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MEASURING ASYMPTOTIC CONVEXITY

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Abstract. We study a class of functions which are almost convex in a certain sense for large values of the argument. For this class of functions an Abel–Tauber theorem is proved.

0. Introduction

The theory of regular variation, including second order regular variation (the class Π) is well established by now. Basic properties were developed by Karamata in the thirties in order to define a suitable class of functions in connection with Tauberian theorems. In the first order theory basically functions f are studied which are slowly varying. These are measurable, eventually positive and satisfy

$$f(tx)/f(t) \to 1 \quad (t \to \infty) \quad \text{for} \quad x > 0. \tag{0.1}$$

The next step is a second order theory: One considers the class of functions f for which there exists a positive function a such that $\lim_{t\to\infty}\{f(tx) - f(t)\}/a(t)$ exists. The most interesting case is the class Π, for which

$$\{f(tx) - f(t)\}/a(t) \to \log x \quad (t \to \infty) \quad \text{for} \quad x > 0. \tag{0.2}$$

A third order class, connected with the class Π, is defined by the relation

$$\{f(tx) - f(t) - a(t)\log x\}/a_1(t) \to \frac{1}{2}(\log x)^2 \quad (t \to \infty) \quad \text{for} \quad x > 0.$$

or equivalently,

$$\{f(txy) - f(tx) - f(ty) + f(t)\}/a_1(t) \to (\log x)(\log y) \quad (t \to \infty) \quad \text{for} \quad x, y > 0. \tag{0.3}$$

The relations (0.1) and (0.2) are discussed in [**3**] and [**4**]. The third order relation (0.3) is discussed in [**2**] and [**6**]. Note the relation with convexity: If f satisfies (0.3), there exists a function f_1 such that $f_1(e^x)$ is convex and $f_1(t) - f(t) = o(a_1(t)) \quad (t \to \infty)$. See the appendix in [**2**].

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If in the defining relation (0.1) for slowly varying functions the existence of the limit is replaced by a boundedness condition one obtains the concept of O-regular variation. This concept was introduced in the paper [**1**] by Aljančić and Aranđelović in 1977. For more recent references the reader is referred to [**3**] and [**4**]. A measurable, eventually positive function f is O-regularly varying ($f \in \mathrm{RO}$) if

$$\overline{\lim_{t\to\infty}} f(tx)/f(t) < \infty \quad \text{for} \quad x > 0. \tag{0.1a}$$

Similarly if in the defining relation (0.2) of the class Π the existence of the limit is replaced by boundedness conditions, one obtains the class AB of asymptotically balanced functions. In the paper by de Haan and Resnick [**5**] this class is used in the study of extreme values in probability theory. For a more restricted definition the reader is referred to [**3**, Ch. 3.11].

Definition 0.1. A measurable function f is asymptotically balanced ($f \in \mathrm{AB}$ or $f \in \mathrm{AB}(\sigma)$) if there exists a positive function σ such that

$$\overline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) < \infty \quad \text{for} \quad x > 1 \tag{0.2a}$$

$$\underline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) > -\infty \quad \text{for} \quad x > 0 \tag{0.2b}$$

and if there exists $x_0 > 1$ such that

$$\underline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) > 0 \quad \text{for all} \quad x > x_0. \tag{0.2c}$$

The class AB is related to the class RO in the sense that if $f \in \mathrm{AB}(\sigma)$, then $\sigma \in \mathrm{RO}$. See [**4**, lemma 3.10]. In the defining relation (0.3) we shall now replace the existence of a limit by appropriate boundedness conditions. The following class of functions results.

Definition 0.2. Suppose the function $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable. The function f is asymptotically balanced of second order and we write $f \in \mathrm{AB}_2$ or $f \in \mathrm{AB}_2(\sigma)$ if there exists a positive function σ and a constant $y_0 > 1$ such that the function $\rho_{x,y}(t)$ defined by

$$\rho_{x,y}(t) := \frac{f(txy) - f(tx) - f(ty) + f(t)}{\sigma(t)}$$

satisfies

$$\overline{\lim_{t\to\infty}} \rho_{x,y}(t) < \infty \qquad \text{for} \quad x > 1,\ y \geq y_0 \tag{0.4}$$

$$\underline{\lim_{t\to\infty}} \rho_{x,y}(t) > -\infty \quad \text{for} \quad x > 0,\ y \geq y_0 \tag{0.5}$$

$$\underline{\lim_{t\to\infty}} \rho_{x,y}(t) > 0 \qquad \text{for} \quad x > y_0,\ y \geq y_0 \tag{0.6}$$

Notation: $f \in \mathrm{AB}_2(\sigma)$ or $f \in \mathrm{AB}_2$.

