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ALMOST PARA-CONTACT FINSLER CONNECTIONS ON VECTOR BUNDLE

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Abstract. We study almost para-contact Finsler connections on the total space of a vector bundle. In a vector bundle of Finsler space we define a Finsler connection compatible with almost para-contact Finsler structure (J, η, ξ) if horizontal and vertical derivatives of all the three elements vanish. We give the family of all Finsler connections compatible with (J, η, ξ) and some interesting special cases.

1. Introduction. Let $EM = (E, \pi, M)$ be a vector bundle with the $(n+m)$-dimensional total space E, n-dimensional base space M and the projection map π, such that $\pi: E \to M$, $u \in E \to \pi(u) = x \in M$, where $u = (x, y)$ and $y = \pi^{-1}(x)$ is the fibre of EM over x. We denote by E_u^v the local fibre of the vertical bundle VE at $u \in E$ and by N_u the complementary space of E_u^v in the tangent space E_u at u to the total space E.

We have

$$E_u = N_u \oplus E_u^v. \tag{1.1}$$

A nonlinear connection on E is a differentiable distribution $N: u \in E \to N_u \subseteq E_u$ with the property (1.1).

We denote by (x^i, y^a) the canonical coordinates of the point $u \in E$. The transformation of canonical coordinates $(x, y) \to (x', y')$ of a point of E are given by

$$x^{i'} = x^{i'}(x^1, \dots, x^n), \qquad y^{a'} = L_b^a(x^1, \dots, x^n) y^b$$
$$\det(L_b^a) \neq 0; \qquad i = 1, 2, \dots n; \qquad a = 1, 2, \dots m$$

Let $\{\partial/\partial x^i, \partial/\partial y^a\}$, $\{dx^i, dy^a\}$ be dual natural basis and $\{\delta/\delta x^i, \partial/\partial y^a\}$, $\{dx^i, \delta y^a\}$ the adapted dual basis on E. These bases are related by the coefficients of nonlinear connections as follows

$$\delta/\delta x^i = \partial/\partial x^i - N_i^a \partial/\partial y^a, \qquad \delta y^a = dy^a + N_i^a dx^i \tag{1.2}$$

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For every vector field X on E there exists the unique decomposition $X = X^H + X^v$; $X_u^H \in N_u$, $X_u^v \in E_u$, $u \in E$, where X^H is called the horizontal part and X^v is called the vertical part of X.

A linear connection ∇ on E is a Finsler connection if and only if it determines a unique decomposition

$$\nabla_X y = \nabla_X^H y + \nabla_X^v y, \quad \forall X, \quad y \in \mathcal{X}(E)$$

where $\mathcal{X}(E)$ is $\mathcal{F}(E)$-module of the vector fields on E.

The coefficients of Finsler connection ∇ in adapted frames are denoted by $F\Gamma = (N, \underset{1}{F}, \underset{2}{F}, \underset{1}{C}, \underset{2}{C})$ and are given by (1.2) and

$$\begin{aligned}
\nabla^H_{\delta/\delta x^k} \delta/\delta x^j &= \underset{1}{F}{}^i{}_{jk}(x,y)\delta/\delta x^i, \qquad & \nabla^H_{\delta/\delta x^k} \partial/\partial y^b &= \underset{2}{F}{}^a{}_{bk}(x,y)\partial/\partial y^a, \\
\nabla^v_{\partial/\partial y^c} \delta/\delta x^j &= \underset{1}{C}{}^i{}_{jc}(x,y)\delta/\delta x^i, \qquad & \nabla^V_{\partial/\partial y^c} \partial/\partial y^b &= \underset{2}{C}{}^a{}_{bc}(x,y)\partial/\partial y^a,
\end{aligned}$$

where $\underset{1}{F}(\equiv \underset{1}{F}{}^i{}_{jk}(x,y))$ and $\underset{2}{F}(\equiv \underset{2}{F}{}^a{}_{bk}(x,y))$ are called the coefficients of h-connection ∇^H and $\underset{1}{C}(\equiv \underset{1}{C}{}^i{}_{jc}(x,y))$ and $\underset{2}{C}(\equiv \underset{2}{C}{}^a{}_{bc}(x,y))$ are called the coefficients of v-connection ∇^v.

