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DEFINABLE ULTRAPOWERS AND THE OMITTING TYPES THEOREM

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Abstract. A proof on the existence of elementary end extensions of certain countable linearly ordered models using definable ultrapowers is given. In the same style the proof of Keisler's two cardinal theorem is presented.

1. Introduction. One of the principal tools in constructing models with specific properties in model theory is the omitting types theorem. The aim of this paper is to show that in particular cases the use of this theorem can be avoided, or as one said "it is possible to omit the use of the omitting types theorem". The theorems in question mainly say something about the existence of elementary end extensions of some types of countable models. This will include also applications to other branches of model theory, as in the proof of Keisler's two cardinal theorem. The technique we shall use are Skolem functions and definable ultrapowers.

We shall assume the usual terminology and notation in model theory, as in [**1**] for example. We shall work with a countable first-order predicate logic L with identity. Models will be denoted by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... and their domains by $A, B, C, \ldots$ respectively. The symbols $\mathfrak{A} \equiv \mathfrak{B}$, $\mathfrak{A} \prec \mathfrak{B}$ mean that $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$, and $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$. Now we shall review some properties of definable subsets of a model.

Let $\mathfrak{A}$ be a model of a first order language L, $\mathcal{D}$ the set of all definable subsets of $\mathfrak{A}$ with parameters in A, $\mathcal{F}$ the set of all definable functions of $\mathfrak{A}$ in one variable with parameters in A. Then

(1) $\mathcal{D}$ is a Boolean algebra under the usual set theoretical operations.
(2) The identity function of A belongs to $\mathcal{F}$.
(3) $\mathcal{F}$ is closed under the substitution, i.e. if $f : A^n \to A$ is definable in $\mathfrak{A}$, and $g_1, \ldots, g_n \in \mathcal{F}$, then h defined by $h(i) = h(g_1(i), \ldots, g_n(i))$ belongs to $\mathcal{F}$.

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(4) $\mathcal{D}$ is closed under the substitution i.e. if $\varphi(y_1, \dots, y_n)$ is a formula of L and $f_1, \dots, f_n \in \mathcal{F}$, then $X = \{i \in A |\ \mathfrak{A} \models \varphi(f_1(i), \dots, f_n(i))\}$ belongs to $\mathcal{F}$.

(5) $\mathcal{F}$ is a domain of a submodel of the power $\mathfrak{A}^A$.

For example, in order to prove (4), let θ_i define f_i in $\mathfrak{A}$. Then

$$i \in X \quad \text{iff} \quad \mathfrak{A} \models \exists y_1 \dots y_n (\varphi(\vec{y}) \wedge \theta_1(i, y_1) \wedge \dots \wedge \theta_n(i, y_n)).$$

Thus $X \in \mathcal{D}$. Statement (5) is an immediate consequence of (4), while the other properties are proved in a similar way. These properties of definable subsets we will use in the following often and without explicit mention.

2. Definable ultrapowers. In this part we shall review Skolem ultrapower construction with definable functions over a countable model, and we shall see that in certain circumstances Łoš theorem holds. An appropriate choice of an ultrafilter will enable us to omit a particular type, too. In the following we shall use the usual notions connected with the ultrapower construction. For example, if $\mathcal{F}$ is a set of functions with a domain M, and $\mathcal{U}$ is an ultrafilter of a Boolean algebra of subsets of M, then we can introduce an equivalence relation on $\mathcal{F}$ defined by $f \sim g$ iff $f = g$ mod $\mathcal{U}$ i.e. $\{i \in M |\ f(i) = g(i)\} \in \mathcal{U}$. In this case, if $f \in \mathcal{F}$ then $f_{\mathcal{U}}$ denotes the class of equivalence of f, while $\mathcal{F}/\mathcal{U}$ is the set of all equivalence classes. If M is a domain of a model $\mathfrak{M}$ of a language L and if $\mathcal{F}$ is a domain of the submodel of the power $\mathfrak{M}^M$, then the same letter shall denote this model. In this case $\mathcal{F}/\mathcal{U}$ is a domain of a model defined in the usual way, and this model we shall denote also by $\mathcal{F}/\mathcal{U}$. Finally, it is said that a model $\mathfrak{M}$ has built in Skolem functions if its complete theory has built in Skolem functions.

