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EXACT ASYMPTOTIC BEHAVIOR OF THE SINGULAR VALUES OF INTEGRAL OPERATORS WITH THE KERNEL HAVING SINGULARITY ON THE DIAGONAL

Milutin Dostanić

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Abstract. Starting with a given function k, as the kernel of a convolution operator, the auxiliary function H is constructed, which is the kernel of a normal operator. Establishing the connection between this operator and the previous one, the exact asymptotic of singular values is obtained. The method is used to find the exact asymptotic of the singular values of integral operators with the kernel of the form $T(x,y)k(x = y)$, where k is not necessary a homogeneous function.

0. Introduction. We study the asymptotic singular value behavior of integral operators defined by kernels of the form

$$(*) \qquad K(x,y) = T(x,y)k(\|x = y\|^m), \qquad x,y \in \Omega$$

where Ω is a Jordan measurable set in R^m, and T, k are some suitably chosen functions.

Asymptotic properties of the spectrum of operators with convolution kernels are considered in many papers [**1**], [**2**], [**3**], [**8**], [**9**], [**12**], [**13**], [**14**], [**17**], [**18**]. The exact asymptotics are obtained under the condition that the Fourier transform of the kernel satisfies some conditions concerning the growth rate.

Kac [**8**], obtained the exact asymptotic of eigenvalues of the operators with kernel $\rho(y)|x-y|^{\alpha-1}$ ($0<\alpha<1$, $\varrho \in C[a,b]$, $\varrho>0$ on $[a,b]$). He used a probabilistic method and Karamata Tauberian theorem.

Birman, Solomjak, Kostometov and Rotfeld in [**1**], [**2**], [**9**], [**14**] considered the asymptotic of the spectrum of operators with the kernel of the form $(*)$. They assumed that k is a homogeneous function from the class $C^\infty(R\backslash\{0\})$ and that T is a function which is smooth of some order. They obtained the exact asymptotic

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AMS Subject Classification (1991): Primary 47B10 or the upper estimate of eigenvalues, depending on the assumptions mode on the smoothness.

Cobos and Kuhn [**3**] treated the problem of estimating the singular values of operators with the kernel of the form $(*)$ where

$$k(x) = |x|^{\alpha-1}(1 + m^{-1}|\ln|x||)^{\gamma}, \quad \gamma \in R, \quad 0 < \alpha \le 1/2.$$

They found an upper estimate for singular values of such operators and proved their optimality (in the sense of the growth order) in the case $m = 1$, $\Omega = [-1/2, 1/2]$ and

$$T(x,y) = \begin{cases} |x-y|^{\alpha-1} \cdot (1 - \ln|x-y|)^{\gamma}; & |x-y| \le 1/2 \\ 0; & |x-y| > 1/2 \end{cases}.$$

Oehring [**11**] proved the convergence of series of singular values (with weights defined by regularly varying sequences) for Hilbert Schmidt operators with the kernel which is 2π periodical function on the second variable.

In the cited papers the problem of determining the exact singular values asymptotic of integral operators with the kernel of the form $(*)$ (where k is not a homogeneous function) is not considered.

Here we propose a new method for solving such a problem. In the special case when

$$k(x) = |x|^{\alpha-1}\left(1 + \frac{1}{m}|\ln|x||\right)^{\gamma}, \quad \gamma \in R, \quad \frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$$

the exact asymptotic of singular values of operators considered in [**3**] is obtained.

1. Preliminaries. Suppose $\mathcal{H}$ is a complex Hilbert space and T is a compact operator on $\mathcal{H}$. The singular values of T ($s_n(T)$) are the eigenvalues of $(T^*T)^{1/2}$ (or $(TT^*)^{1/2}$). The eigenvalues of $(T^*T)^{1/2}$ arranged in a decreasing order and repeated according to their multiplicity, form a sequence $s_1, s_2, \ldots$ tending to zero.

Denote the set of compact operators on $\mathcal{H}$ by C_∞.

The operator T is a Hilbert–Schmidt one ($T \in C_2$) if $(\sum_{n=1}^{\infty} s_n^2(T))^{1/2} = |T|_2 < \infty$.

If $T \in C_2$ is an integral operator on $L^2(\Omega)$ defined by

$$Tf(x) = \int_\Omega M(x,y) f(y)\,dy$$

then [**7**]

$$|T|_2^2 = \int_\Omega \int_\Omega |M(x,y)|^2 dx dy.$$

Denote by $\int_\Omega K(x,y) \cdot dy$ the integral operator on $L^2(\Omega)$ with the kernel K.

Let $\mathcal{N}_t(T)$ be the singular value distribution function

$$\mathcal{N}_t(T) = \sum_{s_n(T) \geq t} 1 \qquad (t > 0).$$

A positive function L is a slowly varying function on $[a, +\infty)$ if it is measurable and for each $\lambda > 0$ the equality

$$\lim_{x \to +\infty} L(\lambda x)/L(x) = 1$$

holds. It is well known [**15**] that for every $\gamma > 0$ we have

$$\lim_{x \to +\infty} x^{\gamma} L(x) = +\infty \qquad \lim_{x \to +\infty} x^{\gamma} L(x) = 0.$$

In what follows we need some lemmas

Lemma 1. *Suppose L is a slowly varying function such that $\varphi(x) = x^{-r}L(x)$ and $\psi(x) = x^r L(x)$ $(r > 0)$ are monotone for $x \geq x_0$ and*

$$\lim_{x \to +\infty} \frac{L(x(L(x))^{\pm 1/r})}{L(x)} = 1 \tag{0}$$

Then

$$\varphi^{-1}(y) \sim \Big(L(y^{-1/r})/y\Big)^{1/r} \qquad y \to 0^+,$$
$$\psi^{-1}(y) \sim \Big(y/L(y^{1/r})\Big)^{1/r} \qquad y \to +\infty,$$

where φ^{-1}, ψ^{-1} are the inverses of φ and ψ.

Proof. Directly follows from (0) by substitution.

We observe that the functions

$$L(x) = \prod_{i=1}^{s} (\ln_{m_i}(x))^{\alpha_i} \qquad (\ln_{m_i}(x) = \underbrace{\ln \ln \ldots \ln}_{m_i} x)$$

satisfy conditions of Lemma 1.

Lemma 2. *Suppose the operator $H \in C_\infty$ is such that for every $\varepsilon > 0$ there exist a decomposition $H = H'_\varepsilon + H''_\varepsilon$ $(H'_\varepsilon, H''_\varepsilon \in C_\infty)$ with the following properties:*

1° *There exists* $\lim_{t \to 0+} \left(\frac{t}{L(t^{-1/r})}\right)^{1/r} \mathcal{N}_t(H'_\varepsilon) = c(H'_\varepsilon)$ 2° $\overline{\lim_{n \to \infty}} \frac{n^r}{L(n)} s_n(H''_\varepsilon) < \varepsilon$.