In this paper we study the relation between the classes AB and AB_2. In section 1 we consider the case where $\psi(t) := f(e^t)$ is convex. Note that the relation

$$f(txy) - f(tx) - f(ty) + f(t) > 0 \quad \text{for} \quad x, y > 1,\ t > 0$$

is equivalent to ψ being convex. This is not implied by definition 0.2, but relation (0.6) in the definition can be seen as a form of asymptotic convexity. The size of the function σ (which is an element of RO) in the denominator can be seen as a measure of the asymptotic convexity of ψ. We show that if ψ is convex, the condition $f \in \mathrm{AB}_2$ is equivalent to: $tf'(t) \in \mathrm{AB}$. This is similar to the connection between the class Π and slow variation: If f is concave, then $f \in \Pi$ if and only if $tf'(t)$ is slowly varying. In section 2 the convexity condition on the function ψ is replaced by the weaker condition of asymptotic convexity (see (2.2) below). In that case the connection between the classes AB and AB_2 runs via fractional integrals rather than derivatives. More specifically, using the function γ_r defined by

$$\gamma_r(t) := f(t) - rt^{-r} \int_{t_0}^{t} s^{r-1} f(s)\, ds$$

it follows that $f \in \mathrm{AB}_2(\sigma)$ if and only if $\gamma_r \in \mathrm{AB}(\sigma)$ for r sufficiently large. We close the section with an Abel–Tauber theorem for the class AB_2.

1. Asymptotic balance with convexity

In this section we assume that $\psi(t) := f(e^t)$ is convex. Then ψ has a non-decreasing Radon–Nikodym derivative $\varphi = \psi'$. We shall prove

THEOREM 1.1. *Suppose ψ is convex with derivative φ. The function $f(s) := \psi(\log s)$ is asymptotically balanced of second order if and only if $g(s) := \varphi(\log s)$ is asymptotically balanced (of first order).*

For the proof of the theorem we need two propositions in which it is shown that the convexity assumption allows us to describe the concepts AB and AB_2 in terms of the asymptotic behaviour of certain sequences.

PROPOSITION 1.1. *Suppose φ is a non-decreasing function. Equivalent are:*

1. *The function g defined by $g(s) := \varphi(\log s)$ is asymptotically balanced.*
2. *There exists a constant $c > 0$ such that $\log(a_n/a_{n+1})$ is bounded $(n \to \infty)$ where $a_n := \varphi((n+1)c) - \varphi(nc)$.*

Proof. For the proof of $1 \Rightarrow 2$ note that in definition 0.1 we may replace $\sigma(t)$ by either $\sigma(ty)$ for any $y > 0$ (since σ is RO, see [**4**, Lemma 3.10]) or by $f(tz) - f(t)$ for any $z > x_0$ (obvious from the definition). It follows that $\{\varphi(t+z) - \varphi(t)\}/\{\varphi(t+z_0) - \varphi(t)\}$ is bounded away from zero and infinity $(t \to \infty)$ for $z > 0$ and z_0 sufficiently large.

Next we prove the implication $2 \Rightarrow 1$. Note that for $n = [t/c]$ and $k \geq 1$ integer we have

$$\frac{\sum_{i=1}^{k-1} a_{n+i}}{\sum_{i=0}^{2} a_{n+i}} \leq \frac{\varphi(t+kc) - \varphi(t)}{\varphi(t+2c) - \varphi(t)} \leq \frac{\sum_{i=0}^{k} a_{n+i}}{a_{n+1}}.$$

Now (0.2c) follows by taking the $\liminf$ as $t \to \infty$. Moreover (0.2a) follows with $x = \exp(kc)$ and by monotonicity (0.2a) is also true for any $x > 1$. Similarly (0.2b) is a consequence of the inequality

$$\frac{\varphi(t-kc)-\varphi(t)}{\varphi(t+2c)-\varphi(t)} \geq -\frac{\sum_{i=0}^{k} a_{n-i}}{a_{n+1}},$$

valid for $n = [t/c]$ and $k \geq 1$.