For a tensor field K, for instance of type (1.1) on E, there are four Finsler tensor fields of types $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Their components are denoted by $K^i{}_j$, $K^a{}_j$, $K^i{}_a$, $K^a{}_b$; h- and v-covariant derivatives are given by

$$\begin{aligned}
K^i{}_{j|k} &= \delta K^i{}_j/\delta x^k + F^i{}_{nk} K^h{}_j - F^h{}_{jk} K^i{}_h, \\
K^i{}_{j\|a} &= \partial K^i{}_j/\partial y^a + C^i{}_{ha} K^h{}_j - C^h{}_{ja} K^i{}_h,
\end{aligned}$$

etc.

If $F\Gamma = (N, \underset{1}{F}, \underset{2}{F}, \underset{1}{C}, \underset{2}{C})$ and $F\overline{\Gamma} = (\overline{N}, \underset{1}{\overline{F}}, \underset{2}{\overline{F}}, \underset{1}{\overline{C}}, \underset{2}{\overline{C}})$ are two Finsler connections on E, then a unique system of Finsler tensor fields $(A^a{}_i, \underset{1}{B}{}^i{}_{jk}, \underset{2}{B}{}^a{}_{bk}, \underset{1}{D}{}^i{}_{jc}, \underset{2}{D}{}^a{}_{bc})$ is determined such that

$$\begin{aligned}
&\overline{N}^a{}_i = N^a{}_i - A^a{}_i, \\
&\underset{1}{\overline{F}}{}^i{}_{jk} = \underset{1}{F}{}_{ijk} - \underset{1}{B}{}^i{}_{jk}, \qquad \underset{1}{\overline{C}}{}^i{}_{jc} = \underset{1}{C}{}^i{}_{jc} - \underset{1}{D}{}^i{}_{jc}, \\
&\underset{2}{\overline{F}}{}^a{}_{bk} = \underset{2}{F}{}^a{}_{bk} - \underset{2}{B}{}^a{}_{bk}, \qquad \underset{2}{\overline{C}}{}^a{}_{bc} = \underset{2}{C}{}^a{}_{bc} - \underset{2}{D}{}^a{}_{bc}
\end{aligned} \tag{1.3}$$

Conversely, given the Finsler connection $F\Gamma = (N, \underset{1}{F}, \underset{2}{F}, \underset{1}{C}, \underset{2}{C})$ and a system $(A, \underset{1}{B}, \underset{2}{B}, \underset{1}{D}, \underset{2}{D})$ of Finsler tensor fields, the connection $F\overline{\Gamma}$ given by (1.3) is a Finsler connection on E.

2. Almost para-contact Finsler connection on vector bundle

Definition 2.1. An almost para-contact structure on E is defined by the triple (J, η, ξ) where

$$J = J^i{}_j(x,y)\delta/\delta x^i \otimes dx^j + J^a{}_b(x,y)\partial/\partial y^a \otimes \delta y^b,$$
$$\eta = \eta_i dx^i + \eta_a \delta y^a; \qquad \xi = \xi^i \delta/\delta x^i + \xi^a \partial/\partial y^a,$$

satisfying the following conditions

$$\begin{aligned} J^i{}_j J^k{}_i &= \delta^k{}_j - \eta_j \xi^k; \qquad J^i{}_j \xi^j = 0, \qquad J^i{}_j \eta_i = 0 \\ J^a{}_b J^c{}_a &= \delta^c{}_b - \eta_b \xi^c; \qquad J^a{}_b \xi^b = 0, \qquad J^a{}_b \eta_a = 0 \end{aligned} \tag{2.1}$$

where $\det(J^i{}_j) \neq 0$ and $\det(J^a{}_b) \neq 0$.

We now associate to the almost para-contact Finsler structure, the following Finsler tensor fields, called Obata operators [**3**]

$$\begin{aligned} \underset{1}{\phi}{}^{ij}_{kl} &= (\delta^i{}_k \delta^j{}_l + J^i{}_k J^j{}_l)/2, \qquad \underset{2}{\phi}{}^{ij}_{kl} = (\delta^i{}_k \delta^j{}_l - J^i{}_k J^j{}_l)/2, \\ \underset{1}{\psi}{}^{ac}_{bd} &= (\delta^a{}_b \delta^c{}_d + J^a{}_b J^c{}_d)/2, \qquad \underset{2}{\psi}{}^{ac}_{bd} = (\delta^a{}_b \delta^c{}_d - J^a{}_b J^c{}_d)/2, \end{aligned} \tag{2.2}$$