Then there exists $\lim_{\varepsilon \to 0+} C(H'_\varepsilon) = C(H)$ *and* $\lim_{t \to 0} \left(\frac{t}{L(t^{-1/r})}\right) \mathcal{N}_t(H) = C(H)$

Proof. Lemma 2 can be proved by a slight modification of the proof of Ky Fan theorem [**7**].

2. Main result. Suppose k is an even complex valued function from $C^r(R\backslash\{0\})$ having a compact support. Let $K(\xi) = \int_R e^{it\xi} k(t)\,dt$. Consider the operator $A\colon L^2(-1,1) \to L^2(-1,1)$ defined by

$$Af(x) = \int_{-1}^{1} k(x-y)f(y)\,dy$$

THEOREM 1. *Let the function $|K(\xi)|$ be decreasing for ξ large enough, and $|K(\xi)| \sim \xi^{-r}L(\xi)$ $(r \in N)$ and L is some slowly varying function). If the operator $B\colon L^2(0,2) \to L^2(0,2)$ defined by*

$$Bf(x) = \int_0^2 k(x+y)f(y)\,dy$$

satisfies the condition

$$\lim_{n\to\infty} \frac{n^r}{L(n)} s_n(B) = 0; \tag{1}$$

then $s_n(A) \sim \dfrac{L(n)}{(n\pi/2)^r}$.

Proof. Consider the function

$$H(x,y) = \sum_{n=-\infty}^{\infty} [k(x-y+4n) - k(x+y+4n+2)], \qquad x,y \in [-1,1].$$

Let $\varphi_n(x) = \sin n\pi(1+x)/2$, $n \in N$. The system of functions $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(-1,1)$. By a direct computation we get

$$\int_{-1}^{1} H(x,y)\varphi_n(y)\,dy = K\left(\frac{n\pi}{2}\right)\varphi_n(x), \quad x \in [-1,1], \quad n = 1,2,3,\dots$$

The operator $A_0\colon L^2(-1,1) \to L^2(-1,1)$ defined by $A_0 f(x) = \int_{-1}^{1} H(x,y)f(y)\,dy$ is a normal one and $\{|\lambda_n(A_0)|\}_{n\ge 1}$ are its singular values.

By assumption of Theorem 1 we have

$$s_n(A_0) \sim (n\pi/2)^{-r}L(n). \tag{2}$$

Let

$$H_1(x,y) = k(x-y-4) + \sum_{\substack{n\neq 0\\ n\neq -1}} [k(x-y+4n) - k(x+y+4n+2)],$$

$$H_2(x,y) = -k(x+y+2), \qquad H_3(x,y) = -k(x+y-2)$$

and let A_i be linear operators on $L^2(-1,1)$ defined by

$$A_i f(x) = \int_{-1}^{1} H_i(x,y) f(y)\, dy \qquad (i = 1,2,3).$$

Then $A_0 = A + A_1 + A_2 + A_3$. Since $\partial^r H_1/\partial y^r$ is a bounded function on $[-1,1]^2$, we have [**7**] $s_n(A_1) = o(n^{-r-1/2})$ and consequently

$$s_n(A_1) = o(L(n)/n^r). \tag{3}$$

From the properties of singular values [**7**] and (1) it follows

$$s_n(A_i) = o(L(n)/n^r) \qquad i = 2,3. \tag{4}$$

Finallly, from $A = A_0 - A_1 - A_2 - A_3$, (2), (3), (4) and Ky Fan theorem we get

$$s_n(A) \sim (n\pi/2)^{-r} L(n).$$

Remark 1. Let Δ be an interval in R. If the function k satisfy the conditions of Theorem 1, then

$$s_n\left(\int_\Delta k(x-y)\cdot dy\right) \sim \left(\frac{n\pi}{|\Delta|}\right)^{-r} L(n),$$

where $|\Delta|$ denotes the length of the interval Δ (Consequence of Theorem 1).

Remark 2. It is not necessary to assume that k has the compact support; it is enough to suppose that the series defining H is convergent and r times differentiable by y term by term. It is easy to prove Theorem 1 in some cases when $r \notin \mathbf{N}$. Namely, in the case $0 < r \le 1/2$ the condition (1) can be substituted by

$$\int_0^2 \int_0^2 |k(x+y)|^2\, dx\, dy < \infty \qquad (\text{i.e. } B \in C_2).$$

Then it is not necessary to assume that k is smooth. It is enough that $k \in C(R\backslash\{0\})$ (with compact support). The function k is not supposed to be homogeneous, and so the method from [**1**], [**2**] cannot be used to obtain the asymptotic behavior of the associated operators.

Example 1. Let $k(x) = |x|^{\alpha-1}$, $0 < \alpha < 1$. The series

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{|x-y+4n|^{1-\alpha}} - \frac{1}{|x+y+4n+2|^{1-\alpha}}\right)$$

is convergent and

$$\int_R e^{itx} |t|^{\alpha-1}\, dt = 2\Gamma(\alpha) \cos\frac{\alpha\pi}{2} |x|^{-\alpha}.$$

So, $K(\xi) = 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}|\xi|^{-\alpha}$. If we prove

$$s_n\left(\int_0^2 (x+y)^{\alpha-1}\cdot dy\right) = o(n^{-\alpha}) \tag{5}$$

then by Theorem 1 we obtain

$$s_n\left(\int_{-1}^1 |x-y|^{\alpha-1}\cdot dy\right) \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\left(\frac{2}{n\pi}\right)^{\alpha}.$$

The operator $\int_{-1}^1 |x-y|^{\alpha-1}\cdot dy$ $(0<\alpha<1)$ is a positive one. From the previous relation we get

$$\lambda_n\left(\int_{-1}^1 |x-y|^{\alpha-1}\cdot dy\right) \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\left(\frac{2}{n\pi}\right)^{\alpha}.$$

Observe that the direct application of Theorem from [**18**] to the kernel $k(x) = |x|^{\alpha-1}$ can not give the corresponding asymptotic formula, because the function K is not bounded on $R\backslash\{0\}$.

Now, we prove (5).

Lemma 3. *For the operator $C\colon L^2(0,2)\to L^2(0,2)$ defined by*

$$Cf(x) = \int_0^2 (x+y)^{\alpha-1} f(y)\,dy \qquad (0<\alpha<1)$$

we have $\lim_{n\to\infty} n^\alpha s_n(C) = 0$.