PROPOSITION 1.2. *Suppose ψ is convex. Equivalent are*

1. *The function f defined by $f(s) := \psi(\log s)$ is asymptotically balanced of second order.*
2. *There exists a constant $c > 0$ such that*

$$\overline{\lim_{t\to\infty}} \frac{\Delta(nc,2c)}{\Delta(nc,c)} < \infty \tag{1.1}$$

where $\Delta(t,x) := \psi(t+x) - 2\psi(t) + \psi(t-x)$.

Proof. We prove the implication $1 \Rightarrow 2$. Set $s(t) = \sigma(e^t)$. The conditions (0.4) and (0.6) imply the conditions

$$\overline{\lim_{t\to\infty}}\, \Delta(t,x)/s(t) < \infty \quad \text{for} \quad x \geq x_0$$
$$\underline{\lim_{t\to\infty}}\, \Delta(t,x)/s(t) > 0 \quad \text{for} \quad x \geq x_0.$$

It follows that we may choose $s(t) = \Delta(t, x_1)$ for any fixed $x_1 > x_0$. This gives (1.1) with $c = x_0$. For the converse implication one can use similar arguments as in the proof of Proposition 1.1.

Proof of Theorem 1.1. Suppose $f \in \mathrm{AB}_2$. Set $d_n = \Delta(nc, c)$. Then

$$\Delta(nc,2c) = d_{n+1} + 2d_n + d_{n-1}. \tag{1.2}$$

Divide by d_n. Proposition 1.2 ensures that d_{n+1}/d_n and d_{n-1}/d_n are bounded. Since $\varphi = \psi'$ it follows that

$$\begin{aligned} d_{n+1} &= \Delta((n+1)c, c) = \psi((n+2)c) - 2\psi((n+1)c) + \psi(nc) \qquad (1.3)\\ &= \int_{(n+1)c}^{(n+2)c} \varphi(s)\,ds - \int_{nc}^{(n+1)c} \varphi(s)\,ds = \int_0^c \{\varphi(c(n+1)+s) - \varphi(nc+s)\}\,ds. \end{aligned}$$

Using monotonicity of φ gives

$$c\{\varphi((n+2)c) - \varphi(nc)\} \geq d_{n+1}. \tag{1.4}$$

Similarly we find

$$c\{\varphi((n+2)c) - \varphi(nc)\} \leq \int_{(n+2)c}^{(n+3)c} \varphi(s)\,ds - \int_{(n-1)c}^{nc} \varphi(s)\,ds = d_n + d_{n+1} + d_{n+2}. \tag{1.5}$$

If we replace n by $2n$ and c by $c/2$ in (1.4) and (1.5) we see that the conditions of Proposition 1.1 are satisfied, hence g is asymptotically balanced. The proof of the converse statement is an immediate consequence of the inequalities (1.4) and (1.5) and Propositions 1.1 and 1.2.

2. Asymptotic balance with asymptotic convexity

In this section we do not assume that $\psi(s) = f(e^s)$ is convex. However in order to obtain non-trivial results we have to impose condition (2.2) below which can be seen as an asymptotic convexity condition. First we consider the possible order of growth of the function σ in definition 0.2.

LEMMA 2.1. *If $f \in \mathrm{AB}_2(\sigma)$, then $\overline{\lim}_{t\to\infty} \sigma(at)/\sigma(t) < \infty$ for all $a > 0$. Moreover we may take σ measurable, hence $\sigma \in \mathrm{RO}$.*

Proof. Take $a > 0$ arbitrary. Observe that

$$\sigma(at)/\sigma(t) = \{\rho_{ay,x}(t) - \rho_{a,x}(t)\}/\rho_{x,y}(at). \tag{2.1}$$

Note that $\overline{\lim}_{t\to\infty} \sigma(at)/\sigma(t) < \infty$ if we choose $x > y_0$, $y > \max(a^{-1}, y_0)$ and use definition 0.2. We may choose $\sigma(t) = f(ty_0^2) - 2f(ty_0) + f(t)$ which is measurable.

