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# ON SOME PROBLEMS IN COMBINATORIAL SET THEORY

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**Abstract**. Some problems and conjectures are discused pertaining to combinatorial set theory and graph theory.

In this paper dedicated to the memory of Professor Đ. Kurepa I just state a few of my old and new problems in combinatorial set theory and related topics. I will use the arrow notation introduced by R. Rado.

**1.** In one of our early papers R. Rado and I conjectured that:

$$\omega^2 \to (\omega^2, 3)^2 \tag{1}$$

In other words if you have a sequence of type $\omega^2$ and a graph defined on it, then either this graph contains a triangle or if not it contains an independent set of type $\omega^2$. This conjecture was proved by Specker in 1954 who also proved that for every $n > 2$

$$\omega^n \not\to (\omega^n, 3)^2.$$

Specker also proved that for every $k < \omega$

$$\omega^2 \to (\omega^2, k)^2$$

The problem remained: is it true that

$$\omega^\omega \to (\omega^\omega, 3)^2 \tag{2}$$

In a very difficult paper Chang proved (2). The proof was simplified by E. Milner who also proved

$$\omega^\omega \to (\omega^\omega, k)^2, \text{ for every } k < \omega \tag{3}$$

Jean Larson obtained a much simpler proof of (3) and obtained several further results. Her first unsolved problem was: Is it true that $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$ holds?

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I conjectured some time ago that if for an ordinal number

$$\alpha \to (\alpha, 3)^2$$

holds, then also for every finite

$$\alpha \to (\alpha, k)^2$$

also holds.

Very recently R. Laver in a letter informed me that Darby disproved my conjecture. He certainly proved that there is a countable ordinal $\alpha$ for which $\alpha \to (\alpha, 3)^2$ holds but $\alpha \to (\alpha, 6)^2$ does not hold. He obtained several further interesting results which I hope will soon be published.

**2.** A. Hajnal and I have the following type of problems which we think are new and startling. Let $G$ be a triangle free graph whose vertices are the integers. Determine the smallest $n$ if it exists for which every $G$ whose vertices are the integers $1 \leq t \leq n$ and which is triangle free contains three independent points $a$, $b$, $a+b$. It was a great surprise to us that this problem does not seem to be trivial and so far we could not prove the existence of such an $n$. Hajnal in fact thought that there is an independent set which is a Hindman set i.e. there is a sequence $a_1 < a_2 < \dots$ for which all the finite sums $\sum \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1 (not all $\varepsilon_i = 0$) are independent. The same result could hold if we assume that our $G$ contains no complete graph $k(r)$ of size $r$. Many generalisations are clearly possible.

**3.** A paper of Hajnal, Szemerédi and myself contains I think very many nice and simple problems in combinatorial set theory which I think have been undeservedly neglected in our paper on almost bipartite large chromatic graphs, Annals of Discrete Mathematics **12** (1982) 117–123.

Let $f(n)$ be a function which tends to infinity arbitrarily slowly. Is there a graph $G$ of infinite chromatic number every subgraph of $n$ vertices of which can be made bipartite by the omission of at most $f(n)$ edges? If the conjecture is false try to determine the slowest growing function $f(n)$, for which the conjecture holds. In particular does it hold for $f(n) < n^{\varepsilon}$?

Let $G$ have infinite chromatic number. A well known theorem of de Bruijn and myself implies that for every $n$, $G$ has a finite subgraph of chromatic number $n$. Let $f(n)$ tend to infinity arbitrarily fast. Hajnal and I showed that there is a graph $G$ of infinite chromatic number, every subgraph of chromatic number $n$ of which has more than $f(n)$ vertices. Does this result remain true if $G$ has uncountable chromatic number? We expect that the answer is negative and perhaps if $f(n)$ increases faster then the $k$ times iterated exponential function for every $k$, then for every $G$ of uncountable chromatic number for $n > n_0(G)$ there is a subgraph of chromatic number $n$ which has fewer than $f(n)$ vertices. Unfortunately we are very far from being able to prove this conjecture but we proved that if true it is best possible.