Proof. Let $\varrho>0$ be an arbitrary real number $(\varrho<2)$ and let $P_\varrho\colon L^2(0,2)\to L^2(0,2)$ be an operator defined by

$$P\rho f(x) = \begin{cases} f(x); & x\in[0,\varrho] \\ 0; & x\in(\varrho,2]\end{cases}.$$

Then $C = C(I-P_\varrho) + (I-P_\varrho)CP_\varrho + P_\varrho CP_\varrho$. Form Krein Theorem [**7**] it follows

$$\begin{aligned} s_n(C(I-P_\varrho)) &= o(n^{-3/2}) \\ s_n((I-P_\varrho)CP_\varrho) &= o(n^{-3/2}) \end{aligned} \tag{6}$$

Since $P_\varrho CP_\varrho f(x) = \int_0^\varrho (x+y)^{\alpha-1} f(y)\,dy$, then applying the partial integration of order α and the Hardy-Littlewood inequality [**16**] we get

$$n^\alpha s_n(P_\varrho CP_\varrho) \le C_0\varrho, \tag{7}$$

where the constant C_0 does not depend on ϱ and n. (This method we use in [**5**]).

From (6), (7) and the properties of singular values we obtain $n^\alpha s_n(C) \le V_0 \cdot \varrho$ for $n \ge n_0$, where the constant V_0 does not depend on n and ϱ. Lemma is proved.

Now, consider the case $\alpha > 1$, $\alpha \notin \mathbf{N}$. Start with the function,

$$G_\alpha(x) = \frac{2^{(1-\alpha)/2}}{\sqrt{\pi}\Gamma(\alpha/2)} \cdot \frac{K_{(1-\alpha)/2}}{|x|^{(1-\alpha)/2}} \quad (\in C^\infty(R\backslash\{0\}))$$

where K_ν is McDonald function [**16**]. It is well known [**16**] that $G_\alpha \in L^1(R)$ for each $\alpha > 0$,

$$\int_{\mathbf{R}} e^{itx} G_\alpha(t)\, dt = (1+x^2)^{-\alpha/2} \quad G_\alpha(x) \sim \frac{|x|^{(\alpha-2)/2} e^{-|x|}}{2^{\alpha/2}\Gamma(\alpha/2)}, \qquad |x| \to +\infty.$$

By Theorem 1 we have

$$s_n\left(\int_{-1}^{1} G_\alpha(x-y)\cdot dy\right) \sim \left(\frac{\pi n}{2}\right)^{-\alpha}. \tag{8}$$

Since

$$G_\alpha(x) = \frac{|x|^{\alpha-1}}{2\Gamma(\alpha)\cos\alpha\pi/2} + |x|^{\alpha-1}\varphi(x) + \psi(x), \qquad \varphi(0) = 0,$$

where φ and ψ are even entire functions, then from (8), Ky Fan and Krein theorem [**7**] it follows

$$s_n\left(\int_{-1}^{1} \frac{|x-y|^{\alpha-1}}{2\Gamma(\alpha)\cos\alpha\pi/2}\cdot dy\right) \sim \left(\frac{\pi n}{2}\right)^{-\alpha}, \qquad \text{i.e.}$$

$$s_n\left(\int_{-1}^{1} |x-y|^{\alpha-1}\cdot dy\right) \sim 2\Gamma(\alpha)\left|\cos\frac{\alpha\pi}{2}\right| \cdot \left(\frac{2}{n\pi}\right)^{-\alpha}.$$

Example 2. Let $k(x) = \dfrac{(\ln \gamma/|x|)^m}{|x|^{1-\alpha}}$, $0 < \alpha < 1$, $\gamma > 0$, $m = 0, 1, 2\ldots$ Since

$$\int_0^2 \int_0^2 \frac{\left|\ln \frac{\gamma}{x+y}\right|^{2m}}{(x+y)^{2-2\alpha}}\, dx\, dy < \infty \qquad (\text{for } \alpha > 0,\ m = 0, 1, 2\ldots)$$

we have that for $0 < \alpha \le 1/2$

$$\lim_{n\to\infty} \frac{n^\alpha}{(\ln n)^m} s_n\left(\int_0^2 \frac{(\ln \gamma/(x+y))^m}{(x+y)^{1-\alpha}}\cdot dy\right) = 0 \tag{9}$$

holds. Since

$$\int_R e^{itx} k(t)\, dt \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\frac{(\ln|x|)^m}{|x|^\alpha} \qquad (x \to +\infty)$$

(which is obtained by differentiating $\int_R e^{itx}|t|^{\alpha-1}\,dt = 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}|x|^{-\alpha}$ by α), then from (9) and Theorem 1 we obtain

$$s_n\left(\int_{-1}^{1}\frac{(\ln\gamma/|x-y|)^m}{|x-y|^{1-\alpha}}\cdot dy\right)\sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\frac{(\ln n)^m}{(n\pi/2)^{\alpha}}.$$

THEOREM 2. *Suppose k is an even complex valued function satisfying the following conditions:*

1° *$k\in L^1(0,\infty)$, $k\in L^2(1/\alpha,+\infty)$ for each $a>0$ and*

$$\int_0^{1/a}|k(t)|\,dt + a^{-1/2}\left(\int_{1/a}^{+\infty}|k(t)|^2\,dt\right)^{1/2} = O\left(\frac{L(a)}{a^r}\right)$$

($0<r<1/2$, L is a function from Lemma 1).

2° *The function $|K(\xi)|$, where $K(\xi)=\int_R e^{it\xi}k(t)\,dt$, is decreasing for ξ large enough.*

3° *The series $\sum_{\substack{n\neq 0\\ n\neq -1}}[k(a(x-y+4n))-k(a(x+y+4n+2))]$ is convergent and its sum is a bounded function on $[-a,a]^2$ for each $a>0$.*

If $T\in L^\infty((-1,1)^2)$ is a continuous function in some neighborhood of the diagonal $y=x$ and $T(x,x)>0$ on $[-1,1]$, then

$$s_n\left(\int_{-1}^{1}T(x,y)k(x-y)\cdot dy\right)\sim\frac{L(n)}{n^r}\left(\frac{1}{\pi}\int_{-1}^{1}(T(x,x))^{1/r}\,dx\right)^r$$

Proof. Let $\Delta_i=[-1+2(i-1)/N,-1+2i/N]$, $i=1,2,\dots N$ and let x_i be the midpoint of Δ_i. From the assumptions of the theorem, by Theorem 1 (having in mind Remarks 1 and 2) we have

$$s_n\left(\int_{\Delta_i}k(x-y)\cdot dy\right)\sim\left(\frac{n\pi}{|\Delta_i|}\right)^{-r}L(n)$$

Condition 1° (for $a=n$) and Lemma 1 from [**3**] imply

$$s_n\left(\int_{-1}^{1}T(x,y)k(x-y)\cdot dy\right)\le C\|T\|_\infty\frac{L(n)}{n^r}\tag{10}$$

for each $T\in L^\infty((-1,1)^2)$ where the constant C does not depend on n and T.

