mathcal{A}PI}$ and $r, s \in \mathcal{A} \cap [0,1] \cap Q$.

2.4. *Definition.* The Axioms for $L_{\mathcal{A}PI}$ are as follows:

A.1. All axioms of finitary intuitionistic first order logic.

A.2. Infinitary axioms

$\bigwedge \Phi \Rightarrow \varphi$ where $\varphi \in \Phi$, $\qquad \varphi \Rightarrow \bigvee \Phi$ where $\varphi \in \Phi$,

$$\varphi \vee \left(\bigvee \Phi\right) \iff \bigvee_{\psi \in \Phi} (\varphi \vee \psi), \qquad \varphi \wedge \left(\bigvee \Phi\right) \iff \bigvee_{\psi \in \Phi} (\varphi \wedge \psi),$$

$$\varphi \wedge \left(\bigwedge \Phi\right) \iff \bigwedge_{\psi \in \Phi} (\varphi \wedge \psi), \qquad \varphi \vee \left(\bigwedge \Phi\right) \iff \bigwedge_{\psi \in \Phi} (\varphi \vee \psi),$$

$$\bigwedge_{\psi \in \Phi} (\varphi \Rightarrow \psi) \Rightarrow \left(\varphi \Rightarrow \bigwedge \Phi\right), \qquad \bigwedge\nolimits_{\psi \in \Phi} (\psi \Rightarrow \varphi) \iff \left(\bigvee \Phi \Rightarrow \varphi\right).$$

A.3. Monotonicity:

$(Px \geq r)\varphi \Rightarrow (Px \geq s)\varphi$, where $r \geq s$, $\qquad (Px > r)\varphi \Rightarrow (Px \geq r)\varphi$,

$(Px \leq r)\varphi \Rightarrow (Px \leq s)\varphi$, where $r \leq s$, $\qquad (Px < r)\varphi \Rightarrow (Px \leq r)\varphi$.

A.4. $(Px \geq r)\varphi \Rightarrow (Py \geq r)\varphi$, $\qquad (Px \leq r)\varphi \Rightarrow (Py \leq r)\varphi$,

$(Px > r)\varphi \Rightarrow (Py > r)\varphi$, $\qquad (Px < r)\varphi \Rightarrow (Py < r)\varphi$.

A.5. Finite additivity:

(i) $(Px \leq r)\varphi \wedge (Px \leq s)\varphi \Rightarrow ((Px \leq r + s)(\varphi \vee \psi))$,

(ii) $(Px \geq r)\varphi \wedge (Px \geq s)\psi \wedge (Px \leq 0)(\varphi \wedge \psi) \Rightarrow (Px \geq r + s)(\varphi \vee \psi)$,

A.6. $(Px \geq 0)\varphi$.

A.7. The Archimedean property:

(i) $(Px > r)\varphi \iff \bigvee_{n\in\mathbf{N}} (Px \geq r + 1/n)\varphi$,

(ii) $\big(\bigwedge_{n\in\mathbf{N}} (Px \geq r - 1/n)\varphi\big) \iff (Px \geq r)\varphi$.

A.8. $(\forall x)\varphi \Rightarrow (Px \geq 1)\varphi$.

2.5. *Definition.* The rules of inference for L_{API} are as follows.

R.1. Modus Ponens: $\varphi, \varphi \Rightarrow \psi \vdash \psi$.

R.2. Conjunction: $\{\varphi \Rightarrow \psi \mid \psi \in \Phi\} \vdash \varphi \Rightarrow \bigwedge \Phi$.

R.3. Disjunction: $\{\psi \Rightarrow \varphi \mid \psi \in \Phi\} \vdash \bigvee \Phi \Rightarrow \varphi$.

R.4. Generalization: $\varphi \Rightarrow \psi(x) \vdash \varphi \Rightarrow (\forall x)\varphi(x)$ provided x is not free in φ.

2.6. *Definition.* A Kripke model is a (complex) structure

$$\mathcal{K} = \big\langle \langle \mathfrak{A}_t, \mu_o^t, \mu_i^t \rangle, \langle T_0, \leq \rangle \big\rangle_{t\in T_0},$$

where:

(a) $\langle T_0, \leq \rangle$ is an ordered tree with the least element t_0

(b) $\mathfrak{A}_t \subseteq \mathfrak{A}_s$, for $t \leq s$

(c) $\mu_i^t \leq \mu_i^s \leq \mu_o^s \leq \mu_o^t$, for $t \leq s$

(d) μ_o^t is subadditive, i.e. $\mu_o^t(A) + \mu_o^t(B) \geq \mu_o^t(A \cup B)$

(e) μ_i^t is superadditive, i.e. $\mu_i^t(A) + \mu_i^t(B) \leq \mu_i^t(A \cup B)$ for $\mu_o^t(A \cap B) = 0$.

