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COHERENT STATES AND FRAMES IN THE BARGMAN SPACE OF ENTIRE FUNCTIONS

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Abstract. A conjecture was given in [**3**] about the possibility of decomposition of an arbitrary f in $L^2(\mathbf{R})$ in terms of the family of functions

$$\varphi_{mn}(x) = \pi^{-1/4}\exp\{-(1/2)\,imnab + imxa - (1/2)\,(x-nb)^2\}, \qquad a,b>0; \quad ab<2\pi.$$

We prove this conjecture for $ab < 2\pi$ and b sufficiently large. Also, we give some applications for the Bargman space of entire functions.

1. Preliminaries. Coherent states are L^2 functions labeled by phase space points. If we want to treat functions depending on n-dimensional Cartesian variables the associated phase space is $\mathbf{R}^n \times \mathbf{R}^n$. To construct a family of coherent states, one starts by choosing one vector (see [**6**]) Φ in $L^2(\mathbf{R}^n)$. For any phase space point $(p,q) \in \mathbf{R}^n \times \mathbf{R}^n$ the associated coherent state Φ_{pq} is defined by

$$\Phi_{pq}(x) = e^{ipx}\Phi(x-q).$$

Perhaps the most important property of the coherent states is the "resolution of identity" (see [**6**]) i.e. for any function $\Phi \in L^2(\mathbf{R}^n)$ which satisfies the condition $\int |\Phi(x)|^2\,dx = 1$ and any function $f \in L^2(\mathbf{R}^n)$ we have

$$f(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} dp \int_{\mathbf{R}^n} (f,\Phi_{pq})\Phi_{pq}(x)\,dq \tag{1}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $L^2(\mathbf{R}^n)$.

This representation of functions has been used in quantum mechanics, quantum optics and signal theory (see [**4**], [**5**], [**6**], [**7**]). In the case when $\Phi(x) = e^{-x^2/2}$ and $n=1$ the properties of associated coherent states were studied in the papers [**2**], [**3**].

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Let F (see [**1**]) denote the vector space of entire functions of one complex variable $z = x + iy$ such that

$$\|f\|^2 = \frac{1}{2\pi} \iint_{\mathbf{R}^2} |f(x+iy)|^2 e^{-(x^2+y^2)/2}\, dxdy < \infty.$$

F is a Hilbert space with inner product

$$(f,g) = \frac{1}{2\pi} \iint_{\mathbf{R}^2} f(x+iy)\, \overline{g(x+iy)} e^{-(x^2+y^2)/2}\, dxdy.$$

For any $z \in \mathbf{C}$, the function $e_z(\,\cdot\,)$ is defined by

$$e_z(z') = \exp\{-|z|^2/4 - \bar{z}z'/2\} \qquad \text{(reproducing kernel)}.$$

Then $e_z \in F$, and for every $f \in F$

$$(f, e_z) = \exp\{-|z|^2/4\} f(z). \tag{2}$$

An unitary map from $L^2(\mathbf{R})$ onto F is given by the Bargman transform U_B. It is defined by

$$(U_B\varphi)(z) = \pi^{-1/4} e^{z^2/4} \int_{\mathbf{R}} e^{-x^2/2} \varphi(x) e^{-ixz}\, dx \qquad \text{for } \ \varphi \in L^2(\mathbf{R}).$$

Its inverse is given by

$$(U_B^{-1} f)(x) = \pi^{-1/4} e^{-x^2/2} \int_{\mathbf{C}} f(z) e^{\bar{z}^2/4} e^{ix\bar{z}}\, d\mu(z) \qquad \text{for } \ f \in F$$

where $d\mu(z) = (1/2\pi) e^{-|z|^2/2}\, d(\operatorname{Re} z)\, d(\operatorname{Im} z)$.

It is well known (see [**2**]) that the family

$$\varphi_{mn}(x) = \exp\Big\{-\frac{1}{2} imnab + imxa - \frac{1}{2}(x - nb)^2\Big\}$$

is not complete in $L^2(\mathbf{R})$ if $ab > 2\pi$; only if $ab \le 2\pi$ can φ_{mn} give rise to an expansion formula for arbitrary $f \in L^2(\mathbf{R})$.