$$\underset{1}{\Omega}{}^{ij}_{kl} = (\eta_k \xi^i \delta^j{}_l + \delta^i{}_k \eta_l \xi^j - \eta_k \xi^i \eta_l \xi^j)/2,$$
$$\underset{2}{\Omega}{}^{ac}_{bd} = (\eta_b \xi^a \delta^c{}_d + \delta^a{}_b \eta_d \xi^c - \eta_b \xi^a \eta_d \xi^c)/2.$$

with properties

$$\phi_1 + \phi_2 = I \otimes I,$$
$$\phi_2 \cdot \phi_1 = \phi_1 \cdot \phi_2 = \underset{1}{\Omega}/2 = \phi_2 \cdot \underset{1}{\Omega} = \underset{1}{\Omega} \cdot \phi_2 = \phi_1 \cdot \underset{1}{\Omega} = \underset{1}{\Omega} \cdot \phi_1 = \underset{1}{\Omega} \cdot \underset{1}{\Omega}$$
$$(\phi_2 - \underset{1}{\Omega}) \cdot (\phi_1 + \underset{1}{\Omega}) = (\phi_1 + \underset{1}{\Omega}) \cdot (\phi_2 - \underset{1}{\Omega}) = 0$$

$$\psi_1 + \psi_2 = I \otimes I,$$
$$\psi_2 \cdot \psi_1 = \psi_1 \cdot \psi_2 = \underset{2}{\Omega}/2 = \psi_2 \cdot \underset{2}{\Omega} = \underset{2}{\Omega} \cdot \psi_2 = \psi_1 \cdot \underset{2}{\Omega} = \underset{2}{\Omega} \cdot \psi_1 = \underset{2}{\Omega} \cdot \underset{2}{\Omega}$$
$$(\psi_2 - \underset{2}{\Omega}) \cdot (\psi_1 + \underset{2}{\Omega}) = (\psi_1 + \underset{2}{\Omega}) \cdot (\phi_2 - \underset{2}{\Omega}) = 0$$

LEMMA 2.1. [**3**] *A system of tensor equations*

$$\begin{array}{c|c} (\phi_1 + \underset{1}{\Omega}) \cdot X = A & (\psi_1 + \underset{2}{\Omega}) \cdot X = A \\ [\text{resp. } (\phi_2 - \underset{1}{\Omega}) \cdot X = A] & [\text{resp. } (\psi_2 - \underset{2}{\Omega}) \cdot X = A] \end{array} \tag{2.3}$$

with X as unknowns, has solutions if and only if

$$\begin{array}{c|c} (\phi_2 - \underset{1}{\Omega}) \cdot A = 0 & (\psi_2 - \underset{2}{\Omega}) \cdot A = 0 \\ [\text{resp. } (\phi_1 + \underset{1}{\Omega}) \cdot A = 0] & [\text{resp. } (\psi_1 + \underset{2}{\Omega}) \cdot A = 0] \end{array} \tag{2.4}$$

If the conditions (2.4) *hold, then the general solution of the system* (2.3) *is*

$$X = A + (\phi_2 - \underset{1}{\Omega}) \cdot Y \qquad X = A + (\psi_2 - \underset{2}{\Omega}) \cdot Y$$
$$[\text{resp. } X = A + (\phi_1 + \underset{1}{\Omega}) \cdot Y] \qquad [\text{resp. } X = A + (\psi_1 + \underset{2}{\Omega}) \cdot Y]$$

where Y is an arbitrary Finsler tensor field of the same type as X.

Definition 2.2. A Finsler connection ∇ on E is called almost para-contact Finsler connection or connection compatible with almost para-contact Finsler structure (J, η, ξ) if and only if

$$\begin{aligned} &J^i_{j|k} = 0, \quad J^a_{b|k} = 0, \quad J^i_{j\|c} = 0, \quad J^a_{b\|c} = 0, \\ &\eta_{i|k} = 0, \quad \eta_{a|k} = 0, \quad \eta_{a\|b} = 0, \quad \eta_{i\|b} = 0 \end{aligned} \tag{2.5}$$

Remark. From (2.1) and (2.5), it follows that

$$\xi^i{}_{|k} = 0, \quad \xi^i{}_{\|b} = 0, \quad \xi^a{}_{|k} = 0, \quad \xi^a{}_{\|b} = 0 \tag{2.6}$$

From (2.2), (2.5) and (2.6) it follows

THEOREM 2.1. *For any almost para-contact Finsler connection on E, the operators $\underset{1}{\phi}$, $\underset{2}{\phi}$, $\underset{1}{\psi}$, $\underset{2}{\psi}$, $\underset{1}{\Omega}$ and $\underset{2}{\Omega}$ are h- and v-covariant constants.*

The family of all almost para-contact Finsler connections on the total space of vector bundle can be determined by a well known method [**2**] based on Lemma 2.1.

