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PARAQUATERNIONIC PROJECTIVE SPACE AND PSEUDO-RIEMANNIAN GEOMETRY

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Abstract. A natural and geometrical definition of projective space $(P_n(\mathbb{B}), g_0)$, based on the algebra of paraquaternionic numbers $\mathbb{B}$, is given. Using the technique of pseudo-Riemannian submersions, we determine the curvature of the paraquaternionic space $(P_n(\mathbb{B}), g_0)$. Moreover, the properties of this curvatures are studied.

0. Introduction. Studying differential geometry of spaces related to the algebra of paracomplex numbers was initiated by Rasevskii [**R**] and Libermann [**L**] who defined paracomplex and para-Kählerian manifolds. This topic was discussed later by several authors (see [**B,CFG**] for details).

Also, the geometry based on the paraquaternionic numbers $\mathbb{B}$ is interesting. Etayo [**E**] studied $P_{n,n}(\mathbb{C})$, the space of paraquaternionic projective type, as a para-Hermitian symmetric space related to Kaneyuki-Kozai classification. He studied the principal bundle $\pi : GL(n+1,\mathbb{C})/GL(1,\mathbb{C}) \longrightarrow P_{n,n}$ whose structure group is $GL(1,\mathbb{C}) = \mathbb{C}^*$. An integrable paraquaternionic structure appeared naturally in the study of Osserman pseudo-Riemannian manifolds (see [**BBR, Rk**]).

Here we consider a pseudo-Riemannian submersion $\pi : S^{4n+3}_{2n+1} \longrightarrow P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1$ with totally geodesic fibres S^3_1. The submersion π is used to give a natural and geometrically oriented definition of paraquaternionic projective space $P_n(\mathbb{B})$. It is simply connected and complete with respect to the metric induced by submersion. Using the well known result of O'Neill we determine the sectional curvature of a paraquaternionic projective space $\mathbb{B}$. It has constant paraquaternionic sectional curvature and unbounded sectional curvature. Some other properties of the paraquaternionic projective spaces are also studied.

1. Paraquaternionic numbers and projective spaces. Let $\mathbb{B}$ denote the algebra of paraquaternionic numbers generated by $\{1, i, e, f = ie\}$ over $\mathbb{R}$ where

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$e^2 = 1$, $i^2 = -1$, $ie = -ei$. This is an associative, noncomutative and unitary algebra over $\mathbb{R}$ of rank 3. To an arbitrary paraquaternionic number $q \in \mathbb{B}$, $q = x_1 + x_2 i + x_3 e + x_4 f$, $x_i \in \mathbb{R}$, $1 \leq i \leq 4$, we can associate its conjugate $\bar{q} = x_1 - x_2 i - x_3 e - x_4 f$, real part $\operatorname{Re} q = x_1$, and the norm

$$\|q\|^2 = q\bar{q} = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

Then we have $\|qp\|^2 = \|q\|^2 \|p\|^2$ and clearly q is invertible if and only if $\|q\| \neq 0$.

Let $\tilde{\mathbb{B}} = \{q \in \mathbb{B} \mid \|q\| \neq 0\}$ be the multiplicative group of invertible paraquaternionic numbers. Now we can define the corresponding projective space. The set

$$\mathbb{B}^{n+1} = \{(q_0, \dots, q_n) \mid q_i \in \mathbb{B}, 0 \leq i \leq n\}$$

is a paraquaternionic unitary module. On $\mathbb{B}^{n+1}$ we have the endomorphisms I,E,F defined by $I(q) = iq$, $E(q) = eq$, $F(q) = fq$ respectively. Then $E^2 = F^2 = -I^2 = \mathrm{id}$ and $IE = -EI = F$. A vector $q = (q_0, \dots, q_n) \in \mathbb{B}^{n+1}$ is a singular vector if and only if there exists a $\lambda \in \mathbb{B}, \lambda \neq 0$, such that $\lambda q = 0$. Clearly, then $\|q_0\|^2 = \cdots = \|q_n\|^2 = 0$ and $q_0, \dots, q_n$ are mutually proportional with same real factors. Let $\tilde{\mathbb{B}}^n$ denote the nonsingular vectors in $\mathbb{B}^n$. On the set $\mathbb{B}^{n+1} \setminus \{0\}$ the equivalence relation $\sim$ is defined as usual $u \sim v$ if and only if $v = zu$ for some $z \in \tilde{\mathbb{B}}$. The corresponding quotient space:

$$P(\mathbb{B}^{n+1}) = (\mathbb{B}^{n+1} \setminus \{0\})/\tilde{\mathbb{B}}$$

is called the algebraic paraquaternionic projective space associated with $\mathbb{B}^{n+1}$.