It is important at this point to remark that for, $ab \le 2\pi$ the vectors $\{\varphi_{mn}\}$ are not ω "independent" in the sense that one vector of the family lies in the closed linear span of the other vectors.

If $ab = 2\pi$, then removing one φ_{mn} transforms the remaining family into an ω-independent set.

If $ab < 2\pi$ then the family $\{\varphi_{mn}\}$ remains ω-independent even after the removal of any finite number of φ_{mn}'s.

It is proved in [**3**] that every function $f \in L^2(\mathbf{R})$ can be expanded into a series with respect to a system $\{\varphi_{mn}\}$ if $a, b > 0$, $ab = 2\pi/n$ and $n \in \mathbf{N}$, $n \ge 2$.

From that result it follows that

$$m\|f\|^2 \le \sum_{p,q \in \mathbf{Z}} |f(z_{pq})|^2 e^{-|z_{pq}|^2/2} \le M\|f\|^2 \tag{3}$$

where $z_{pq} = pa + iqb$, $m, M > 0$, and does not depend on f.

In [**3**] an open problem is given: is it possible to expand a function $f \in L^2(\mathbf{R})$ into a series with respect to a system $\{\varphi_{mn}\}$ if $a, b > 0$ and $ab < 2\pi$?

In that paper the authors stated that they have proved the conjecture for $ab < 2\pi \cdot 0.996$.

We give a simple proof of this conjecture if $ab < 2\pi \cdot k$ $(k < 1)$ and b is sufficiently large.

2. Main result. THEOREM 1. *If* $a, b > 0$ *and*

$$2\sum_{m=1}^{\infty} e^{-m^2\pi^2/a^2} < \frac{\min_{x\in\mathbf{R}} g(x)}{\max_{x\in\mathbf{R}} g(x)}, \qquad \text{where} \quad g(x) = \sum_{n=-\infty}^{\infty} e^{-(x-nb)^2},$$

then every function $f \in L^2(\mathbf{R})$ *can be expanded into a series with respect to a system of functions* $\{\varphi_{mn}\}$.

Proof. Let $\varphi \in C_0^\infty$. We define the operator T on C_0^∞ by

$$T\varphi = \sum_{m,n} (\varphi, \varphi_{mn})_{L^2(\mathbf{R})} \cdot \varphi_{mn} \tag{4}$$

The kernel operator T is

$$K(x,y) = \theta(x,y) \cdot \sum_{m\in\mathbf{Z}} e^{ima(x-y)}, \tag{5}$$

where

$$\theta(x,y) = \sum_{n\in\mathbf{Z}} e^{-(x-nb)^2/2-(y-nb)^2/2}$$

The convergence in (5) is in the sense of distributions. Since

$$\sum_{m\in\mathbf{Z}} e^{ima(x-y)} = \frac{2\pi}{a}\sum_{m\in\mathbf{Z}} \delta\Big(y - \Big(x - \frac{2m\pi}{a}\Big)\Big),$$

(see [**8**]), the operator T is acting on C_0^∞ in the following way

$$(T\varphi)(x) = (K(x,\,\cdot\,), \varphi) = \frac{2\pi}{a}\sum_{m\in\mathbf{Z}} \varphi\Big(x - \frac{2m\pi}{a}\Big)\theta\Big(x, x - \frac{2m\pi}{a}\Big)$$

i.e.

$$(T\varphi)(x) = \frac{2\pi}{a}\sum_{m\in\mathbf{Z}} \theta\Big(x, x + \frac{2m\pi}{a}\Big)\varphi\Big(x + \frac{2m\pi}{a}\Big).$$