The basic result in this section relates second order asymptotic balance of a function f to first order asymptotic balance of the transform γ_r of f.

THEOREM 2.1. *Suppose $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable and suppose there exists a positive function σ such that*

$$\lim_{t\to\infty} \rho_{x,y}(t) = \lim_{t\to\infty} \frac{f(txy) - f(tx) - f(ty) + f(t)}{\sigma(t)} \geq 0 \quad \text{for all} \quad x, y > 1. \tag{2.2}$$

Define the functions $\gamma_r(t)$ and $s_t(x)$ by

$$\gamma_r(t) := f(t) - rt^{-r} \int_{t_0}^{t} s^{r-1} f(s)\, ds \qquad (t > t_0) \tag{2.3}$$

$$s_t(x) := \frac{f(tx) - f(t) - r \log x \gamma_r(t)}{\sigma(t)} \tag{2.4}$$

Consider the following statements:

(i) $f \in \mathrm{AB}_2(\sigma)$

(ii) *there exist t_0, r such that $\gamma_r(t)$ is well defined for $t > t_0$ and $\gamma_r(t) \in \mathrm{AB}(\sigma)$*

(iii) *there exist t_0, r such that the function $\gamma_r(t)$ is well defined for $t > t_0$ and $s_t(x)$ satisfies the conditions*

$$\overline{\lim_{t\to\infty}} |s_t(x)| < \infty \quad \text{for all} \quad x > 0 \tag{2.5}$$

$$\underline{\lim}_{t\to\infty} \{s_t(y) - s_t(x)\} \geq 0 \quad \text{for all} \quad y > x \geq 1 \tag{2.6}$$

and there exists a constant $x_0 > 0$ *such that*

$$\varliminf_{t\to\infty} s_t(x) > 0 \quad \text{for all} \quad x > x_0. \tag{2.7}$$

Moreover $\sigma \in \mathrm{RO}$.

Statement (i) *implies* (ii) *for all sufficiently large* r. *For fixed* $r > 0$ *the statements* (ii) *and* (iii) *are equivalent and imply statement* (i).

In order to be able to formulate the proof of this theorem we need the following class of functions: a measurable, eventually positive function f is of bounded and positive increase ($f \in \mathrm{BI} \cap \mathrm{PI}$) if $f \in \mathrm{RO}$ with lower Matuszewska index positive. See [**3**, Chapter 2.1] or [**4**, Chapter 3]. In order to prove the theorem we need an auxiliary result on ordinary AB functions which is an analogue of Theorem 3.13 in [**4**].

LEMMA 2.2. *Suppose* $f : \mathbf{R}^+ \to \mathbf{R}$ *is measurable. Consider the following statements:*

(i) *There is a (positive) function* σ *such that* $f \in \mathrm{AB}(\sigma)$ *and*

$$\varliminf_{t\to\infty} \frac{f(tx) - f(t)}{\sigma(t)} \geq 0 \quad \text{for all} \quad x > 1, \tag{2.8}$$

(ii) *For some* $t_0 > 0$

$$g_r(t) := t^r f(t) - r \int_{t_0}^{t} s^{r-1} f(s)\, ds$$

is well defined for $t > t_0$ *and in BI* $\cap$ *PI. Moreover*

$$\varliminf_{t\to\infty} \frac{f(tx) - f(t)}{t^{-r} g_r(t)} \geq 0 \quad \text{for all} \quad x > 1. \tag{2.9}$$

Statement (i) *implies* (ii) *for all sufficiently large* r. *If statement* (ii) *is true for some* $r > 0$, *then* (i) *holds with* $\sigma(t) := t^{-r} g_r(t)$.