Let $A_i^N, A_{ij}\colon L^2(-1,1)\to L^2(-1,1)$ $(i,j=1,2,\dots N)$ be the linear operators defined by

$$A_i^N f(x)=\chi_{\Delta_i}(x)\int_{-1}^{1}k(x-y)\chi_{\Delta_i}(y)T(x_i,x_i)f(y)\,dy$$

$$A_{ij}f(x)=\chi_{\Delta_j}(x)\int_{-1}^{1}k(x-y)\chi_{\Delta_i}(y)T(x,y)f(y)\,dy$$

($\chi_\Delta(\cdot)$ is the characteristic function of the set Δ). Let $A_N = \sum_{i=1}^N A_i^N$ and let $B_N\colon L^2(-1,1) \to L^2(-1,1)$ be the linear operator defined by $B_N f(x) = \int_{-1}^1 G_N(x,y)k(x-y)f(y)\,dy$, where

$$G_N(x,y) = \sum_{i-1}^{N} \chi_{\Delta_i}(x)\chi_{\Delta_i}(y)(T(x,y) - T(x_i,x_i)).$$

Then $A = A_N + B_N + \sum_{i\neq j} A_{ij}$. Suppose $\varepsilon > 0$. Then from continuity of T in the neighborhood of the diagonal $y = x$, it follows that for N large enough $|T(x,y) - T(x_i,x_i)| < \varepsilon$ for $(x,y) \in \Delta_i \times \Delta_i$. Then

$$|G_N(x,y)| < \varepsilon, \qquad (x,y) \in [-1,1]^2. \tag{11}$$

Since $B_N f(x) = \int_{-1}^1 G_N(x,y)k(x-y)f(y)\,dy$, then from (10) (for $T = G_N$) and (11) it follows

$$s_n(B_N) \leq C \cdot \varepsilon \cdot L(n)/n^r \tag{12}$$

where the constant C does not depend on ε and n.

Now, we prove that $A_{ij} \in C_2$ $(i \neq j)$. From the condition 1° of Theorem 2 it follows

$$\frac{1}{a}\int_{1/a}^{\infty} |k(t)|^2\,dt \leq \text{const}\,(L(a))^2/a^{2r} \qquad (a \text{ large enough})$$

and therefore

$$\int_y^\infty |k(t)|^2\,dt \leq \text{const}\, y^{2r-1}\left(L\left(\frac{1}{y}\right)\right)^2. \tag{13}$$

To prove $A_{ij} \in C_2$ $(i \neq j)$ it is enough to prove

$$\int_{\Delta_i}\int_{\Delta_j} |k(x-y)|^2\,dx\,dy < \infty \tag{14}$$

because $T \in L^\infty((-1,1)^2)$. In the case $\Delta_i \cap \Delta_j = \emptyset$ (14) is true because $k \in L^\infty(\Delta_i \times \Delta_j)$. Now, suppose that the intervals Δ_i and Δ_j are neighbours (for example $j = i+1$).

Since

$$\begin{aligned}\int_{\Delta_i} dx \int_{\Delta i+1} |k(x-y)|^2\,dy &= \int_0^{2/N} dy \int_0^{2/N} \left|k\left(x-y-\frac{2}{N}\right)\right|^2 dx \\ &= \int_0^{2/N} dy \int_y^{y+\frac{2}{N}} |k(t)|^2\,dt \leq \int_0^{2/N} dy \int_y^\infty |k(t)|^2\,dt,\end{aligned}$$

then from (13) it follows

$$\int_{\Delta_i}\int_{\Delta_j} |k(x-y)|^2\,dx\,dy \le \text{const}\int_0^{2/N} y^{2r-1}\left(L\left(\frac{1}{y}\right)\right)^2 dy$$
$$= \text{const}\int_{N/2}^{\infty} t^{-1-2r}(L(t))^2\,dt < \infty.$$

Since $A_{ij} \in C_2$ $(i \neq j)$, then $C_N = \sum_{i,j=1}^{N}{}_{i\neq j} A_{ij}$ is Hilbert-Schmidt operator and $\lim_{n\to\infty} n^{1/2} s_n(C_N) = 0$, i.e.

(15) $$s_n(C_N) = o(L(n)/n^r) \qquad (\text{because } 0 < r < 1/2)$$

From (12) and (15) it follows that for each $\varepsilon > 0$ there exists a sufficiently large positive integer N such that

(16) $$\overline{\lim_{n\to\infty}}\ \frac{n^r}{L(n)} s_n(B_N + C_N) \le \varepsilon.$$

Since the operator A_N is the orthogonal sum of the operators A_i^N, we have

(17) $$\mathcal{N}_t(A_N) = \sum_{i=1}^{N} \mathcal{N}_t(A_i^N)$$

From conditions 2° and 3° of Theorem 2 and Theorem 1 it follows

$$s_n(A_i^N) \sim T(x_i, x_i)\frac{|\Delta_i|^r}{\pi^r}\frac{L(n)}{n^r} \qquad (i = 1, 2, \ldots N).$$

Applying Lemma 1 we obtain

$$\lim_{t\to 0^+}\left(\frac{t}{L(t^{-1/r})}\right)^{1/r}\mathcal{N}_t(A_i^N) = \frac{|\Delta_i|}{\pi}(T(x_i, x_i))^{1/r}.$$

Combining this with (17) we get

$$\lim_{t\to 0}\mathcal{N}_t(A_N)\cdot\left(\frac{t}{L(t^{-1/r})}\right)^{1/r} = \sum_{i=1}^{N}\frac{|\Delta_i|}{\pi}(T(x_i, x_i))^{1/r}.$$

Having in mind (16) and $A = A_N + B_N + C_N$ by Lemma 2 we obtain

$$\lim_{t\to 0^+}\mathcal{N}_t(A)\cdot\left(\frac{t}{L(t^{-1/r})}\right)^{1/r}$$
$$= \frac{1}{\pi}\lim_{N\to+\infty}\sum_{i=1}^{N}|\Delta_i|(T(x_i, x_i))^{1/r} = \frac{1}{\pi}\int_{-1}^{1}(T(x, x))^{1/r}\,dx$$

Substituting $t = s_n(A)$ we get

$$\frac{n}{\frac{1}{\pi}\int_{-1}^{1}(T(x,x))^{1/r}\,dx} \sim \left(\frac{L((s_n(A))^{-1/r})}{s_n(A)}\right)^{1/r}.$$

Applying Lemma 1 we obtain

$$s_n(A) \sim \left(\frac{1}{\pi}\int_{-1}^{1}(T(x,x))^{1/r}\,dx\right)^{r}\frac{L(n)}{n^r}.$$

which ends the proof of Theorem 2.