Since $\varphi \in C_0^\infty$, only a finite number of terms in this series is nonzero. Then

$$\begin{aligned}(T\varphi,\varphi) = {} & \frac{2\pi}{a}\int_{\mathbf{R}} \theta(x,x)|\varphi(x)|^2\,dx \\ & + \frac{2\pi}{a}\sum_{m\neq 0}\int_{\mathbf{R}} \theta\Big(x, x + \frac{2m\pi}{a}\Big)\varphi\Big(x + \frac{2m\pi}{a}\Big)\overline{\varphi(x)}\,dx.\end{aligned} \tag{6}$$

Since $\theta(x,x) = g(x)$, it follows from (6) that

$$\begin{aligned}(T\varphi,\varphi) \geq \frac{2\pi}{a}\min_{x\in\mathbf{R}} g(x)\|\varphi\|^2 \\ - \frac{2\pi}{a}\sum_{m\neq 0}\int_{\mathbf{R}} \theta\Big(x-\frac{m\pi}{a}, x+\frac{m\pi}{a}\Big)\Big|\varphi\Big(x+\frac{m\pi}{a}\Big)\Big|\,\Big|\varphi\Big(x-\frac{m\pi}{a}\Big)\Big|\,dx \\ = \frac{2\pi}{a}\min_{x\in\mathbf{R}} g(x)\|\varphi\|^2 \\ - \frac{2\pi}{a}\sum_{m\neq 0}\int_{\mathbf{R}} g(x)e^{-m^2\pi^2/a^2}\Big|\varphi\Big(x+\frac{m\pi}{a}\Big)\Big|\,\Big|\varphi\Big(x-\frac{m\pi}{a}\Big)\Big|\,dx.\end{aligned}$$

Hence, we have the estimate

$$(T\varphi,\varphi) \geq \frac{2\pi}{a}\|\varphi\|^2\min_{x\in\mathbf{R}} g(x) - \frac{2\pi}{a}\sum_{m\neq 0}\max_{x\in\mathbf{R}} g(x)\cdot e^{-m^2\pi^2/a^2}\|\varphi\|^2$$

i.e.

$$(T\varphi,\varphi) \geq \frac{2\pi}{a}\Big(\min_{x\in\mathbf{R}} g(x) - 2\max_{x\in\mathbf{R}} g(x)\sum_{m=1}^{\infty} e^{-m^2\pi^2/a^2}\Big)\|\varphi\|^2. \tag{7}$$

From the assumption in Theorem 1 and from (7) we get

$$(T\varphi,\varphi) \geq C\|\varphi\|^2, \qquad \text{for every } \varphi\in C_0^\infty \tag{8}$$

where $C>0$ and C does not depend on φ. From (6) in a similar way we get

$$(T\varphi,\varphi) \leq D\|\varphi\|^2, \qquad D = \max_{x\in\mathbf{R}} g(x)\cdot\frac{2\pi}{a}\sum_{m=-\infty}^{\infty} e^{-m^2\pi^2/a^2} \tag{9}$$

for every $\varphi\in C_0^\infty$. From (9) it follows that the operator T, defined by (4) on C_0^∞, can be extended for all $\varphi\in L^2(\mathbf{R})$ ($C_0^\infty$is dense in $L^2(\mathbf{R})$).

Then (8) and (9) hold for every $\varphi\in L^2(\mathbf{R})$. That means the operator T is invertible and its inverse T^{-1} is bounded. Let $f\in L^2(\mathbf{R})$ and $g=T^{-1}f$. Then $Tg = \sum_{m,n}(g,\varphi_{mn})\varphi_{mn}$. Since $g=T^{-1}f$, i.e. $f=Tg$, we obtain $f=\sum_{m,n}(T^{-1}f,\varphi_{mn})\varphi_{mn}$ and the theorem is proved.

COROLLARY 1. *If $ab<2\pi\cdot k$ ($k<1$) and $b>b_0(k)=2k\sqrt{\pi}/\sqrt{1-k^2}$ then every function $f\in L^2(\mathbf{R})$ can be expanded into a series with respect to a system of functions $\{\varphi_{mn}\}$.*

Proof. We consider the function $g(x)=\sum_{n=-\infty}^{\infty} e^{-(x-nb)^2}$. Clearly $g\in C_0^\infty$, it is even and periodic with a period b. So it is enough to estimate $\min_{-b/2\leq x\leq b/2} g(x)$ and $\max_{-b/2\leq x\leq b/2} g(x)$.