THEOREM 2.1. *Let $\mathfrak{A}$ be a model of a language L with built in Skolem functions, $\mathcal{D}$ the Boolean algebra of definable subsets of $\mathfrak{A}$ with parameters, $\mathcal{F}$ the set of all definable functions of $\mathfrak{A}$ with parameters in A, and $\mathcal{U}$ an ultrafilter of $\mathcal{D}$. Then $\mathcal{F}/\mathcal{U}$ is a model of L and it satisfies Łoš theorem i.e.*

For all formulas ψ of L and $\vec{f} \in \mathcal{F}$

$$\mathcal{F}/\mathcal{U} \models \varphi[f_{1\mathcal{U}}, \dots, f_{n\mathcal{U}}] \quad \textit{iff} \quad \{i \in A |\ \mathfrak{A} \models \varphi[f_1(i), \dots, f_n(i)]\} \in X$$

Proof. The proof is by induction on the complexity of formulas as in the original form of Łoš theorem. We present here only the main step: the case of the existential quantifier. So let $\varphi = \exists x \psi$.

($\Rightarrow$) This part is obvious as the sets $\{i \in A |\ \mathfrak{A} \models \psi[g(i), f_1(i), \dots, f_n(i)]\}$ and $\{i \in A |\ \mathfrak{A} \models \varphi[f_1(i), \dots, f_n(i)]\}$ belong to $\mathcal{D}$ for definable $g, \vec{f}$.

($\Leftarrow$) Let $X = \{i \in A |\ \mathfrak{A} \models \varphi[f_1(i), \dots, f_n(i)]\}$, and θ_i be formulas of L which define f_i in $\mathfrak{A}$. By assumption, $X \in \mathcal{U}$, so let $\theta_X(y)$ be a formula which defines X in $\mathfrak{A}$. Further

$$\mathfrak{A} \models \forall y_1 \dots y_n \forall i (\theta_X(i) \wedge \theta_1(i, y_1) \wedge \dots \wedge \theta_n(i, y_n) \rightarrow \exists x \varphi(x, \vec{y})).$$

Therefore, there is a Skolem function $h(x, \vec{y})$ such that

$$\mathfrak{A} \models \forall y_1 \dots y_n \forall i(\theta_X(i) \wedge \theta_1(i, y_1) \wedge \dots \wedge \theta_n(i, y_n) \to \varphi(h(x, \vec{y}), \vec{y})).$$

Taking $g(i) = h(i, f_1(i), \dots, f_n(i))$ we have:

$$i \in X \quad \text{implies} \quad \mathfrak{A} \models \psi[g(i), f_1(i), \dots, f_n(i)]$$

and therefore, by the induction hypothesis, $\mathcal{F}/\mathcal{U} \models \psi[g_{1\mathcal{U}}, f_{1\mathcal{U}}, \dots, f_{n\mathcal{U}}]$, hence $\mathcal{F}/\mathcal{U} \models \varphi[f_{1\mathcal{U}}, \dots, f_{n\mathcal{U}}]$. ♢

3. Definable ultrapowers of ordered structures. Now we shall consider definable ultrapowers of a countable ordered model. It will appear that under some assumptions on the model and the proper choice of an ultrafilter, the ultrapower is an end extension of the model.

LEMMA 3.1. *Let $\mathfrak{M} = (M, \leq, \dots)$ be a linearly ordered model without the greatest element, $\mathcal{D}$ the Boolean algebra of all definable subsets of $\mathfrak{M}$ with parameters in M, and $S = \{g_0, g_1, \dots\}$ a countable family of bounded definable functions in $\mathfrak{M}$ with parameters in M. If for all formulas of L*

(R) $$\mathfrak{M} \models \forall x \leq z \exists y \varphi \to \exists u \forall x \leq z \exists y \leq u \varphi$$

then there is an ultrafilter $\mathcal{U}$ of $\mathcal{D}$ such that

— *For all $g \in S$, $g = \text{const} \mod \mathcal{U}$.*

— *For all $a \in M$, $(a, \infty) \in \mathcal{U}$, where $(a, \infty) = \{x \in M \mid a < x\}$.*

First we prove the following

CLAIM 3.2. *If $X \in \mathcal{D}$ is unbounded and $f : M \to M$ is definable and bounded, then there is an unbounded $Y \subseteq X$, $Y \in \mathcal{D}$ such that $f|Y = \text{const}$.*

Proof of Claim. There are two possibilities:

Case 1. There is an $a \in M$ such that $f^{-1}[a] \cap X$ is unbounded. Then we can take $Y = f^{-1}[a] \cap X$.