This definition is appropriate from an algebraic point of view. But difficulties are coming from the equivalence classes of singular elements of $\mathbb{B}^{n+1}$. From a geometrical point of view it is better to consider only equivalence classes of nonsingular paraquaternionic lines and we define the paraquaternionic projective space as

$$P(\tilde{\mathbb{B}}^{n+1}) = P_n(\mathbb{B}) = \tilde{\mathbb{B}}^{n+1}/\tilde{\mathbb{B}}.$$

For the algebra $\mathbb{A}$ of paracomplex numbers, $\mathbb{A} = \mathbb{R}\langle 1, i\rangle$, the situation is similar. A broad survey of geometries based on paracomplex numbers is given in [**CFG**]. We are using some notations and notions from there, modified to the paraquaternionic numbers. The algebraic paracomplex projective space defined in [**CFG**] coincides with Rosenfeld's paracomplex projective spaces P_n^0 (see [**Ro**]). The geometrical definition of a paracomplex projective space $P_n(\mathbb{A})$ was given by Libermann [**Lb**]. In some sense similar to paracomplex projective spaces are the paracomplex projective models $P_n(B)$ in [**GM**] (open subsets of $\tilde{\mathbb{A}}^{n+1}/\tilde{\mathbb{A}}_{++}$, the four-fold covering space of the Liberman projective space $P_n(\mathbb{A})$, where $\tilde{\mathbb{A}}_{++}$ is the identity component of the group $\tilde{\mathbb{A}}$).

Etayo Gordejuela [**E**] defined and studied the spaces $P_{n,n}(\mathbb{C})$, similar to paracomplex projective models, and called them paraquaternionic projective spaces.

2. Pseudo-Riemannian geometry and submersions. In $\mathbb{R}^{4n+4} = \mathbb{B}^{n+1}$ we have the following natural scalar product $\langle\cdot,\cdot\rangle$ defined as

$$\langle u, v\rangle = \operatorname{Re}\sum_{i=0}^{n} u_i\bar{v}_i$$

where $u = (u_0, \dots , u_n)$, $v = (v_0, \dots , v_n)$, $u_i, v_i \in \mathbb{B}$. Clearly, the scalar product, of the signature $(2n + 2, 2n + 2)$, is anti-invariant with respect to endomorphisms E and F, and invariant with respect to I, i.e.,

$$\langle Eu, Ev\rangle = -\langle u, v\rangle, \qquad \langle Fu, Fv\rangle = -\langle u, v\rangle, \qquad \text{and} \quad \langle Iu, Iv\rangle = \langle u, v\rangle.$$

The pseudosphere and the pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}^{4n+4}$, are the hyperquadrics

$$S^{4n+3}_{2n+1}(r) = \Big\{u \in \mathbb{B}^{n+1} \mid \langle u, u\rangle = \sum \|u_i\|^2 = r^2\Big\},$$
$$H^{4n+3}_{2n+1}(r) = \{u \in \mathbb{B}^{n+1} \mid \langle u, u\rangle = -r^2\}$$

respectively. On the tangent spaces $T_pS^{4n+3}_{2n+1}(r)$ and $T_pH^{4n+3}_{2n+2}(r)$ the induced metrics are of the signature $(2n + 1, 2n + 2)$ and $(2n + 2, 2n + 1)$. The pseudosphere and the pseudohyperbolic space are anti-isometric [**O**]. For $r = 1$ we set $S^{4n+3}_{2n+1}(r) = S^{4n+3}_{2n+1}$ and $H^{4n+3}_{2n+2}(r) = H^{4n+3}_{2n+2}$. The pseudosphere S^{4n+3}_{2n+1} is diffeomorphic to $\mathbb{R}^{2n+2} \times S^{2n+1}$ and the pseudohyperbolic space H^{4n+3}_{2n+2} is also diffeomorphic to $\mathbb{R}^{2n+2} \times S^{2n+1}$ (see [**O**, p. 110]).