In the following the nonlinear connection N is fixed. A Finsler connection with fixed N will be denoted by $F\Gamma(N)$. Let $(\underset{1}{B}, \underset{2}{B}, \underset{1}{D}, \underset{2}{D})$ be the difference tensors of the pair $F\overline{\Gamma}, F\Gamma)$. Then any $F\overline{\Gamma}(N)$ on E can be expressed as (1.3) with $A^a{}_i = 0$.

Requiring $F\overline{\Gamma}(N)$ to be an almost para-contact Finsler connection, we obtain for the Finsler tensor fields $\underset{1}{B}$, $\underset{2}{B}$, $\underset{1}{D}$, $\underset{2}{D}$ the following expressions:

$$\begin{aligned} \underset{1}{B}{}^i{}_{jk} &= (J^i{}_{m|k} J^m{}_j)/2 + (\eta_{j|k}\xi^i)/2 + (\eta_j \xi^m{}_{|k} \eta_m \xi^i)/2 + (\underset{1}{\phi}{}^{ri}{}_{mj} + \underset{1}{\Omega}{}^{ri}{}_{mj})\, \underset{1}{Y}{}^m{}_{rk}, \\ \underset{2}{B}{}^a{}_{b|k} &= (J^a{}_{g|k} J^g{}_b)/2 + \xi^a(\eta_{b|k} + \eta_b \eta_g \xi^g{}_{|k})/2 + (\underset{1}{\psi}{}^{ga}{}_{cb} + \underset{2}{\Omega}{}^{ga}{}_{cb})\, \underset{2}{Y}{}^c{}_{gk}, \\ \underset{1}{D}{}^i{}_{jc} &= (J^i{}_{m\|c} J^m{}_j)/2 + \xi^i(\eta_{j\|c} + \eta_j \eta_m \xi^m{}_{\|c})/2 + (\underset{1}{\phi}{}^{mi}{}_{kj} + \underset{1}{\Omega}{}^{mi}{}_{kj})\, \underset{1}{Z}{}^k{}_{mc}, \\ \underset{2}{D}{}^a{}_{bc} &= (J^a{}_{g\|c} J^g{}_b)/2 + \xi^a(\eta_{b\|c} + \eta_b \eta_g \xi^g{}_{\|c})/2 + (\underset{1}{\psi}{}^{ga}{}_{db} + \underset{2}{\Omega}{}^{ga}{}_{db})\, \underset{2}{Z}{}^d{}_{gc}, \end{aligned}$$

where $\underset{1}{Y}$, $\underset{2}{Y}$, $\underset{1}{Z}$, $\underset{2}{Z}$ are arbitrary Finsler tensor fields.

Hence we have

THEOREM 2.2. *The general family of the almost para-contact Finsler connection $F\overline{\Gamma}(N) = (\underset{1}{\overline{F}}, \underset{2}{\overline{F}}, \underset{1}{\overline{C}}, \underset{2}{\overline{C}})$ relative to the almost para-contact Finsler structure*

(J,η,ξ) on the total space E of vector bundle EM is given by

$$\overline{\underset{1}{F}}{}^i{}_{jk} = \underset{1}{F}{}^i{}_{jk} - (J^m{}_j J^i{}_{m|k})/2 - \xi^i(\eta_{j|k} + \eta_j\eta_m\xi^m{}_{|k})/2 - (\underset{1}{\phi}{}^{ri}{}_{mj} + \underset{1}{\Omega}{}^{ri}{}_{mj})\, \underset{1}{Y}{}^m{}_{rk},$$