Proof. (i) $\to$ (ii) Since $f \in \mathrm{AB}(\sigma)$ we may choose $\sigma \in \mathrm{RO}$ (see [**4**, Lemma 3.10]). Then $t^r\sigma(t) \in \mathrm{BI} \cap \mathrm{PI}$ for any $r > r_0 := -\beta(\sigma)$, the lower Matuszewska index of σ (see [**3**, Chapter 2.2] or [4, Chapter 3]). We prove that $t^r\sigma(t) \asymp g_r(t)$ for $r > r_0$ as $t \to \infty$. Note that this proves $g_r \in \mathrm{BI} \cap \mathrm{PI}$ ($r > r_0$) and the implication (2.8) $\to$ (2.9). Since $f \in \mathrm{AB}(\sigma)$ there exist c, α, $t_0 > 0$ such that $|f(t)| < ct^\alpha$ for $t > t_0$ (see [**4**, Lemma 3.12]). We have

$$\frac{g_r(t)}{t^r\sigma(t)} = r \int_{t_0/t}^{1} \frac{f(t) - f(ts)}{\sigma(ts)} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds + \frac{t_0^r f(t)}{t^r\sigma(t)}.$$

Hence $g_r(t)$ is finite for $t > t_0$ and if we choose r sufficiently large, then $f(t)/t^r\sigma(t) \to 0$ as $t \to \infty$. Since $\sigma \in \mathrm{RO}$ we can use Lemma 3.12 in [**4**] together with Fatou's lemma to find that

$$\varliminf_{t\to\infty} \frac{g_r(t)}{t^r\sigma(t)} \geq r \int_0^1 \varliminf_{t\to\infty} \frac{f(t/s) - f(t)}{\sigma(t)} \varliminf_{t\to\infty} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds.$$

Now by (2.8) and the definition of AB(σ)

$$\varliminf_{t\to\infty} \frac{f(t/s)-f(t)}{\sigma(t)} \begin{cases} \geq 0 & \text{for all } 0<s<1 \\ >0 & \text{for } s<x_0^{-1} \end{cases}$$

It follows that $\varliminf_{t\to\infty} g_r(t)/\{t^r\sigma(t)\} > 0$.

Similarly using the inequality

$$\left|\frac{f(t)-f(ts)}{\sigma(ts)}\right| \frac{\sigma(ts)}{\sigma(t)} \leq c_1 s^{-\alpha_1} c_2 s^{\alpha_2} \qquad \text{(see [4])}$$

for $t_0/t<s<1$ where c_i, α_i are positive constants, we have $\varlimsup_{t\to\infty} g_r(t)/\{t^r\sigma(t)\} < \infty$ if we choose $r>\alpha_1-\alpha_2$. This proves $g_r \in \mathrm{BI}\cap\mathrm{PI}$ for r sufficiently large.

(ii) $\to$ (i) From the definition of g_r, for $x>1$ we have

$$\frac{g_r(tx)-g_r(t)}{g_r(t)} = \frac{f(tx)-f(t)}{t^{-r}g_r(t)} + r\int_1^x \frac{f(tx)-f(ts)}{(ts)^{-r}g_r(ts)} \frac{(ts)^{-r}g_r(ts)}{t^{-r}g_r(t)} s^{r-1}ds.$$

Application of Fatou's lemma (using again Lemma 3.12 in [4] and (2.9)) shows that

$$\varliminf_{t\to\infty} g_r(tx)/g_r(t) \geq 1 \quad \text{for} \quad r>r_0,\ x>1. \tag{2.10}$$

From the definition of $g_r(t)$ it follows that

$$f(t) = t^{-r}g_r(t) + r\int_{t_0}^t g_r(s)s^{-r-1}ds$$

for $t>t_0$, hence

$$\frac{f(tx)-f(t)}{t^{-r}g_r(t)} = r\int_1^x \frac{g_r(tu)}{g_r(t)}u^{-r-1}du + \frac{(xt)^{-r}g_r(tx)}{t^{-r}g_r(t)} - 1. \tag{2.11}$$