2.7. *Definition.* Let $\mathcal{K}$ be a Kripke model. The relation $p \Vdash \varphi$, where $p \in T_0$ and φ is a sentence, is defined by induction on the formation of φ as follows:

(i) If φ is an atomic formula, then $t \Vdash \varphi$ iff $\mathfrak{A}_t \models \varphi$.

(ii) If Φ is a set of formulas, then

$$t \Vdash \bigwedge \Phi \quad \text{iff for each} \quad \varphi \in \Phi,\ t \Vdash \varphi$$
$$t \Vdash \bigvee \Phi \quad \text{iff for some} \quad \varphi \in \Phi,\ t \Vdash \varphi$$

(iii) If φ is $\psi \Rightarrow \theta$ then $p \Vdash \varphi$ iff for each $q \geq p$, if $q \Vdash \psi$ then $q \Vdash \theta$.

(iv) If φ is $\neg\psi$, then $p \Vdash \varphi$ iff for no $q \geq p$, $q \Vdash \psi$.

(v) If φ is $(Px \geq r)\varphi$, then $t \Vdash (Px \geq r)\varphi$ iff $\mu_i^t\{x \in A_t : t \Vdash \varphi(x)\} \geq r$ and similarly for other quantifiers. $\mathcal{K} \models \varphi$ iff $t_0 \Vdash \varphi$.

In order to prove a completeness theorem we need the notion of saturated theory.

2.8. *Definition*. The theory Δ is saturated if:

(i) $\mathrm{Cn}(\Delta) = \Delta$ i.e. $\Delta \vdash \varphi$ iff $\varphi \in \Delta$

(ii) $\bigwedge \Phi \in \Delta$ iff for each $\varphi \in \Phi$, $\varphi \in \Delta$

(iii) $\bigvee \Phi \in \Delta$ iff for some $\varphi \in \Phi$, $\varphi \in \Delta$

(iv) $(\exists x)\varphi(x) \in \Delta$ iff there is a $c \in \mathrm{Ind}(\Delta)$ such that $\varphi(c) \in \Delta$.

Here, $\mathrm{Ind}(\Delta)$ is the set of all constants which appear in some formula of Δ.

We now derive the necessary Rasiowa-Sikorski type result. The proof is very much in the spirit of an elementary proof of the Rasiowa-Sikorski Lemma for Boolean algebras, except that certain simplifications for Boolean algebras are not permissible in the more general setting.

2.9. Lemma. *If $T \nvdash \varphi \Rightarrow \psi$, then there is a saturated theory $\Delta \supseteq T$ such that $\varphi \in \Delta$ and $\psi \notin \Delta$.*

Proof. Let $\psi_1, \psi_2, \psi_3, \ldots,$ be a sequence of all formulas from L_{API}, such that $\psi_{3n+1} = \bigvee_j \varphi_j^{3n+1}$, $\psi_{3n+2} = \bigwedge_j \varphi_j^{3n+2}$, and $\psi_{3n+3} = (\exists x_j)\ \varphi^{3n+3}(x_j)$.

We will recursively construct sequences $T \subseteq T_0 \subseteq T_1 \subseteq \ldots$ and $A_0 \subseteq A_1 \subseteq \ldots$. At each stage we will have $T_n - T$ and A_n finite and if $\alpha^n = \bigwedge(T_n - T)$ and $\beta^n = \bigvee A_n$ at stage n, then $T \nvdash \alpha^n \Rightarrow \beta^n$. Let $T_0 = T \cup \{\varphi\}$ and $A_n = \{\psi\}$. A typical stage falls into one of the following six cases.

Case a). We consider $n = 3k$, $\psi_{n+1} = \bigvee_j \varphi_j^{n+1}$ and $T \nvdash \big(\bigvee_j \varphi_j^{n+1}\big) \wedge \alpha^n \Rightarrow \beta^n$. Now, since $\alpha^n \wedge \big(\bigvee_j \varphi_j^{n+1}\big) \iff \bigvee_j(\alpha^n \wedge \varphi_j^{n+1})$ it follows that for some j_0, $T \nvdash \alpha^n \wedge \varphi_{j_0}^{n+1} \Rightarrow \beta^n$ and we put $A_{n+1} = A_n$ and $T_{n+1} = T_n \cup \{\varphi_{j_0}^{n+1}\}$.