$$\overline{\underset{2}{F}}{}^a{}_{bk} = \underset{2}{F}{}^a{}_{bk} - (J^g{}_b J^a{}_{g|k})/2 - \xi^a(\eta_{b|k} + \eta_b\eta_g\xi^g{}_{|k})/2 - (\underset{1}{\psi}{}^{ga}{}_{cb} + \underset{2}{\Omega}{}^{ga}{}_{cb})\, \underset{2}{Y}{}^c{}_{gk},$$

$$\overline{\underset{1}{C}}{}^i{}_{jc} = \underset{1}{C}{}^i{}_{jc} - (J^i{}_{m\|c} J^m{}_j)/2 - \xi^i(\eta_{j\|c} + \eta_j\eta_m\xi^m{}_{\|c})/2 - (\underset{1}{\phi}{}^{mi}{}_{kj} + \underset{1}{\Omega}{}^{mi}{}_{kj})\, \underset{1}{Z}{}^k{}_{mc},$$

$$\overline{\underset{2}{C}}{}^a{}_{bc} = \underset{2}{C}{}^a{}_{bc} - (J^a{}_{g\|c} J^g{}_b)/2 - \xi^a(\eta_{b\|c} + \eta_b\eta_g\xi^g{}_{\|c})/2 - (\underset{1}{\psi}{}^{ga}{}_{db} + \underset{2}{\Omega}{}^{ga}{}_{db})\, \underset{2}{Z}{}^d{}_{gc},$$

where $|$ (resp. $\|$) is the h- (resp. v-) covariant derivatives with respect to an arbitrary initial Finsler connection $F\Gamma(N)$ on E and $\underset{1}{Y}$, $\underset{2}{Y}$, $\underset{1}{Z}$, $\underset{2}{Z}$ are arbitrary Finsler tensor fields.

If we take $\underset{1}{Y}{}^i{}_{jk} = 0 = \underset{2}{Y}{}^a{}_{bk} = \underset{1}{Z}{}^i{}_{jc} = \underset{2}{Z}{}^a{}_{bc}$, then we have

THEOREM 2.3. *If the initial connection is $F\Gamma(N)$, then the following Finsler connection*

$$\begin{aligned}
\overset{k}{\underset{1}{F}}{}^i{}_{jk} &= \underset{1}{F}{}^i{}_{jk} - (J^m{}_j J^i{}_{m|k})/2 - \xi^i(\eta_{j|k} + \eta_j\eta_m\xi^m{}_{|k})/2,\\
\overset{k}{\underset{2}{F}}{}^a{}_{bk} &= \underset{2}{F}{}^a{}_{bk} - (J^g{}_b J^a{}_{g|k})/2 - \xi^a(\eta_{b|k} + \eta_b\eta_g\xi^g{}_{|k})/2,\\
\overset{k}{\underset{1}{C}}{}^i{}_{jc} &= \underset{1}{C}{}^i{}_{jc} - (J^i{}_{m\|c} J^m{}_j)/2 - \xi^i(\eta_{j\|c} + \eta_j\eta_m\xi^m{}_{\|c})/2,\\
\overset{k}{\underset{2}{C}}{}^a{}_{bc} &= \underset{2}{C}{}^a{}_{bc} - (J^a{}_{g\|c} J^g{}_b)/2 - \xi^a(\eta_{b\|c} + \eta_b\eta_g\xi^g{}_{\|c})/2,
\end{aligned} \tag{2.7}$$

is an almost para-contact Finsler connection.

The Finsler connection $K\Gamma(N) = (\overset{k}{\underset{1}{F}}, \overset{k}{\underset{2}{F}}, \overset{k}{\underset{1}{C}}, \overset{k}{\underset{2}{C}})$ given by (2.7) is the almost para-contact connection on E derived from $F\Gamma(N)$. We may call it Kawaguchi connection on E.

Next we find an interesting particular case, if we take $\underset{1}{Y}{}^i{}_{jk} = 0 = \underset{2}{Y}{}^a{}_{bk} = \underset{1}{Z}{}^i{}_{jc} = \underset{2}{Z}{}^a{}_{bc}$ and the initial condition $F\Gamma(N)$ as $F\overset{m}{\Gamma}(N)$, where

$$\overset{m}{\underset{1}{F}}{}^i{}_{jk} = \underset{1}{F}{}^i{}_{jk}, \quad \overset{m}{\underset{2}{F}}{}^a{}_{bk} = \partial N^a{}_k/\partial Y^b, \quad \overset{m}{\underset{1}{C}}{}^i{}_{jc} = \underset{1}{C}{}^i{}_{jc}, \quad \overset{m}{\underset{2}{C}}{}^a{}_{bc} = \underset{2}{C}{}^a{}_{bc}.$$