Using the inequalities $c^{-1}x^\beta \leq g_r(tx)/g_r(t) \leq cx^\alpha$ for $x\geq 1$, $t>t_0$ where $\alpha,\beta>0$, $c>1$ (see [4, Theorem 3.5]) we see that (2.8) holds,

$$\varlimsup_{t\to\infty} \frac{f(tx)-f(t)}{t^{-r}g_r(t)} < \infty \quad \text{for } x>1 \quad \text{and} \quad \varliminf_{t\to\infty} \frac{f(tx)-f(t)}{t^{-r}g_r(t)} > -\infty \quad \text{for } x>0.$$

It remains to prove that $\varliminf_{t\to\infty}\{f(tx)-f(t)\}/\{t^{-r}g_r(t)\} > 0$ for $x>x_0$. By (2.10)

$$\varliminf_{t\to\infty} r\int_1^{x_0} \frac{g_r(tu)}{g_r(t)}u^{-r-1}du \geq r\int_1^{x_0} u^{-r-1}du. \tag{2.12}$$

Moreover, since

$$\varliminf_{t\to\infty} g_r(tx)/g_r(t) > 1 \quad \text{for} \quad x>x_0, \quad r>r_0,$$

we get for $x > x_0$

$$\underline{\lim}_{t\to\infty} r\int_{x_0}^{x} \frac{g_r(tu)}{g_r(t)} u^{-r-1}du + \frac{(xt)^{-r}g_r(tx)}{t^{-r}g_r(t)} - 1 > r\int_{x_0}^{x} u^{-r-1}du + x^{-r} - 1 \qquad (2.13)$$

Combination of (2.11), (2.12) and (2.13) gives the claimed result.

Remark. Note that the lemma fails if the assumptions (2.8) and (2.9) are omitted. Take e.g. $f(t) = \log t + \sin t$.

Proof of Theorem 2.1. Without loss of generality we may assume that $f(t) = 0$ on a neighborhood of zero.

(i) $\Leftrightarrow$ (ii) Define $\tilde{f}_y(t) := f(ty) - f(t)$. From the definitions of AB and AB_2 it follows that (i) is equivalent to: $\tilde{f}_y \in AB(\sigma)$ for all $y > y_0$. Application of Lemma 2.3 shows that (i) holds if and only if there exists y_0 such that for $y \geq y_0$, $r \geq r_0(y)$

$$t^r\{\gamma_r(ty) - \gamma_r(t)\} = t^r\tilde{f}y(t) - r\int_{t_0}^{t} \tilde{f}_y(s)s^{r-1}ds \quad \text{is in BI} \cap \text{PI}$$
$$\gamma_r(ty) - \gamma_r(t) \asymp \sigma(t) \quad (t\to\infty). \qquad (2.14)$$

Since σ is positive, the convexity condition (2.2) implies that the functions

$$\psi_y(x) := \underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)} \quad \text{and} \quad \Psi_y(x) := \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)}$$

are non-decreasing in x and y for all $x, y > 0$. Indeed this follows since $\Psi_y(x) = \Psi_x(y)$ and $\Psi_y(x)$ is non-decreasing in x since for $x \in (0, u)$ we have

$$\Psi_y(x) \leq \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(tu)}{\sigma(tx)}\frac{\sigma(tx)}{\sigma(t)} + \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tu) - \tilde{f}_y(t)}{\sigma(t)}. \qquad (2.15)$$

Note that the convexity condition (2.2) is equivalent to

$$\underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)} \geq 0 \quad \text{for} \quad x > 1.$$

Hence (2.15) is at most

$$-\underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tu) - \tilde{f}_y(tx)}{\sigma(tx)} \overline{\lim}_{t\to\infty} \frac{\sigma(tx)}{\sigma(t)} + \Psi_y(u) \leq \Psi_y(u) \leq \infty$$

and a similar argument for $\psi_y(x)$. Hence for all $x, y > 1$ we have $0 \leq \psi_y(x) \leq \Psi_y(x) \leq \Psi_{\max(y_0,y)}(x) < \infty$. Applying Lemma 3.12 in [**4**] we get

$$\left|\frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)}\right| \leq c_1(y)x^{\alpha_1(y)} \quad \text{for} \quad x \geq 1,\ t \geq t_0.$$