Since

$$\begin{aligned}g(x) &= e^{-x^2} + \sum_{n=1}^{\infty}\big(e^{-(x-nb)^2}+e^{-(x+nb)^2}\big) \\ &\geq e^{-x^2} + 2\sum_{n=1}^{\infty} e^{-x^2-n^2b^2} = e^{-x^2}\sum_{n=-\infty}^{\infty} e^{-n^2b^2}.\end{aligned}$$

We get

$$\min_{-b/2\le x\le b/2} g(x) \ge e^{-b^2/4} \sum_{n=-\infty}^{\infty} e^{-n^2 b^2}.$$

Now, we prove

$$\max_{-b/2\le x\le b/2} g(x) = g(0) = \sum_{n=-\infty}^{\infty} e^{-n^2 b}.$$

Since

$$\sum_{n\in\mathbf{Z}} e^{-\pi(n+\alpha)^2/y} = \sqrt{y}\sum_{n\in\mathbf{Z}} e^{-\pi n^2 y + 2\pi i n\alpha} \qquad \text{for } y>0$$

(Poisson-formula, see [**8**]), we get

$$g(x) = \frac{\sqrt{\pi}}{b}\sum_{n\in\mathbf{Z}} e^{-n^2\pi^2/b^2 - 2\pi i n x/b} \le g(0).$$

Hence

$$\Big(\min_{-b/2\le x\le b/2} g(x)\Big) \Big/ \Big(\max_{-b/2\le x\le b/2} g(x)\Big) \ge e^{-b^2/4}.$$

Since $ab < 2\pi\cdot k$ $(k<1)$, to complete the proof of Corollary 1 it is enough to show the inequality

$$\sum_{n=1}^{\infty} e^{-n^2 b^2/(4k^2)} < \frac{1}{2} e^{-b^2/4}.$$

Because of

$$\sum_{n=1}^{\infty} e^{-\frac{b^2}{4k^2}(n^2-k^2)} < \sum_{n=1}^{\infty} e^{-\frac{b^2}{4k^2}(1-k^2)n^2} < \int_0^{\infty} e^{-x^2\frac{b^2(1-k^2)}{4k^2}}\,dx = \frac{\sqrt{\pi}\,k}{b\sqrt{1-k^2}} < \frac{1}{2}$$

$$\text{for} \quad b > b_0(k) = \frac{2k\sqrt{\pi}}{\sqrt{1-k^2}},$$

the corollary is proved.

Remark 1. The preceding corollary shows that the conjecture is true in the case $ab < 2\pi\cdot k$ and $1 > k > 0.996$ if b is sufficiently large.

COROLLARY 2. *Under the assumption of Theorem* 1 *we have*:

$$m\|f\|^2 \le \sum_{p,q\in\mathbf{Z}} |f(z_{pq})|^2 e^{-|z_{pq}|^2/2} \le M\|f\|^2,$$

for every $f\in F$, *where* $z_{pq} = pa + iqb$. (M, m *are positive constants which do not depend on* $f\in F$).

Proof. Let $\varphi\in L^2(\mathbf{R})$. Then $f(z) = (U_B\varphi)(z)\in F$. By direct computation we get $(U_B\varphi_{pq})(z) = e_{z_{pq}}(z)$. Since U_B is an unitary operator then from (2) it follows that

$$f(z_{pq})e^{-|z_{pq}|^2/2} = (f, e_{z_{pq}}) = (U_B\varphi, U_B\varphi_{pq}) = (\varphi,\varphi_{pq}). \tag{10}$$

From (8), (9) and (10) Corollary 2 follows.

Now, we consider a more general system of functions $\psi_{mn}(x) = e^{imax}\cdot\varphi_0\cdot(x-nb)$. If φ_0 is not an entire function, then the completeness of the system $\{\psi_{mn}\}$ can not be proved by entire functions method. But under some conditions it is possible to use the method from Theorem 1.