To illustrate the difference between graphs of countable and uncountable chromatic number observe that Hajnal and I proved that there are graphs of infinite chromatic number which have arbitrarily large girth, but every graph of chromatic number $\geq \aleph_1$ must contain all finite bipartite graphs and in fact it must contain a complete bipartite graph of $n$ white and $\aleph_1$ black vertices (for every $n$). Also Hajnal, Shelah and I proved that it also must contain all large odd cycles, but for every fixed $k$ it does not have to contain odd cycles of length $\leq 2k+1$.

In our triple paper with Hajnal and Szemerédi we define $f_G^{(1)}(n)$ and $f_G^{(2)}(n)$ as follows: $f_G^{(1)}(n)$ is the smallest integer for which every induced subgraph of $n$ vertices of $G$ contains an independent set of $f_G^{(1)}(n)$ vertices. $f_G^{(2)}(n)$ is the smallest integer for which every induced subgraph of $n$ vertices of $G$ contains an induced bipartite graph of $f_G^{(2)}(n)$ vertices.

Clearly $f_G^{(1)}(n) \geq \frac{1}{2} f_G^{(2)}(n)$. We prove that for every $\varepsilon > 0$ and every cardinal $\kappa \geq \aleph_0$ there is a graph $G$ of chromatic number $\kappa$ such that for all $n < \omega$

$$f_G^{(2)}(n) > (1-\varepsilon)n. \tag{1}$$

It is easy to see that (1) is best possible, if $G$ has uncountable chromatic number, it must contain $\aleph_1$ vertex disjoint odd cycles of size $2l+1$ for some $l$. On the other hand the following beautiful problem is open: Does there exist a graph $G$ and a constant $c$ for which $G$ has chromatic number and power $\aleph_1$ and

$$f_G^{(1)}(n) > cn?$$

On the other hand we know that if $G$ has chromatic number $\aleph_0$, then there is a sequence $\varepsilon_n \to 0$ for which

$$f_G^{(2)}(n) \geq n(1-\varepsilon_n)$$

but we do not know how fast $\varepsilon_n$ can tend to 0.

A nice result of Folkman implies that if

$$\frac{n}{2} - f_G^{(1)}(n) \leq k$$

then the chromatic number of $G$ is at most $2k+2$.

I offer 1000 dollars for the complete solution of these problems and a generous reward for any significant partial results.

**4.** Hajnal and I conjectured that for every cardinal number $m$ and every integer $r$ there is an $f_r(m)$ for which every graph $G$ of chromatic number $\geq f_k(m)$ contains a subgraph of chromatic number $m$ the smallest odd cycles of which is $> 2r+1$. (By our old result with Hajnal, $G$ must contain all even cycles). This nice conjecture is as far as I know open even for $r = 1$, i.e. does every graph of sufficiently large chromatic number contains a triangle free graph of chromatic number $m$?

The finite version of our conjecture is also interesting: Is it true that for every integer $r$ and $n$ there is an integer $g(r,n)$ so that every graph $G$ of chromatic number $\geq g(r,n)$ has a subgraph of girth $> r$ and chromatic number $n$. Rödl proved our conjecture for $r = 3$ but his bound for $g(3,n)$ is probably very far from being best possible. V. Rödl, *On the chromatic number of subgraph of a given graph*, Proc. Amer. Math. Soc. **64** (1977), 370–371.

**5.** Sierpinski proved in 1933 that $c \not\to (\aleph_1, \aleph_1)^2$. In other words, there is a graph on the real numbers which contains no uncountable complete graph or an uncountable independent set. The same result was obtained independently by Kurepa. In 1954 I asked can one divide the pairs of real numbers into three classes so that every uncountable subset should contain a pair from each of the classes? I proved that if $c = \aleph_1$, then the answer is affirmative, but I could not decide what happens if the continuum hypothesis is not assumed. I offered a reward of 100 dollars for a decision. Shelah proved that it is consistent that the answer is negative. In his model $c$ is very large. It is not yet known if it is consistent to have a negative answer for $c = \aleph_2$.