Example 3. Let $k(x) = |x|^{\alpha-1}(1+|\ln|x||)^{\gamma}$, $0<\alpha<1/2$, $\gamma\in R$. The function k satisfies conditions of Theorem 2 and

$$K(\xi) = \int_R e^{it\xi}k(t)\,dt = 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\frac{(\ln\xi)^{\gamma}}{\xi^{\alpha}}\cdot(1+o(1))\quad \xi\to+\infty \qquad [6].$$

According to Theorem 2 we have

$$(18)\quad s_n\left(\int_{-1}^{1}T(x,y)\frac{(1+|\ln|x-y||)^{\gamma}}{|x-y|^{1-\alpha}}\cdot dy\right) \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\left(\frac{1}{\pi}\int_{-1}^{1}(T(x,x))^{1/\alpha}\,dx\right)^{\alpha}\cdot\frac{(\ln n)^{\gamma}}{n^{\alpha}}.$$

Remark 3. In [**3**] Cobos and Kühn obtained the asymptotic order of the singular values of the operator

$$\int_{-1}^{1}T(x,y)\frac{(1+|\ln|x-y||)^{\gamma}}{|x-y|^{1-\alpha}}\cdot dy$$

for a special case of T. Exact asymptotic behavior is not derived.

Remark 4. If we put $T(x,y)=\varrho(y)$ in (18) where $\varrho\in C[-1,1]$, $\varrho>0$ on $[-1,1]$ and $\gamma=0$, then for $0<\alpha<1/2$ we get

$$(19)\quad s_n\left(\int_{-1}^{1}\varrho(y)|x-y|^{\alpha-1}\cdot dy\right) \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\left(\frac{1}{\pi}\int_{-1}^{1}(\varrho(x))^{1/\alpha}\,dx\right)^{\alpha}\cdot n^{-\alpha}.$$

Having in mind the proof of Theorem 2, as in Lemma 5 we get that asymptotic formula (19) is also valid in the case $1/2\le\alpha<1$. From [**4**, Theorem 1] it follows

$$\lambda_n\left(\int_{-1}^{1}\varrho(y)|x-y|^{\alpha-1}\cdot dy\right) \sim 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\left(\frac{1}{\pi}\int_{-1}^{1}(\varrho(x))^{1/\alpha}\,dx\right)^{\alpha}\cdot n^{-\alpha}.$$

Kac [8] obtained the asymptotic behavior of eigenvalues of the operator $\int_{-1}^{1} \varrho(y)|x-y|^{\alpha-1} \cdot dy$ using a probabilistic method and Karamata Tauberian Theorem.

3. Multidimensional case Suppose $k_0\colon (0,\infty) \to R$ is a rapidly enough decreasing function (for example one having a compact support) and let $k(x) = k_0(\|x\|^m)$ $(\|x\| = (\sum_{i=1}^{m} x_i^2)^{1/2}$ $x = (x_1, x_2, \dots x_m))$.

It is known [16] that

$$\int_{R^m} e^{ix\cdot y} k_0(\|y\|^m)\,dy = \frac{(2\pi)^{\frac{m}{2}}}{\|x\|^{(m-2)/2}} \int_0^{\infty} k_0(\varrho^m)\varrho^{m/2} J_{m/2-1}(\varrho\|x\|)d\varrho \tag{20}$$

where J_ν is Bessel function.

We introduce the functions $k_1, k_2, \dots k_{m-1}$ in the following way:

$$\begin{aligned} k_1(t_1,t_2,\dots t_{m-1}) &= \sum_{n_m\in\mathbf{Z}} [k(t_1,\dots t_{m-1}, x_m - y_m + 4n_m) \\ &\qquad - k(t_1,\dots t_{m-1}, x_m + y_m + 4n_m + 2)] \\ k_2(t_1,t_2,\dots t_{m-2}) &= \sum_{n_{m-1}\in\mathbf{Z}} [k_1(t_1,\dots t_{m-2}, x_{m-1} - y_{m-1} + 4n_{m-1}) \\ &\qquad - k_1(t_1,\dots t_{m-2}, x_{m-1} + y_{m-1} + 4n_{m-1} + 2)] \\ &\vdots \\ k_{m-1}(t_1) &= \sum_{n_2\in\mathbf{Z}} [k_{m-2}(t_1, x_2 - y_2 + 4n_2) - k_{m-2}(t_1, x_2 + y_2 + 4n_2 + 2)]. \end{aligned}$$

Define the function H by

$$H(x,y) = \sum_{n\in\mathbf{Z}} [k_{m-1}(x - y + 4n) - k_{m-1}(x + y + 4n + 2)] \tag{21}$$

Suppose that the function k_0 is chosen such that all the series defining functions $k_1, k_2, \dots k_{m-1}$, H are convergent. By a direct computation we obtain

$$\int_{I^m} H(x,y)\varphi_{n_1 n_2\dots n_m}(y)\,dy = K\left(\frac{n_1\pi}{2}, \frac{n_2\pi}{2}, \dots \frac{n_m\pi}{2}\right)\varphi_{n_1 n_2\dots n_m}(x) \tag{22}$$

where $I = [-1,1]$,

$$\begin{aligned} \varphi_{n_1 n_2\dots n_m}(x) &= \prod_{i=1}^{m} \sin\frac{\pi n_i(1+x_i)}{2} \\ K(t_1,t_2,\dots t_m) &= \int_{R^m} e^{it\cdot\xi} k(\xi)d\xi, \quad t = (t_1,t_2,\dots t_m). \end{aligned}$$

Let

$$K_0(\lambda) = \frac{(2\pi)^{m/2}}{\lambda^{(m-2)/2}} \int_0^{+\infty} k_0(\varrho^m)\varrho^{m/2} J_{m/2-1}(\varrho\lambda)d\varrho \qquad (\lambda > 0).$$

Then from (20) it follows that $K(t_1, t_2, \dots t_m) = K_0(\|t\|)$.