Case 2. For all $a \in M$, $f^{-1}[a] \cap X$ is bounded. Thus, as X and f are definable, we can write informally

$$\mathfrak{M} \models \forall x \leq m \exists y f^{-1}[x] \cap X \subseteq \{v \in M \mid v \leq y\}$$

where $m \in M$ is a bound of f. Since $\mathfrak{M}$ satisfies the scheme (R) there is a $u \in M$ such that

$$\mathfrak{M} \models \forall x \leq m \exists y \leq u f^{-1}[x] \cap X \subseteq \{v \in M \mid v \leq y\}.$$

Since $X = \bigcup_{x \leq m} (f^{-1}[x] \cap X) \subseteq \bigcup_{x \leq u} \{x \in M \mid x \leq y\} \subseteq \{x \in M \mid x \leq u\}$ it follows that $X \subseteq \{v \in M \mid v \leq u\}$, and this contradicts the assumption that X is unbounded. Thus, Case 2 is impossible, i.e. Claim holds.

Proof of Lemma. By the above claim we can construct a sequence of unbounded definable subsets of A such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots \quad \text{and} \quad g_i | X_i = \text{const}.$$

Since X_i is unbounded, $(a, \infty) \cap X_i \neq \emptyset$. Therefore, the family $\{X_i |\ i \in \omega\} \cup \{(a, \infty)|\ a \in M\}$ has the finite intersection property, hence there is an ultrafilter $\mathcal{U}$ of $\mathcal{D}$ containing all sets X_i and (a, ∞). $\diamondsuit$

Let $\mathfrak{M}, \mathcal{D}$, and $\mathcal{U}$ be as in the previous lemma, and let $\mathcal{F}$ be the set of all functions definable in $\mathfrak{M}$ with parameters in M. Observe that the identity function $i : M \to M$ and $\hat{a} = \langle a|\ i \in M\rangle$ belong to $\mathcal{F}$. Further, $\mathcal{F}/\mathcal{U}$ is a model of the language L, and it has the following additional properties:

(1) By Łoš theorem (Theorem 2.1), the mapping $\mu : a \mapsto \hat{a}_{\mathcal{U}}$, $a \in M$, is an elementary embedding of $\mathfrak{M}$ into $\mathcal{F}/\mathcal{U}$.

(2) For every $a \in M$, $\hat{a} \leq i \mod \mathcal{U}$ since $(a, \infty) \in \mathcal{U}$. Thus $i_{\mathcal{U}} \in \mathcal{F}/\mathcal{U} \backslash M$, i.e. $\mathcal{F}/\mathcal{U}$ is a proper extension of $\mathfrak{M}$.

(3) If for $f \in \mathcal{F}$ there is a $b \in M$ such that $f_{\mathcal{U}} \leq \hat{b}_{\mathcal{U}}$, then there is a $g \in \mathcal{F}$ such that $f = g \mod \mathcal{U}$, and $g(i) \leq b$ for all $i \leq a$. Hence $g = \text{const} \mod \mathcal{U}$. Thus, there is an $a \in M$ such that $f_{\mathcal{U}} = \hat{a}$. Therefore, M is an initial segment of $\mathcal{F}/\mathcal{U}$, i.e. $\mathcal{F}/\mathcal{U}$ is an end extension of $\mathfrak{M}$.

The above consideration is summarized in the following

COROLLARY 3.3. *Let $\mathfrak{M} = (M, \leq, \dots)$ be a countable, linearly ordered model without the greatest element of a countable language L. If $\mathfrak{M}$ has built in Skolem functions and $\mathfrak{M}$ satisfies*

$$\forall x \leq z \exists y \varphi \to \exists u \forall x \leq z \exists y \leq u \varphi,$$

where φ is a formula of L and u is a variable not occurring in φ, then $\mathfrak{M}$ has a proper elementary end extension.

This corollary can be derived under weaker assumptions, i.e. it is not necessary to assume that the model $\mathfrak{A}$ has built in Skolem functions. Namely, as it is easily seen, a linear ordering without the greatest element which satisfies scheme (R) is a regular relation in the sense of [**5**], and according to the main theorem of [**5**] then $\mathfrak{A}$ has an elementary end extension. However, the theorem in [**6**] is proved by use of a quite different technique, i.e. using the omitting types theorem.