It follows that for arbitrary $y > 1$ there exist c, α such that

$$\overline{\lim_{t\to\infty}} \frac{\gamma_r(ty)-\gamma_r(t)}{\sigma(t)} = \overline{\lim_{t\to\infty}}\, r \int_{t_0/t}^{1} \frac{\tilde{f}_y(t)-\tilde{f}_y(ts)}{\sigma(t)} s^{r-1} ds$$

$$\leq r \int_0^1 \Psi_y(1/s)\, \overline{\lim_{t\to\infty}} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds \leq c \int_0^1 s^{\alpha+r-1} ds \leq \infty \tag{2.16}$$

if $r > -\alpha$. The proof of $\underline{\lim}_{t\to\infty}\{\gamma_r(ty)-\gamma_r(t)\}/\sigma(t) > -\infty$ for $y > 0$, $r > r_0$ is similar. Hence (i) implies (ii) for all sufficiently large r. The implication (ii) $\to$ (2.14) is trivial.

(ii) $\to$ (iii) From (2.3) it follows that

$$f(t) = \gamma_r(t) + r \int_{t_0}^{t} \gamma_r(s) \frac{ds}{s}, \quad t > t_0$$

hence

$$s_t(x) = \frac{f(tx)-f(t)-r\gamma_r(t)\log x}{\sigma(t)} = \frac{\gamma_r(tx)-\gamma_r(t)}{\sigma(t)} + r \int_1^x \frac{\gamma_r(ts)-\gamma_r(t)}{\sigma(t)} \frac{ds}{s}. \tag{2.17}$$

The last expression together with application of Lemma 3.12 in [**4**] and

$$\underline{\lim_{t\to\infty}} \{\gamma_r(tx)-\gamma_r(t)\}/\sigma(t) \geq 0 \quad \text{for} \quad x > 1$$

(which follows as in (2.16)), shows that (ii) implies (iii).

(iii) $\to$ (ii) Define $q_t(x) := \{\gamma_r(tx)-\gamma_r(t)\}/\sigma(t)$. From (2.4) it follows that for $y > x > 0$

$$q_t(x) = \frac{s_t(y)-s_t(x)-s_{tx}(y/x)\sigma(tx)/\sigma(t)}{r\log y/x}. \tag{2.18}$$

Hence by the assumptions on the functions $s_t(x)$ and σ it follows that $\overline{\lim}_{t\to\infty} |q_t(x)| < \infty$ for $x > 0$. Application of Lemma 3.12 in [**4**] then shows that

$$|q_t(x)| \leq cx^{\varepsilon} \quad \text{for} \quad x > 1,\ t > t_0, \tag{2.19}$$

where $\varepsilon, c > 0$. Hence using (2.17), i.e.

$$s_t(x) = q_t(x) + r \int_1^x q_t(s) \frac{ds}{s}, \tag{2.20}$$

it follows that $s_t(x)$ satisfies the inequality

$$|s_t(x)| < c_0 x^{\varepsilon_0} \quad \text{for} \quad x \geq 1,\ t \geq t_0, \tag{2.21}$$

where c_0 and ε_0 are constants. From (2.20) it follows that the function $q_t(x)$ satisfies the relation

$$q_t(x) = rx^{-r} \int_1^x (s_t(x) - s_t(u))u^{r-1}du + x^{-r}s_t(x). \tag{2.22}$$

The proof of $\underline{\lim}_{t\to\infty} q_t(x) > 0$ for $x > x_0$ follows by application of Fatou's lemma to the integral in (2.14) (use (2.6) and (2.7)). Note that by (2.6) for $x > 1$ we have $\underline{\lim}_{t\to\infty} s_t(x) = \underline{\lim}_{t\to\infty}\{s_t(x) - s_t(1)\} > 0$.

In order to formulate our next result we need the following notion. The functions $f, f_0 : \mathbf{R}^+ \to \mathbf{R}$ are O-inversely asymptotic if there exist constants $a > 1$ and t_0 such that $f(t) \le f_0(at)$ and $f_0(t) < f(at)$ for $t \ge t_0$. Notation: $f \overset{O}{\sim} f_0$ or $f(t) \overset{O}{\sim} f_0(t)$ $(t \to \infty)$. Observe that if f, f_0 are increasing and unbounded, then $f \overset{O}{\sim} f_0$ if and only if the inverse functions satisfy $f^{\leftarrow} \asymp f_0^{\leftarrow}$, which explains the terminology.