THEOREM 3. *Let the function* k_0 *satisfy following conditions*

1° k_0 *is measurable and bounded on* $(\varepsilon, +\infty)$ *for each* $\varepsilon > 0$ *and all the series defining the function* $k_1, \dots k_{m-1}$, H *are convergent.*

2° *The function* $|K_0(\xi)|$ *is monotone if* ξ *is large enough and* $|K_0(\xi)| \sim \xi^{-r}L(\xi)$ $(\xi \to +\infty)$, $0 < r < m/2$, *where* L *is a slowly varying function from Lemma 1.*

3° *All the integrals*

$$\int_{[0,2]^m} \int_{[0,2]^m} \left| k_0\left(\Big(\sum_{i=1}^{m} (x_i \pm y_i)^2 \Big)^{m/2} \right) \right|^2 dx\, dy$$

are finite for all the combinations of $+$ *and* $-$, *except for the one with all signs* $-$.

Then for the operator $A\colon L^2(I^m) \to L^2(I^m)$ *defined by*

$$Af(x) = \int_{I^m} k(x-y) f(y)\, dy \quad \textit{we have}$$

$$s_n(A) \sim \left(\frac{2}{\pi C_0} \right)^r \frac{L(n^{1/m})}{n^{r/m}} \quad \textit{where} \quad C_0 = \frac{2}{\sqrt{\pi}} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{1/m}.$$

Proof. From (21) and assumptions 1° and 3° of Theorem 3 it follows that

$$\int_{I^m} \int_{I^m} |H(x,y) - k(x-y)|^2\, dx\, dy < +\infty$$

and therefore the operator $Bf(x) = \int_{I^m} (H(x,y) = k(x-y)) f(y)\, dy$ is a Hilbert-Schmidt one. So,

$$\lim_{n \to \infty} n^{1/2} s_n(B) = 0 \tag{23}$$

Let $D\colon L^2(I^m) \to L^2(I^m)$ be a linear operator defined by

$$Df(x) = \int_{I^m} H(x,y) f(y)\, dy.$$

From (22) it follows that

$$s_{n_1, n_2, \dots n_m}(D) = \left| K_0\left(\frac{\pi}{2} \sqrt{n_1^2 + n_2^2 + \cdots + n_m^2} \right) \right|.$$

(by $s_{n_1 \dots n_m}(D)$ we denote singular values of D). Clearly $A = D = B$. Let $m_0(\xi) = |K_0(\xi)|$. From condition 1° it follows that m_0 is a monotone function for $\xi \geq \xi_0$ and $m_0(\xi) \sim \xi^{-r} L(\xi)$ $\xi \to \infty$, $0 < r < m/2$.

Then $s_{n_1\ldots n_m}(D) = m_0\left(\frac{\pi}{2}\sqrt{n_1^2+\cdots n_m^2}\right)$. For large enough $n_1, n_2, \ldots, n_m$ we have

$$n_1^2 + n_2^2 + \cdots + n_m^2 = 4\pi^{-2}(m_0^{-1}(s_{n_1,\ldots,n_m}(D)))^2,$$

where m_0^{-1} is the inverse function of m_0 $(\xi \geq \xi_0)$.

Let the sequence $\{s_{n_1 n_2 \ldots n_m}\}$ be arranged in a nonincreasing order $s_1 \geq s_2 \geq \ldots$. The sequence $\{m_0^{-1}(s_n)\}$ is nondecreasing for n large enough and hence $m_0^{-1}(s_n) = m_0^{-1}(s_{n_1\ldots n_m}(D))$ for n and $n_1 \ldots n_m$ large enough.

Let N be a positive integer such that

$$n_1^2 + n_2^2 + \cdots + n_m^2 = 4\pi^{-2}(m_0^{-1}(s_{n_1\ldots n_m}(D)))^2 = 4\pi^{-2}(m_0^{-1}(s_n))^2 = N^2$$

Denote by ν_1 and ν_2 the smallest and largest values of such that

$$n_1^2 + n_2^2 + \cdots + n_m^2 = 4\pi^{-2}(m_0^{-1}(s_n))^2 = N^2$$

It is known [**10**] that

$$\begin{aligned} \nu_1 &= \frac{\pi^{m/2}}{m2^{m-1}\Gamma(m/2)} N^m + o(N^m) \\ \nu_2 &= \frac{\pi^{m/2}}{m2^{m-1}\Gamma(m/2)} N^m + o(N^m) \end{aligned} \tag{24}$$

From $N^2 = 4\pi^{-2}(m_0^{-1}(s_n))^2$ we get

$$s_n = m_0(\pi N/2). \tag{25}$$

Since $\nu_1 \leq n \leq \nu_2$, from (24) it follows that

$$n = \frac{\pi^{m/2}}{m2^{m-1}\Gamma(m/2)} N^m + o(N^m), \quad \text{i.e.}$$

$$N = C_0 n^{1/m}(1+o(1)), \quad \text{where} \quad C_0 = \frac{2}{\sqrt{\pi}}\left(\Gamma\left(1+\frac{m}{2}\right)\right)^{1/m}. \tag{26}$$

From (25), (26) and condition 2° (Theorem 3) it follows that

$$s_n(D) \sim \left(\frac{2}{\pi C_0}\right)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

Combining this with (23) and Ky Fan Theorem, we get

$$s_n(A) \sim \left(\frac{2}{\pi C_0}\right)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

Theorem 3 is proved.

Remark 5. Let $\Delta = [-a,a]^m$. If the function $t \to k(at)$ satisfies conditions of Theorem 3, then

$$s_n\left(\int_\Delta k(x-y)\cdot dy\right) \sim \left(\frac{|\Delta|^{1/m}}{\pi C_0}\right)^r \cdot \frac{L(n^{1/m})}{n^{r/m}},$$

where $|\Delta|$ denote the measure of the cube Δ.