In $\mathbb{R}^4 = \mathbb{B}$, we have $S^3_1 = \{\beta \in \mathbb{B} \mid \langle\beta, \beta\rangle = \|\beta\|^2 = 1\}$. The pseudosphere S^3_1 is the group of the unit paraquaternions. Then there is an action of S^3_1 on S^{4n+3}_{2n+1} given by

$$T_\beta(u_0, \dots , u_n) = (\beta u_0, \dots , \beta u_n),$$

which are obviously isometries with respect to $\langle\cdot,\cdot\rangle$. Then, the paraquaternionic projective space can be represented as

$$P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1 = H^{4n+3}_{2n+2}/S^3_1.$$

This means that the projection $\pi : S^{4n+3}_{2n+1} \longrightarrow P_n(\mathbb{B})$ is a submersion with fiber S^3_1. Moreover, $P_n(\mathbb{B})$ is homotopically equivalent to the complex projective space $P_n(\mathbb{C})$. Therefore, $P_n(\mathbb{B})$ is simply connected.

The action of these isometries on S^{4n+3}_{2n+1} provides for $P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1$ a unique pseudo-Riemannian metric of signature $(2n, 2n)$. More precisely, similarly as in the Riemannian case, the following lemma can be proved.

Lemma 2.1. *Let M be a pseudo-Riemannian manifold of signature (p, q) and G closed group of isometries such that $\pi : M \longrightarrow B = M/G$ is a submersion. If G admits a metric of signature (p_1, q_1), $p_1 \le p$, $q_1 \le q$, then there is a unique* *pseudo-Riemannian metric on $B = M/G$ of signature $(p - p_1, q - q_1)$ such that π is a pseudo-Riemannian submersion. If M and G are complete, then the metric induced by the submersion, is also complete on M/G.* $\square$

We fix a metric g_0 of signature $(2n, 2n)$ on $P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1$ such that $\pi : S^{4n+3}_{2n+1} \longrightarrow P_n(\mathbb{B})$ is a pseudo-Riemannian submersion. The endomorphisms I, E and F induce corresponding endomorphisms on the tangent space $T_pP_n(\mathbb{B})$. Now we want to determine sectional curvature and the curvature tensor of the paraquaternionic projective space equiped with this metric g_0.

Let us recall some basic definitions concerning the Levi-Civita connection of pseudo-Riemannian manifold (P, h) of signature (p, q). We will also establish important notions for paraquaternionic manifolds.

For a vector $X \in T_pP$, $\|X\|^2 = h(X, X)$, X is isotropic if $\|X\|^2 = 0$ and $\epsilon(X) = \operatorname{sign}(h(X, X))$. Let TP be the tangent bundle of P and X, Y, Z, etc., arbitrary vector fields. If ∇ is the corresponding Levi-Civita connection, $R(X, Y) : T_pP \longrightarrow T_pP$ is a curvature operator defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

If Π is a nondegenerate 2-dimensional plane, $\Pi = \langle X, Y\rangle$, the sectional curvature of Π, $K(\Pi)$, is defined as

$$K(\Pi) = K(X, Y) = \frac{h(R(X, Y)Y, X)}{Q(X, Y)},$$

where $Q(X, Y) = h(X, X)h(Y, Y) - h^2(X, Y)$. Π is nondegenerate if and only if $Q(X, Y) \neq 0$. The sectional curvature does not depend on the choice of the base $\{X, Y\}$ for Π. For an orthonormal base, $K(\Pi) = h(R(X, Y)Y, X)/\epsilon(X)\epsilon(Y)$.

P is a paraquaternionic manifold if we have locally defined endomorphisms I, E and F of the tangent bundle such that $E^2 = F^2 = -I^2 = \text{id}$ and $IE = -EI = F$. (P, h) is a paraquaternionic hermitian manifold if (P, h) is a paraquaternionic manifold and I is an isometry and endomorphisms E, F are antiisometries, i.e.,

$$h(IX, IY) = h(X, Y), \quad h(EX, EY) = -h(X, Y), \quad h(FX, FY) = -h(X, Y),$$

for tangent vectors X and Y. Then we say that a paraquaternionic hermitian manifold P is of constant paraquaternionic sectional curvature if $K(\Pi) = const$ for all nondegenerate 2 planes contained in the vector subspace generated by vectors X, IX, EX, FX for arbitrary nonisotropic vector X. For notational convenience, from now on, we will use the following notation: $J_1 = I$, $J_2 = E$ and $J_3 = F$.

3. Curvature of the paraquaternionic projective space. We will determine now the sectional curvature operator of $P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1$ with respect to the induced metric g_0 (by submersion).