Then the following Finsler connection $F\overset{q}{\Gamma}(N) = (\overset{q}{\underset{1}{F}}, \overset{q}{\underset{2}{F}}, \overset{q}{\underset{1}{C}}, \overset{q}{\underset{2}{C}})$,

$$\begin{aligned}
\overset{q}{\underset{1}{F}}{}^i{}_{jk} &= \overset{m}{\underset{1}{F}}{}^i{}_{jk} - (J^m{}_j J^i{}_{m|k}) - \xi^i(\eta_{j|k} + \eta_j\eta_m\xi^m{}_{|k})/2,\\
\overset{q}{\underset{2}{F}}{}^a{}_{bk} &= \overset{m}{\underset{2}{F}}{}^a{}_{bk} - (J^g{}_b J^a{}_{g|k})/2 - \xi^a(\eta_{b|k} + \eta_b\eta_g\xi^g{}_{|k})/2,\\
\overset{q}{\underset{1}{C}}{}^i{}_{jc} &= \overset{m}{\underset{1}{C}}{}^i{}_{jc} - (J^i{}_{m\|c} J^m{}_j)/2 - \xi^i(\eta_{j\|c} + \eta_j\eta_m\xi^m{}_{\|c})/2,\\
\overset{q}{\underset{2}{C}}{}^a{}_{bc} &= \overset{m}{\underset{2}{C}}{}^a{}_{bc} - (J^a{}_{g\|c} J^g{}_b)/2 - \xi^a(\eta_{b\|c} + \eta_b\eta_g\xi^g{}_{\|c})/2,
\end{aligned}$$

is an almost para-contact Finsler connection, where $|$ (resp. $\|$) is h- (resp. v-) covariant derivative with respect to $F\overset{m}{\Gamma}(N)$ on E. The connection $F\overset{q}{\Gamma}(N)$ will be called *canonical almost para-contact Finsler connection* derived from $F\overset{m}{\Gamma}(N)$ on E.

We can obtain interesting results as follows.

THEOREM 2.4. *If the initial Finsler connection $F\Gamma(N)$ is an almost para-contact Finsler connection, then the general family of the almost para-contact Finsler connection $F\overline{\Gamma}(N)$ is given by*

$$\begin{aligned} \underset{1}{\overline{F}}{}^i{}_{jk} &= \underset{1}{F}{}^i{}_{jk} - (\underset{1}{\phi}{}^{ri}{}_{mj} + \underset{1}{\Omega}{}^{ri}{}_{mj})\,\underset{1}{Y}{}^m{}_{rk}, & \underset{1}{\overline{C}}{}^i{}_{jc} &= \underset{1}{C}{}^i{}_{jc} - (\underset{1}{\phi}{}^{mi}{}_{kj} + \underset{1}{\Omega}{}^{mi}{}_{kj})\,\underset{1}{Z}{}^k{}_{mc}, \\ \underset{2}{\overline{F}}{}^a{}_{bk} &= \underset{2}{F}{}^a{}_{bk} - (\underset{1}{\psi}{}^{ga}{}_{cb} + \underset{2}{\Omega}{}^{ga}{}_{cb})\,\underset{2}{Y}{}^c{}_{gk}, & \underset{2}{\overline{C}}{}^a{}_{bc} &= \underset{2}{C}{}^a{}_{bc} - (\underset{1}{\psi}{}^{ga}{}_{db} + \underset{2}{\Omega}{}^{ga}{}_{db})\,\underset{2}{Z}{}^d{}_{gc}, \end{aligned} \tag{2.8}$$

where $\underset{1}{Y}$, $\underset{2}{Y}$, $\underset{1}{Z}$ and $\underset{2}{Z}$ are arbitrary Finsler tensor fields.