LEMMA 4. *Let* $k_0(t) = t^{\alpha-1}\left(1+\frac{1}{m}|\ln t|\right)^\gamma$, $\gamma \in R$, $1/2 - 1/2m < \alpha < 1/2$. *Then the function* $|K_0(\cdot)|$ *has the following asymptotic behavior*

$$|K_0(\xi)| \sim \frac{(\ln\xi)^\gamma}{\xi^{\alpha m}}\pi^{m/2}2^{\alpha m}\frac{\Gamma(\alpha m/2)}{\Gamma(m(1-\alpha)/2)}, \quad \xi \to +\infty,$$

where

$$K_0(\xi) = \frac{(2\pi)^{m/2}}{\xi^{(m-2)/2}}\int_0^{+\infty} k_0(\varrho^m)\varrho^{m/2}J_{(m/2)-1}(\varrho\xi)d\varrho.$$

Proof.

$$\begin{aligned}
K_0(\xi) &= (2\pi)^{m/2}\xi^{-\alpha m}(\ln\xi)^\gamma\int_0^\infty\left(\frac{1}{\ln\xi}+\left|\frac{\ln u}{\ln\xi}=1\right|\right)^\gamma u^{\alpha m - m/2}J_{(m/2)-1}(u)du\\
&\sim (2\pi)^{m/2}\xi^{-\alpha m}(\ln\xi)^\gamma\cdot\int_0^\infty u^{\alpha m-m/2}J_{(m/2)-1}(u)du\\
&= (2\pi)^{m/2}\xi^{-\alpha m}(\ln\xi)^\gamma\cdot 2^{\alpha m - m/2}\cdot\Gamma\left(\frac{\alpha m}{2}\right)\Big/\Gamma\left(\frac{m(1-\alpha)}{2}\right).
\end{aligned}$$

$\Big($by Veber formula [**16**], $\int_0^\infty \varrho^\beta J_\nu(\varrho)d\varrho = 2^\beta\Gamma\left(\frac{\nu+\beta+1}{2}\right)\Big/\Gamma\left(\frac{\nu+1=\beta}{2}\right)\Big)$.

If $\frac{1}{2} = \frac{1}{2m} < \alpha < \frac{1}{2}$ then the functions k_0 and K_0 satisfy conditions of Theorem 3 (which can be easily verified). Then we have $r = \alpha m$ and

$$L(\xi) = \pi^{m/2}2^{\alpha m}\frac{\Gamma(\alpha m/2)}{(m(1-\alpha)/2)}(\ln\xi)^\gamma.$$

COROLLARY. *If* $\frac{1}{2} = \frac{1}{2m} < \alpha < \frac{1}{2}$ *then the following holds* (Δ *is cube in* R^m)

$$(27)\quad s_n\left(\int_\Delta \frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}}\cdot dy\right) \sim \left(\frac{|\Delta|^{1/m}}{\pi C_0}\right)^{\alpha m}\frac{\pi^{m/2}\cdot 2^{\alpha m}}{m^\gamma}\frac{\Gamma(\alpha m/2)}{\Gamma(m(1-\alpha)/2)}\cdot\frac{(\ln n)^\gamma}{n^\alpha}.$$

Proof. Directly follows from Theorem 33 and Lemma 4.

Theorem 4. *Let $\Omega \subset R^m$ be a bounded Jordan measurable set. If $T \in L^\infty(\Omega \times \Omega)$ is continuous in the neighborhood of the diagonal $y = x$, $T(x,x) > 0$ on Ω, $\gamma \in R$ and $1/2 - 1/2m < \alpha < 1/2$ then*

$$s_n\left(\int_\Omega T(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}}\cdot dy\right) \sim \left(\frac{2}{\pi C_0}\right)^{\alpha m}\frac{\pi^{m/2}}{m^\gamma}\frac{\Gamma(\alpha m/2)}{\Gamma(m(1-\alpha)/2)}\left(\int_\Omega (T(x,x))^{1/\alpha}\,dx\right)^\alpha \frac{(\ln n)^\gamma}{n^\alpha}$$

Proof. Consider first the case $\Omega = [a,b]^m$. Let

$$A = \int_\Omega T(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}}\cdot dy$$

Divide the cube Ω into N cubes Δ_i and denote by x_i the center of Δ_i.

As in the proof of Theorem 2 we introduce the operators

$$\begin{aligned} &A_i^N : L^2(\Omega) \to L^2(\Omega) \qquad i = 1,2,\dots N \\ &A_{ij} : L^2(\Omega) \to L^2(\Omega) \qquad i \neq j,\ \ i,j = 1,2\dots N \end{aligned}$$

defined by

$$\begin{aligned} A_i^N f(x) &= \int_\Omega \frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}}\chi_{\Delta_i}(x)\chi_{\Delta_i}(y)T(x_i,x_i)f(y)\,dy \\ A_{ij} f(x) &= \int_\Omega \frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}}\chi_{\Delta_i}(x)\chi_{\Delta_j}(y)T(x,y)f(y)\,dy. \end{aligned}$$

Let $A_N = \sum_{i=1}^N A_i^N$. Then $A = A_N + \sum_{i\neq j}^N A_{ij} + B_N$, where B_N is the operator defined by

$$\begin{aligned} B_N f(x) &= \int_\Omega \frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}} G_N(x,y)f(y)\,dy, \\ G_N(x,y) &= \sum_{i=1}^N \chi_{\Delta_i}(x)\chi_{\Delta_i}(y)(T(x,y) - T(x_i,x_i)). \end{aligned}$$

From continuity the function $T(\cdot,\cdot)$ in the neighborhood of the diagonal $y = x$ follows that for an arbitrary $\varepsilon > 0$ and for N large enough we have

$$|\,T(x,y) - T(x_i,x_i)\,| < \varepsilon \qquad \text{for } (x,y) \in \Delta_i \times \Delta_i \ \ (i = 1,2,\dots N)$$

Then for $(x,y) \in \Omega \times \Omega$ we have $|\ G_N(x,y)\ | < \varepsilon$. The previous inequality and Lemma 1 [**3**] give

$$s_n(B_N) \le C\cdot\varepsilon(\ln n)^\gamma/n^\alpha \tag{28}$$

where the constant C does not depend on n and ε.

It can be easily verified that if $\frac{1}{2}-\frac{1}{2m}<\alpha<\frac{1}{2}$ and $i\neq j$ then

$$\int_{\Delta_i} dx \int_{\Delta_j} \frac{(1+|\ln\|x-y\||)^{2\gamma}}{\|x-y\|^{2m(1-\alpha)}}\,dy<+\infty \tag{29}$$

holds. From (29) it follows $A_{ij}\in C_2$ (for $i\neq j$), $\lim_{n\to\infty} n^{1/2}s_n(A_{ij})=0$ and $\lim_{n\to\infty} n^{1/2}s_n\left(\sum_{i\neq j}^N A_{ij}\right)=0$. Combining this with (28) and using the properties of singular values, we conclude that for each $\varepsilon>0$ there exists a natural number N such that

$$\overline{\lim_{n\to\infty}}\frac{n^\alpha}{(\ln n)^\gamma}s_n\left(\sum_{i\neq j}^N A_{ij}+B_N\right)<\varepsilon \tag{30}$$

Since $\frac{1}{2}-\frac{1}{2m}<\alpha<\frac{1}{2}$, from (27) it follows that

$$s_n(A_i^N)\sim T(x_i,x_i)\left(\frac{|\Delta_i|^{1/m}}{\pi C_0}\right)^{\alpha m}\frac{\pi^{m/2}2^{\alpha m}}{m^\gamma}\frac{\Gamma(\alpha m/2)}{\Gamma(m(1-\alpha)/2)}\cdot\frac{(\ln n)^\gamma}{n^\alpha}.$$

Applying Lemma 1 we get

$$\mathcal{N}_t(A_i^N)\sim\alpha^{-\gamma/\alpha}(-\ln t)^{\gamma/\alpha}t^{-1/\alpha}d_0^{1/\alpha}(T(x_i,x_i))^{1/\alpha}|\Delta_i|,\qquad (t\to 0+), \tag{31}$$

where $d_0=\left(\frac{2}{\pi C_0}\right)^{\alpha m}\frac{\pi^{m/2}}{m^\gamma}\frac{\Gamma(\alpha m/2)}{\Gamma(m(1-\alpha)/2)}$.