Case b). We consider $n+3k$, $\psi_{n+1} = \bigvee_j \varphi^{n+1}$ and $T \vdash \big(\bigvee_j \varphi_j^{n+1}\big) \wedge \alpha^n \Rightarrow \beta^n$. In this case put $T_{n+1} = T_n$ and $A_{n+1} = A_n \cup \big\{\bigvee_j \varphi_j^{n+1}\big\}$. We must verify that $T \nvdash \alpha^n \Rightarrow \beta^n \vee \big(\bigvee_j \varphi_j^{n+1}\big)$. If not then

$T \vdash \alpha^n \Rightarrow \beta^n \vee \big(\bigvee_j \varphi_j^{n+1}\big)$,	whence
$T \vdash \alpha^n \wedge \alpha^n \Rightarrow \big(\beta^n \vee \big(\bigvee_j \varphi_j^{n+1}\big)\big) \wedge \alpha^n$,	so by assumption
$T \vdash \alpha^n \Rightarrow (\beta^n \wedge \alpha^n) \vee \beta^n$,	and so
$T \vdash \alpha^n \Rightarrow \beta^n$	contradiction.

Case c). We consider $n = 3k+1$, $\psi_{n+1} = \bigwedge_j \varphi_j^{n+1}$ and $T \nvdash \alpha^n \wedge \big(\bigwedge_j \varphi_j^{n+1}\big) \Rightarrow \beta^n$.

Case d). We consider $n = 3k+1$, $\varphi_{n+1} = \bigwedge_j \varphi_j^{n+1}$ and $T \vdash \alpha^n \wedge \big(\bigwedge_j \varphi_j^{n+1}\big) \Rightarrow \beta^n$. In this case we claim that for some j_0, $T \nvdash \alpha^n \Rightarrow (\beta^n \vee \varphi_{j_0}^{n+1})$. If not

$T \vdash \alpha^n \Rightarrow \beta^n \vee \varphi_j^n$,	for each j, whence
$T \vdash \alpha^n \Rightarrow \bigwedge_j(\beta^n \vee \varphi_j^{n+1})$	
$T \vdash \alpha^n \Rightarrow \beta^n \vee \big(\bigwedge_j \varphi_j^{n+1}\big)$,	therefore

$T \vdash \alpha^n \Rightarrow \big(\beta^n \vee \big(\bigwedge_j \varphi_j^{n+1}\big)\big) \wedge \alpha^n$

$T \vdash \alpha^n \Rightarrow (\beta^n \wedge \alpha^n) \vee \big(\big(\bigwedge_j \varphi_j^{n+1}\big) \wedge \alpha^n\big),$ whence, by assumption

$T \vdash \alpha^n \Rightarrow (\beta^n \wedge \alpha^n) \vee \beta^n$

$T \vdash \alpha^n \Rightarrow \beta^n,$ a contradiction.

Now put $T_{n+1} = T_n$ and $A_{n+1} = A_n \cup \{\varphi_{j_0}^{n+1}\}$.

Case e). We consider $n = 3k + 2$, $\psi_{n+1} = (\exists x_j)\ \varphi^{n+1}(x_j)$ and $T \nvdash (\alpha^n \wedge (\exists x_j)\ \varphi^{n+1}(x_j)) \Rightarrow \beta^n$. In this case we claim that $T \nvdash (\alpha^n \wedge \varphi^{n+1}(c_k) \Rightarrow \beta^n)$ for some c_k. If not

$T \vdash (\forall x_j)((\alpha^n \wedge \varphi^{n+1}(x_j)) \Rightarrow \beta^n)$ x_j new variable

$T \vdash (\alpha^n \wedge (\exists x_j)\varphi^{n+1}(x_j)) \Rightarrow \beta^n$ contradiction.

Let us put $A_{n+1} = A_n$ and $T_{n+1} = T_n \cup \{\varphi^{n+1}(c_k)\}$.

Case f). We consider $n = 3k + 2$, $\psi_{n+1} = (\exists x_j)\ \varphi^{n+1}$ and $T \vdash (\alpha^n \wedge (\exists x_j)\ \varphi^{n+1}(x_j)) \Rightarrow \beta^n$. In this case we claim that $T \nvdash \alpha^n \Rightarrow (\beta^n \vee (\exists x_j)\ \varphi^{n+1}(x_j))$. If not,

$$T \vdash \alpha^n \Rightarrow (\beta^n \vee (\exists x_j)\ \varphi^{n+1}(x_j))$$
$$T \vdash \alpha^n \Rightarrow ((\beta^n \wedge \alpha^n) \vee ((\exists x_j)\ \varphi^{n+1}(x_j) \wedge \alpha^n))$$
$$T \vdash \alpha^n \Rightarrow ((\beta^n \wedge \alpha^n) \vee \beta^n)$$

$T \vdash \alpha^n \Rightarrow \beta^n$ contradiction.

Now put $T_{n+1} = T_n$ and $A_{n+1} = A_n \cup \{(\exists x_j)\ \varphi^{n+1}(x_j)\}$.