THEOREM 2.2. *Suppose $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable and suppose*

$$\hat{f}(s) := s\int_0^\infty e^{-st}f(t)\,dt < \infty \quad \textit{for} \quad s > 0.$$

Then

$$f \in \mathrm{AB}_2(\sigma) \quad \textit{with} \quad \beta(\sigma) > -1 \tag{2.23}$$

implies

$$\hat{f}(1/t) \in \mathrm{AB}_2(\sigma) \quad \textit{with} \quad \beta(\sigma) > -1. \tag{2.24}$$

If there exists t_0 such that

$$f(e^t) \textit{ is convex for } t > t_0 \tag{2.25}$$

then the converse holds: (2.24) *implies* (2.23). *Moreover if the function f in* (2.23) *satisfies* (2.2), *then there exist r_0, x_0 such that the transforms γ_r and γ_r^* satisfy*

$$r\gamma_r(t)\log x \overset{O}{\sim} f(tx) - f(t) \overset{O}{\sim} \hat{f}(1/tx) - \hat{f}(1/t) \overset{O}{\sim} r\gamma_r^*(t)\log x \tag{2.26}$$

as $t \to \infty$ for $r > r_0$, $x > x_0$, where $\gamma_r(t)$ is as defined in theorem 2.2 *and*

$$\gamma_r^*(t) = \hat{f}(t^{-1}) - rt^{-r}\int_{t_0}^t s^{r-1}\hat{f}(s^{-1})\,ds.$$

In particular we have for $r > r_0$

$$\gamma_r(t) - \gamma_r^*(t) = O(\sigma(t)) \quad (t \to \infty). \tag{2.27}$$

Proof. By the definitions of AB and AB_2 it follows that $f \in \mathrm{AB}_2(\sigma)$ is equivalent to $\tilde{f}_y(t) = f(ty) - f(t) \in \mathrm{AB}(\sigma)$ for all $y \ge y_0$. Application of theorem 4.2 in

[4] shows that this implies $\hat{\hat{f}}_y(t) = \hat{f}(1/ty) - \hat{f}(1/t) \in \mathrm{AB}(\sigma)$ for $y \geq y_0$ which is equivalent to $\hat{f}(1/t) \in \mathrm{AB}_2(\sigma)$. A converse statement is true if $\tilde{f}_y(t)$ is eventually non-decreasing in t which is condition (2.25). In order to prove (2.26) note that for $x > x_0$, $r > r_0$, there exists $t_0 = t_0(x, r)$ such that $f(tx) - f(t) > r \log x \gamma_r(t)$ for $t > t_0$ by (2.7).

For a converse inequality, fix $x > x_0$, $r > r_0$. Since $\gamma_r \in \mathrm{AB}(\sigma)$ we have by (2.17) for $y > x$ sufficiently large

$$\begin{aligned}\overline{\lim_{t\to\infty}} \frac{f(tx) - f(t) - r\gamma_r(ty)\log x}{\sigma(t)} &\leq c_1 + r \int_1^x \overline{\lim_{t\to\infty}} \frac{\gamma_r(ts) - \gamma_r(ty)}{\sigma(t)} \frac{ds}{s} \\ &\leq c_1 - rc_2 \int_1^x \frac{ds}{s} \qquad (2.28)\end{aligned}$$

where $c_1, c_2 > 0$ are constants (depending on r, see (2.4)). The right-hand side in (2.28) is negative if we choose $x > x_0$ sufficiently large, then $y > x$ sufficiently large in order to ensure the validity of (2.28). Hence $r\gamma_r(t)\log x \overset{O}{\sim} f(tx) - f(t)$. The statements $f(tx) - f(t) \overset{O}{\sim} \hat{f}(1/tx) - \hat{f}(1/t)$ and (2.27) follow from [**4**, theorem 4.2]. The proof of $\hat{f}(1/tx) - \hat{f}(1/t) \overset{O}{\sim} r\gamma_r^*(t)\log x \quad (t \to \infty)$ follows as above.

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