Since the operator A_N is the orthogonal sum of the operators A_i^N ($i=1,2,\dots N$), we have

$$\mathcal{N}_t(A_N)=\sum_{i=1}^N\mathcal{N}_t(A_i^N)$$

and from (31) it follows that

$$\lim_{t\to 0+}t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A_N)=\alpha^{-\gamma/\alpha}d_0^{1/\alpha}\sum_{i=1}^N(T(x_i,x_i))^{1/\alpha}|\Delta_i|.$$

Using $A=A_N+B_N+\sum_{i\neq j}^N A_{ij}$ and (30), by Lemma 2 we get

$$\lim_{t\to 0+}t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A)=\alpha^{-\gamma/\alpha}d_0^{1/\alpha}\int_\Omega(T(x,x))^{1/\alpha}\,dx. \tag{32}$$

Putting $t=s_n(A)$ in (32) by Lemma 1 we get

$$s_n(A)\sim d_0\left(\int_\Omega(T(x,x))^{1/\alpha}\,dx\right)^\alpha\cdot\frac{(\ln n)^\gamma}{n^\alpha}. \tag{33}$$

the theorem is proved in the case when Ω is a cube.

Remark 6. The equalities (32) and (33) can be obtained in the same way when Ω is the union of some cubes with disjoint interiors.

Suppose $\Omega_1,\ \Omega_2 \subset R^m$ are bounded measurable sets, $\Omega_1 \subset \Omega_2$. Let $A_i\colon L^2(\Omega_i) \to L^2(\Omega_i)$ $i = 1,2$, be the linear operators defined by

$$A_i f(x) = \int_{\Omega_i} T(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}} f(y)\,dy \qquad i = 1,2.$$

LEMMA 5. *The singular value distribution functions of the operators A_i $(i = 1,2)$ satisfy the inequality $\mathcal{N}(A_1) \le \mathcal{N}_t(A_2)$ $(t > 0)$.*

Proof. Let $P\colon L^2(\Omega_2) \to L^2(\Omega_1)$ be orthoprojector $(Pf(x) = \chi_{\Omega_1}(x)f(x))$. Since $A_1 = PA_2P$ we have $s_n(A_1) \le s_n(A_2)$ and hence $\mathcal{N}_t(A_1) \le \mathcal{N}_t(A_2)$.

We continue with the proof of Theorem 4 in the general case. Let Ω be a bounded Jordan measurable set. Let $\underline{\Omega}_N \subset \Omega \subset \overline{\Omega}_N$ where the sets $\underline{\Omega}_N$ and $\overline{\Omega}_N$ are the unions of equal cubes (with disjoint interiors) such that $m(\underline{\Omega}_N) \to m(\Omega)$ $m(\overline{\Omega}_N) \to m(\Omega)$; $N \to +\infty$ (m is Lebesgue measure). Denote by $\widetilde{T}$ the function obtained continuously extending T in some neighborhood of the diagonal $y = x$, so that $\widetilde{T}$ is zero in the other points outside $\Omega \times \Omega$. Let $\underline{A}_N$ and $\overline{A}_N$ be the linear operators acting on $L^2(\underline{\Omega}_N)$ and $L^2(\overline{\Omega}_N)$, defined by

$$\underline{A}_N f(x) = \int_{\underline{\Omega}_N} T(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}} f(y)\,dy,$$
$$\overline{A}_N f(x) = \int_{\overline{\Omega}_N} \widetilde{T}(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{m(1-\alpha)}} f(y)\,dy,$$

respectively. From Lemma 5 we get $\mathcal{N}_t(\underline{A}_N) \le \mathcal{N}_t(A) \le \mathcal{N}_t(\overline{A}_N)$ and therefore

$$t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\underline{A}_N) \le t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A) \le t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\overline{A}_N).$$

Next we have

$$\begin{aligned}\varliminf_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\underline{A}_N) &\le \varliminf_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A)\\ \le \varlimsup_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A) &\le \varlimsup_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\overline{A}_N)\end{aligned} \tag{34}$$

Since the limits

$$\lim_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\underline{A}_N) \qquad \text{and} \qquad \lim_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(\overline{A}_N)$$

exists, (Remark 6) and since they are equal to

$$\alpha^{-\gamma/\alpha}d_0^{1/\alpha}\int_{\underline{\Omega}_N}(T(x,x))^{1/\alpha}\,dx \qquad \text{and} \qquad \alpha^{-\gamma/\alpha}d_0^{1/\alpha}\int_{\overline{\Omega}_N}(\widetilde{T}(x,x))^{1/\alpha}\,dx,$$

respectively, from (34) we get

$$\alpha^{-\gamma/\alpha} d_0^{1/\alpha} \int_{\Omega_N} (T(x,x))^{1/\alpha}\, dx \le \varliminf_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A)$$
$$\le \varlimsup_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A) \le \alpha^{-\gamma/\alpha} d_0^{1/\alpha} \int_{\overline{\Omega}_N} (\widetilde{T}(x,x))^{1/\alpha}\, dx$$

Letting here $N \to +\infty$ we obtain

$$\lim_{t\to 0+} t^{1/\alpha}(-\ln t)^{-\gamma/\alpha}\mathcal{N}_t(A) = \alpha^{-\gamma/\alpha} d_0^{1/\alpha} \int_{\Omega} (T(x,x))^{1/\alpha}\, dx.$$

Putting here $t = s_n(A)$, by Lemma 1 we get

$$s_n(A) \sim d_0 \left(\int_{\Omega} (T(x,x))^{1/\alpha}\, dx \right)^{\alpha} \cdot \frac{(\ln n)^{\gamma}}{n^{\alpha}}. \tag{35}$$

Theorem 4 is proved.

Remark 7. Putting $m = 1$ in (35) and applying Legendre duplication formula we obtain (18).

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

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