We will show that a set $\Delta = \{\psi : (\exists n \in \mathbf{N})(T \vdash \alpha^n \Rightarrow \psi)\}$ satisfies the conditions of the Lemma. It is easy to see that $T \subseteq \Delta$, $\varphi \in \Delta$ and $\Delta \cap \big(\bigcup_{n\in\mathbf{N}} A_n\big) = \varnothing$.

We will show that Δ is deductively closed for two rules only (for other rules it is similar).

Rule 1. Suppose that $\varphi, \varphi \Rightarrow \psi \in \Delta$. Then there is an α^n such that $T \vdash \alpha^n \Rightarrow \varphi$ and $T \vdash \alpha^n \Rightarrow (\varphi \Rightarrow \psi)$. So by rules and axioms:

$$T \vdash (\alpha^n \wedge \varphi) \Rightarrow \psi$$
$$T \vdash (\alpha^n \vee (\alpha^n \wedge \varphi)) \Rightarrow \psi$$
$$T \vdash \alpha^n \Rightarrow (\alpha^n \vee (\alpha^n \wedge \varphi))$$

$T \vdash \alpha^n \Rightarrow \psi$ i.e. $\psi \in \Delta$.

Rule 2. Suppose that $\varphi \Rightarrow \psi \in \Delta$ for each $\psi \in \Phi$. So by construction of the theory Δ, $\bigwedge_{\psi\in\Phi}(\varphi \Rightarrow \psi) \in \Delta$. By axiom $\varphi \Rightarrow \bigwedge \Phi \in \Delta$.

The properites 2), 3) and 4) of Definition 2.8 follow by construction.

2.10. COROLLARY. (1) *If $T \nvdash \varphi$ then there is a saturated theory Δ such that $T \subseteq \Delta$ and $\Delta \nvdash \varphi$.*

(2) *Let Δ be a saturated theory. Then $\varphi \Rightarrow \psi \in \Delta$ iff for each saturated theory $\Delta' \supseteq \Delta$, if $\varphi \in \Delta'$ then $\psi \in \Delta'$.*

(3) *Let Δ be a saturated theory. Then $(\forall x_j)\,\varphi(x_j) \in \Delta$ iff for each saturated theory $\Delta' \supseteq \Delta$ and each $a \in \mathrm{Ind}(\Delta')$, $\varphi(a) \in \Delta'$.*

Now, we are going to prove our main result.

2.11. EXTENDED COMPLETENESS THEOREM. *If φ is a sentence and T is a theory of L_{API}, then $T \vdash \varphi$ if and only if $T \models \varphi$.*

Proof. The trivial part is to prove that $T \vdash \varphi$ implies $T \models \varphi$. In order to prove the theorem in the other direction we will make a Kripke model $\mathcal{K}$ such that $\mathcal{K} \not\models \varphi$ and $\mathcal{K} \models T$, where $T \nvdash \varphi$. By corollary 1) there is a saturated theory Δ_0, such that $T \subseteq \Delta_0$ and $\Delta_0 \nvdash \varphi$.

Let $\mathcal{K}$ have the tree $T_{\Delta_0} = \{\Delta \supseteq \Delta_0 : \Delta_0 \text{ saturated}\}$. Also, let $\mathfrak{A}_\Delta = \langle \mathrm{Ind}(\Delta), R_j^\Delta, \dots \rangle$ be a model which corresponds to saturated theory Δ.

If R_j is an n-ary relation symbol and $c_1, \dots, c_n \in C$, then $R_j^\Delta(c_1, \dots, c_n)$ iff $R_j(c_1, \dots, c_n) \in \Delta$.

Define μ_i^Δ by $\mu_i^\Delta\{c \in \mathrm{Ind}(\Delta) : \varphi(c) \in \Delta\} = \sup\{r : (Px \geq r)\varphi \in \Delta\}$ and define μ_o^Δ by $\mu_o^\Delta\{c \in \mathrm{Ind} : \varphi(c) \in \Delta\} = \inf\{r : (Px \leq r)\varphi \in \Delta\}$. It is easy to see that $\mu_i^\Delta\{c \in \mathrm{Ind} : \varphi(c) \in \Delta\} \geq r$ iff $(Px \geq r)\varphi \in \Delta$ and similar for $(Px > r)$, $(Px \leq r)$ and $(Px < r)$. From this and the properties of a saturated theory, it follows that $\Delta \Vdash \varphi$ iff $\varphi \in \Delta$.

Finally, $\mathcal{K} = \big\langle \langle \mathfrak{A}_\Delta, \mu_o^\Delta, \mu_i^\Delta, \langle T_{\Delta_0}, \leq \rangle \rangle \big\rangle_{\Delta \in T_\Delta}$ is the model we want.

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