**6.** Hajnal and I claimed that if a triple system has chromatic number $\aleph_1$, then it must have two triples which have a common pair. In a triple paper with Rotschild we discovered the error, the result only holds if our system has only $\aleph_1$ triples. In fact we proved that there are edge disjoint triple systems of arbitrarily large chromatic number.

In our giant triple paper with Galvin we systematically investigate these phenomena. We prove many theorems but state even more nice unsolved problems which would deserve careful study. Here I only state one unsolved problem: Characterize the finite triple systems that occur in every triple system of chromatic number $> \aleph_0$. I offer 500 dollars for a solution. For graphs the problem is completely solved. A graph of chromatic number $\geq \aleph_1$ must contain all finite bipartite graphs but does not have to contain any fixed odd cycle.

P. Erdős, F. Galvin, A. Hajnal, *On set systems having large chromatic number and not containing prescribed subsystems, Infinite and finite sets*, Coll. Math. Soc. János Bolyai **10** (1975), 425–513.

**7.** Let $G_1$ and $G_2$ be two graphs of uncountable chromatic number. Is it true that there is a graph $G$ of chromatic number 4 which is a subgraph of both $G_1$ and $G_2$? Perhaps there is a subgraph $G$ of both $G_1$ and $G_2$ which has infinite chromatic number, but perhaps this latest conjecture is too optimistic.

**8.** Here is a perhaps more subtle difference between graphs of countable and uncountable chromatic number. It is easy to see that if $G$ has power $m$ and chromatic number $\geq \aleph_0$ then $G$ contains $m$ vertex disjoint subgraphs of infinite chromatic number. Galvin a few years ago asked: Let $G$ have chromatic number $> \aleph_0$. Does $G$ have always two vertex disjoint subgraphs of chromatic number $> \aleph_0$? Hajnal proved that the answer is affirmative and then went on to ask: Let $G$ have chromatic number $> \aleph_0$; can the vertex set of $G$ be decomposed into $\aleph_1$ sets each of which induces a graph of chromatic number $> \aleph_0$? Péter Komjáth observed that the answer is affirmative if the chromatic number of $G$ is $\aleph_1$, but to my great surprise he then proves that if the existence of a measurable cardinal is consistent, then it is also consistent that there is a graph $G$ of uncountable chromatic number which is not the vertex disjoint union of $\aleph_1$ graphs of uncountable chromatic number. The chromatic number of such a graph is no doubt enormous. This is shown in the paper of Komjáth. Many interesting unsolved problems remain.

P. Komjáth, *A Galvin-Hajnal conjecture on uncountably chromatic graphs, A tribute to Paul Erdős*, Cambridge University Press, Edited by A. Baker, B. Bollobás and A. Hajnal, p.p. 313–316.

**9.** To end the paper I state some problems which are more of finite or countable character. Hajnal and I conjectured: Let $G$ have infinite chromatic number and denote by $n_1 < n_2 < \dots$ the length of the distinct odd cycles which occur in $G$. Is it true that

$$\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty? \tag{1}$$

Perhaps the upper density of the $n_i$ is positive and perhaps it is $\frac{1}{2}$. It is not even impossible that

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n_i<x} \frac{1}{n_i} = \frac{1}{2}.$$

On the other hand it is not impossible that the sequence $n_i$ can be very thin.

Is there a sequence of density 0, $n_1 < n_2 < \dots$ for which every $G$ of infinite chromatic number contains for infinitely many $i$ cycles of length $n_i$? Perhaps in fact every $G$ of unbounded edge density contains for infinitely many $i$ cycles of length $n_i$. Gyárfás and I thought about this problem a great deal, probably $n_i = 2^i$ tends to infinity too fast, but we do not know what happens for $n_i = i^2$ or $n_i = p_i + 1$. Many further interesting related questions can be asked, but I must stop.

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