THEOREM 2. *Let* φ_0 *be a continuous function on* $\mathbf{R}$ *which satisfies the following conditions*:

1° $\sup_{x\in\mathbf{R}} |x^n\varphi_0(x)| < \infty$ *for every* $n \in \mathbf{N}$.

2° $|\varphi_0(x-y)\varphi_0(x+y)| \le K_{\varphi_0}(|\varphi_0(x)|\,|\varphi_0(y)|)^p$ (K_{φ_0} *does not depend on* x *and* y; $p > 0$)

3° $\displaystyle\sum_{m\neq 0,\, m\in\mathbf{Z}} \Big|\varphi_0\Big(\frac{m\pi}{a}\Big)\Big|^p < \frac{1}{K_{\varphi_0}} \frac{\min_{x\in\mathbf{R}} g_2(x)}{\max_{x\in\mathbf{R}} g_1(x)}$,

where $g_1(x) = \sum_{n=-\infty}^{\infty} |\varphi_0(x-nb)|^p$ *and* $g_2(x) = \sum_{n=-\infty}^{\infty} |\varphi_0(x-nb)|^2$. *Then every function* $f \in L^2(\mathbf{R})$ *can be expanded into a series with respect to a system* $\{\psi_{mn}\}_{m,n\in\mathbf{Z}}$.

Proof. Using the method of the proof of Theorem 1 we define (for $\varphi \in C_0^\infty$) the mapping $T\varphi = \sum_{m,n}(\varphi,\varphi_{mn})\varphi_{mn}$ so

$$\begin{aligned}(T\varphi,\varphi) &= \frac{2\pi}{a}\int_{\mathbf{R}} |\varphi|^2\theta_1(x,x)\,dx \\ &+ \frac{2\pi}{a}\sum_{m\neq 0}\int_{\mathbf{R}} \varphi\Big(x+\frac{m\pi}{a}\Big)\overline{\varphi\Big(x-\frac{m\pi}{a}\Big)}\theta_1\Big(x-\frac{m\pi}{a}, x+\frac{m\pi}{a}\Big)\,dx \end{aligned} \tag{11}$$

where $\theta_1(x,y) = \sum_{n\in\mathbf{Z}} \varphi_0(x-nb)\,\overline{\varphi_0(y-nb)}$.

From the condition 2° in Theorem 2 follows that

$$|\theta_1(x-m\pi/a, x+m\pi/a)| \le K_{\varphi_0} g_1(x)|\varphi_0(m\pi/a)|^p. \tag{12}$$

From (11) and (12) we get (for $\varphi \in C_0^\infty$)

$$(T\varphi,\varphi) \ge \frac{2\pi}{a}\min_{x\in\mathbf{R}} g_2(x)\|\varphi\|^2 - \frac{2\pi}{a}\sum_{m\neq 0} K_{\varphi_0}\max_{x\in\mathbf{R}} g_1(x)\Big|\varphi_0\Big(\frac{m\pi}{a}\Big)\Big|^p\|\varphi\|^2,$$

i.e.

$$(T\varphi,\varphi) \ge \frac{2\pi}{a}\Big(\min_{x\in\mathbf{R}} g_2(x) - K_{\varphi_0}\max_{x\in\mathbf{R}} g_1(x)\sum_{m\neq 0}\Big|\varphi_0\Big(\frac{m\pi}{a}\Big)\Big|^p\Big)\|\varphi\|^2.$$

The proof now follows as in Theorem 1 because of 3°.

Remark 2. The assumptions 1° and 2° are fulfilled, for example, for $p = 1$, $\varphi_0(x) = e^{-h(x)}$ where $h(\cdot)$ is an even, rapid growing (on $(0,\infty)$), convex function.

For $p = 2$, $\varphi_0(x) = e^{-x^2/2}$, and from Theorem 2 we get Theorem 1.

Remark 3. The statements given in Theorems 1 and 2 and their consequences can be generalized to the case of a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ and to the case of the Bargman space of entire functions on $\mathbf{C}^n$.

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