Equations (2.8) give the transformations of almost para-contact Finsler connections having common nonlinear connection. Let $t\colon F\overset{0}{\Gamma}(N) \to F\overline{\overline{\Gamma}}(N)$ be a transformation of this form; then we can obtain this transformation law by appropriate changes in (2.8). Now we have

THEOREM 2.5. *The set of the transformations t of the almost para-contact Finsler connections obtained by appropriate changes of (2.8) is an abelian group, relative to the product of mappings, which is isomorphic to the additive group of Finsler tensor fields $[(\phi_1 + \underset{1}{\Omega})\,\underset{1}{\widetilde{Y}},\ (\psi_1 + \underset{2}{\Omega})\,\underset{2}{\widetilde{Y}},\ (\phi_1 + \underset{1}{\Omega})\,\underset{1}{\widetilde{Z}},\ (\psi_1 + \underset{2}{\Omega})\,\underset{2}{\widetilde{Z}}]$.*

3. A particular set of Finsler connections

Here we determine the family of Finsler connections $F\Gamma(N)$ with the property $K\Gamma(N) = F\overset{q}{\Gamma}(N)$. So we consider the set $\mathcal{F} = \{F\Gamma(N)\colon K\Gamma(N) = F\overset{q}{\Gamma}(N)\}$. Let $(\underset{1}{B}, \underset{2}{B}, \underset{1}{D}, \underset{2}{D})$ be the difference tensors of the pair $(F\Gamma(N),\ F\overset{q}{\Gamma}(N))$. Then

$$\begin{aligned} \underset{1}{F}{}^i{}_{jk} &= \underset{1}{\overset{q}{F}}{}^i{}_{jk} - \underset{1}{B}{}^i{}_{jk}, & \underset{1}{C}{}^i{}_{jc} &= \underset{1}{\overset{q}{C}}{}^i{}_{jc} - \underset{1}{D}{}^i{}_{jc}, \\ \underset{2}{F}{}^a{}_{bk} &= \underset{2}{\overset{q}{F}}{}^a{}_{bk} - \underset{2}{B}{}^a{}_{bk}, & \underset{2}{C}{}^a{}_{bc} &= \underset{2}{\overset{q}{C}}{}^a{}_{bc} - \underset{2}{D}{}^a{}_{bc}, \end{aligned}$$

writing the h- and v-convariant derivatives of both $J^i{}_j$ and $J^a{}_b$ with respect to $F\Gamma(N)$. Noticing that $F\overset{q}{\Gamma}(N)$ is almost para-contact Finsler connection and proceeding in the same way as in [**3**], we get

THEOREM 3.1. *All Finsler connections $F\Gamma$ from the set $\mathcal{F}$ are given by*

$$\begin{aligned} \underset{1}{F}{}^i{}_{jk} &= \underset{1}{\overset{q}{F}}{}^i{}_{jk} + 2(\xi^i\eta_{j|k}) + \eta_j\xi^i\eta_m\xi^m{}_{|k} - (\underset{2}{\phi}{}^{ih}{}_{mj} - \underset{1}{\Omega}{}^{ih}{}_{mj})\,\underset{1}{Y}{}^m{}_{hk}, \\ \underset{2}{F}{}^a{}_{bk} &= \underset{2}{\overset{q}{F}}{}^a{}_{bk} + 2(\xi^a\eta_{b|k}) + \eta_b\xi^a\eta_d\xi^d{}_{|k} - (\underset{2}{\psi}{}^{ag}{}_{db} - \underset{2}{\Omega}{}^{ag}{}_{db})\,\underset{2}{Y}{}^d{}_{gk}, \end{aligned}$$

$$\underset{1}{C}{}^{i}{}_{jc} = \overset{q}{\underset{1}{C}}{}^{i}{}_{jc} + 2(\xi^i \eta_{j\|c}) + \eta_j \xi^i \eta_m \xi^m{}_{\|c} - (\underset{2}{\phi}{}^{ih}{}_{mj} - \underset{1}{\Omega}{}^{ih}{}_{mj}) \underset{1}{Z}{}^{m}{}_{hc},$$

$$\underset{2}{C}{}^{a}{}_{bc} = \overset{q}{\underset{2}{F}}{}^{a}{}_{bc} + 2(\xi^a \eta_{b\|c}) + \eta_b \xi^a \eta_d \xi^d{}_{\|c} - (\underset{2}{\psi}{}^{ag}{}_{db} - \underset{2}{\Omega}{}^{ag}{}_{db}) \underset{2}{Z}{}^{d}{}_{gc},$$

where $\underset{1}{Y}, \underset{2}{Y}, \underset{1}{Z}, \underset{2}{Z}$ *are arbitrary Finsler tensor fields.*

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