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MULTIPLIERS OF THE VANISHING HARDY CLASSES

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Abstract. Let $h_0^p = \{u : M_p(u, l^-) = 0\}$ denote the "vanishing" harmonic Hardy class. We prove that a linear operator $L : h_0^p \to h_0^p$ $(0 < p < 1)$ is of the form $Lu = v * u$ for some v harmonic in the unit disc, if and only if there are sequences $\{\lambda_j\}$ and $\{w_j\}$ such that $|w_j| = 1$, $\sum |\lambda_j|^p < \infty$ and

$$(Lu)(z) = \sum \lambda_j u(w_j z) \qquad \text{for } |z| < 1.$$

In other words, v is a multiplier of h_0^p if and only if v is the Poisson integral of a purely atomic measure with l^p-weights.

Throughout the paper we assume that $0 < p < 1$. The harmonic Hardy class h^p consists of all the functions u harmonic in the unit disc and such that

$$\|u\|_p := \sup\{M_p(u, r) : 0 < r < 1\} < \infty$$

where M_p denotes the integral mean,

$$M_p^p(u, r) = \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta/2\pi.$$

The class h^p endowed with the above quasi-norm is complete, and its topological structure is quite different from the structure of the Hardy class H^p. For information we refer to [**5**]. In the present paper we are concerned with the vanishing h^p, i.e., with the class

$$h_0^p = \{u : \lim M_p(u, r) = 0 \ (r \to 1^-)\},$$

which is a closed subspace of h^p (cf. [**5**]). Our main result is a description of the algebra $Mh_0^p = \{u : u * v \in h_0^p \text{ for all } v \in h_0^p\}$. Here $*$ indicates the convolution of harmonic functions,

$$(u * v)(re^{i\theta}) = \sum_{-\infty}^{+\infty} \hat{u}(k)\hat{v}(k) r^{|k|} e^{ik\theta}.$$

Before stating the theorem we introduce some classes of harmonic functions.

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The class A^p. For an integer $n \geq 0$ let $(D^n u)(re^{i\theta}) = \partial^n u/\partial\theta^n$. The class A^p consists of those u for which

$$M_p(D^n u, r) = O\big((1-r)^{1/p-n-1}\big) \qquad (0 < r < 1), \tag{1}$$

where $n > 1/p - 1$. It follows from [**6**] and the classical results of Hardy and Littlewood [**1**, **2**] that A^p is an algebra relative to the convolution. More information can be found in [**7**].

The class A_0^p. A function u is in A_0^p if

$$u(z) = \sum_{j=1}^{\infty} \lambda_j P(w_j z), \qquad |z| < 1, \tag{2}$$

for some sequences $\{\lambda_j\}$ and $\{w_j\}$ such that $|w_j| = 1$ for all j and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Here P denotes the Poisson kernel,

$$P(z) = \frac{1-|z|^2}{|1-z|^2} = \sum_{-\infty}^{+\infty} r^{|k|} e^{ik\theta} \qquad (z = re^{i\theta}).$$

The following fact is proved in [**7**]:

$$A_0^p = h_0^p \cap A^p. \tag{3}$$

The class $h\Lambda^p$. For a positive integer n and a harmonic function u we define the harmonic functions $\Delta_t^n u$ $(t > 0)$ by

$$(\Delta_t^n u)(re^{i\theta}) = (\Delta_t^n u_r)(\theta), \qquad u_r(\theta) = u(re^{i\theta}),$$

where Δ_t^n stands for the n-th difference with step t,

$$(\Delta_t^1 u_r)(\theta) = u_r(\theta + t) - u_r(\theta), \qquad \Delta_t^n = \Delta_t^1 \Delta_t^{n-1} \quad (n \geq 2).$$

The class $h\Lambda^p$ consists of those $u \in h^p$ for which

$$\|\Delta_t^n u\|_p = O(t^{1/p-1}) \qquad (t \to 0^+), \tag{4}$$

where $n > 1/p - 1$. It should be noted that (4), as well as (1), is independent of a particular choice of n.

THEOREM. $Mh_0^p = A_0^p = A^p \cap h_0^p = h\Lambda^p \cap h_0^p$.

Proof. In view of (3) it suffices to prove that $A_0^p \subset Mh_0^p \subset h\Lambda^p$ and

$$A^p = h\Lambda^p. \tag{5}$$

If u is given by (2), then $(u * v)(z) = \sum \lambda_j v(w_j z)$, which implies that $M_p^p(u * v, r) \leq \sum |\lambda_j|^p M_p^p(v, r)$, and this proves that $A_0^p \subset Mh_0^p$.

Before proving (5) we use it to prove that $Mh_0^p \subset h\Lambda^p$. Let $u \in Mh_0^p$. Using the fact that if $\|v_j - v\|_p \to 0$ $(j \to \infty)$, then $\hat{v}_j(k) \to \hat{v}(k)$ $(j \to \infty)$ for all k, which follows, e.g., from Hardy and Littlewood's inequality [**1**]

$$|\hat{v}(k)| \le C(|k|+1)^{1/p-1}\|v\|_p \qquad (v \in h^p),$$

one easily verifies that the operator $v \mapsto u * v$ $(v \in h_0^p)$ has closed graph. Since h_0^p is complete, there is a constant $C < \infty$ such that $\|u * v\|_p \le C\|v\|_p$ for all $v \in h_0^p$. We choose $v = \Delta_t^n P$ with $n > 1/p - 1$. Since $P \in A^p$ [**5**, **7**] we deduce from (5) that

$$\|\Delta_t^n u\|_p = \|u * \Delta_t^n P\|_p \le C t^{1/p-1} \qquad (t > 0),$$

i.e., $u \in h\Lambda^p$, which was to be proved.

In proving (5) we can consider real valued functions. Let $u = \operatorname{Re} f \in A^p$, where f is holomorphic in the unit disc. Then $f \in A^p$, because A^p is "self-conjugate", by the well known result of Hardy and Littlewood [**1**]. Now we use the relation [**3**, **4**]

$$A^p \cap H = h\Lambda^p \cap H \qquad (H = \text{holomorphic functions}) \tag{6}$$

to conclude that $f \in h\Lambda^p$, which implies that $u \in h\Lambda^p$, because $\|\Delta_t^n u\|_p \le \|\Delta_t^n f\|_p$.

To finish the proof of (5) we use another well known result of Hardy and Littlewood:

$$M_p(D^1 g, r) \le C(1-r)^{-1}\|\operatorname{Re} g\|_p \qquad (0 < r < 1), \tag{7}$$

where g is holomorphic and C is independent of g and r. Let $u = \operatorname{Re} f \in h\Lambda^p$ and $f_r(z) = f(rz)$. Applying (7) to $g = \Delta_t^n f$, where $n > 1/p - 1$, we see that

$$\|\Delta_t^n D^1 f_r\|_p = M_p(D^1 \Delta_t^n f, r) \le C(1-r)^{-1} t^{1/p-1}$$

for $0 < r < 1$, $t > 0$. Using (6) (with equivalent quasi-norms) we obtain

$$M_p(D^n D^1 f_r, \rho) \le C_1(1-r)^{-1}(1-\rho)^{1/p-n-1} \qquad (0 < r, \rho < 1).$$

Replacing here n by $n-1$ $(n > 1/p)$ we conclude that u satisfies (1). Thus $u \in A^p$, and the proof is completed. □

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