4. Applications. In all of the following examples the notion of k-like model shall be used. A linearly ordered model $\mathfrak{A} = (A, \leq, \dots)$ is said to be k-like if $|A| = k$, and each $k \in A$ has fewer than k predecessors in respect to $\leq$. If $\mathfrak{M}$ is as in the last corollary, then applying the corollary ω_1-times, we obtain an elementary chain of countable models $\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \dots \prec \mathfrak{M}_\alpha \prec \dots$, $\alpha \leq \omega_1$, so that each member of the chain is an elementary end extension of its predecessors. Then $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{M}_\alpha$ is an ω_1-like model which is an end extension of $\mathfrak{M}$.

1. *Peano arithmetic (PA).* This theory has (almost) built-in Skolem functions, further it satisfies the scheme (R) in respect to the natural ordering of models of PA, therefore every countable model has a proper elementary end extension as well as an ω_1-like end extension. This is a part of McDowell-Specker theorem which holds without restrictions for arbitrary models of PA.

2. *Keisler's two cardinal theorem.* This theorem is stated as follows:

THEOREM (J. Keisler) *Let* $\mathfrak{A} = (A, V, \dots)$ *be a model of a countable language* L *such that* $\aleph_0 \leq |V| \leq |A|$. *Then there are models* $\mathfrak{B} = (B, W, \dots)$ *and* $\mathfrak{C} = (C, W, \dots)$ *such that* $\mathfrak{B} \prec \mathfrak{C}$ *and* $|B| = \aleph_0$, $|C| = \aleph_1$.

According to the downward Löwenheim-Skolem theorem we may assume in the proof of Keisler's theorem that $|A| = k^+$ for some cardinal k. Let $\leq$ be a linear ordering of A of the order type k^+, and let $\mathfrak{A}^S$ be the Skolem expansion of model $(A, \leq, V, \dots)$, where V is an interpretation of the unary predicate symbol $P \in L$. Then $\mathfrak{A}^S$ has built in Skolem functions, and since k^+ is a regular cardinal, $\mathfrak{A}^S$ satisfies (R). By the downward Löwenheim-Skolem theorem there is a countable $\mathfrak{B}^S \prec \mathfrak{A}^S$. Then $\mathfrak{B}^S$ has, obviously, built in Skolem functions as well, and it satisfies (R). Also, W is bounded in $\mathfrak{B}^S$ since $\mathfrak{A} \models \exists x \forall y (P(x) \to y \leq x)$. By Corollary 3.3 and the above remark, there is an ω_1-like elementary end-extension $\mathfrak{C}^S$ of $\mathfrak{B}^S$. Then $\mathfrak{C} = \mathfrak{C}^S | L$ is the required model.

There are several proofs of this theorem in the literature. The proof which uses the completeness theorem for ω-logic [**2**], the omitting type theorem [**1**], Robinson's forcing [**4**], end extensions of regular relations [**5**], and the completeness theorem of logic with the quantifier "there exists uncountably many" [**6**]. However, all these proofs use some form of the omitting types theorem. The proof presented in this paper does not rely on this theorem, accordingly it may lead possibly to other applications in model theory.

3. *k-like models.* A particular case of the following problem raised by Mostowski and Furhken (see [**3**]) can be solved too. This problem is stated as:

Which pairs of cardinals k, λ *have the property that for every* k*-like model* $\mathfrak{A}$ *there exists a* λ*-like model* $\mathfrak{B}$ *which is elementarily equivalent to* $\mathfrak{A}$?

From the above proof of Keisler's two cardinal theorem, it is obvious the following: If k is an infinite regular cardinal, and $\mathfrak{A}$ is an ordered k-like model then there is an ω_1-like model $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$.

REFERENCES

[**1**] C. C. Chang, H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[**2**] H. J. Keisler, *Some model theoretic results for* ω*-logic*, Israel J. Math. **4** (1966), 249–261.

[**3**] H. J. Keisler, *Models with orderings*, in: *Logic, Methodology and Philosophy of Science III*, (van Rootstelar and Staal, eds.), North-Holland, Amsterdam, 1968, 35–62.

[**4**] H. J. Keisler, *Forcing and the omitting types theorem, Studies in Model Theory*, MAA Studies in Mathematics **8**, (ed. M. D. Morley), 1973, 96–133.

[**5**] Ž. Mijajlović, *A note on elementary end extension*, Publ. Inst. Math. (Beograd) **21** (**35**) (1977), 141–144.

[**6**] Ž. Mijajlović, *On the definability of the quantifier "there exists uncountably many"*, Studia Logica **44** (1985), 258–264.

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