Let $\pi : P \longrightarrow B$ be a pseudo-Riemannian submersion. At a point $p \in \pi^{-1}(b)$, $\mathcal{H}$ and $\mathcal{V}$ denote the orthogonal projections of T_pM on its horizontal and vertical subspaces

$$\mathcal{H}_p = T_p(\pi^{-1}b)^{\perp} \quad \text{and} \quad \mathcal{V}_p = T_p(\pi^{-1}b)$$

respectively. The following well known theorem of O'Neill is important for us (see [**O**]):

THEOREM 3.1. *Let $\pi : P \longrightarrow B$ be a pseudo-Riemannian submersion. If the horizontal vector fields X, Y on P span nondegenerate planes, then for their sectional curvatures the following holds*

$$K_B(d\pi X, d\pi Y) = K_p(X,Y) + \frac{3}{4}\frac{\langle \mathcal{V}[X,Y], \mathcal{V}[X,Y]\rangle}{Q(X,Y)}. \qquad \square \tag{3.1}$$

We can now prove:

THEOREM 3.2. *The sectional curvature of some nondegenerate 2-plane in the paraquaternionic projective space $(P_n(\mathbb{B}) = S^{4n+3}_{2n+1}/S^3_1, g_0)$ is determined by*

$$K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = 1 + 3(\langle X, J_1Y\rangle^2 - \langle X, J_2Y\rangle^2 - \langle X, J_3Y\rangle^2)/Q(X,Y). \tag{3.2}$$

The paraquaternionic projective space is of constant paraquaternionic sectional curvature 4.

Proof. We will apply O'Neill's theorem to the submersion $\pi : S^{4n+3}_{2n+1} \longrightarrow P_n(\mathbb{B})$ with fiber S^3_1. That means

$$K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = K_{S^{4n+3}_{2n+1}}(X,Y) + \frac{3}{4}\frac{\|\mathcal{V}[X,Y]\|^2}{Q(X,Y)} \tag{3.3}$$

for horizontal vector fields X and Y on S^{4n+3}_{2n+1}. To continue computation we need to determine the vertical subspace at the point N of S^{4n+3}_{2n+1}. Using the hyperbolic rotation we find the curve $\alpha(\theta) = \cosh\theta \cdot 1 + \sinh\theta \cdot e$ on S^3_1, and the corresponding orbit curve is $\alpha(\theta)N$. Its tangent vector at N is J_2N. Also, J_3N lies in the vertical subspace of $T_N S^{4n+3}_{2n+1}$. Finally we can see that J_1N is a vertical vector in a similar way, using euclidean rotations. Hence

$$\mathcal{V}[X,Y] = \langle [X,Y], J_1N\rangle J_1N - \langle [X,Y], J_2N\rangle J_2N - \langle [X,Y], J_3N\rangle J_3N. \tag{3.4}$$

Since: ∇ is a torsion-free connection, $[X,Y] = \nabla_X Y - \nabla_Y X$, and J_1, J_2 and J_3 are parallel with repect to ∇, we have

$$\mathcal{V}[X,Y] = 2(\langle X, J_1Y\rangle J_1N - \langle X, J_2Y\rangle J_2N - \langle X, J_3Y\rangle J_3N)$$

and

$$\|\mathcal{V}[X,Y]\|^2 = 4(\langle X, J_1Y\rangle^2 - \langle X, J_2Y\rangle^2 - \langle X, J_3Y\rangle^2). \tag{3.5}$$

Since the pseudosphere S^{4n+3}_{2n+1} is of constant sectional curvature 1, (3.2) follows from (3.3) and (3.5).

For horizontal vectors $X, Y \in T_p S^{4n+3}_{2n+1}$, the vectors J_1Y, J_2Y, J_3Y are also horizontal and

$$\begin{aligned}\mathrm{Pr}_{\langle Y,J_1Y,J_2Y,J_3Y\rangle}X = \epsilon(Y)\langle X,Y\rangle Y + \epsilon(J_1Y)\langle X,J_1Y\rangle J_1Y \\ + \epsilon(J_2Y)\langle X,J_2Y\rangle J_2Y + \epsilon(J_3Y)\langle X,J_3Y\rangle J_3Y\end{aligned}$$

where $\mathrm{Pr}_{\langle Y,J_1Y,J_2Y,J_3Y\rangle}X$ is normal projection of the vector X on the subspace $\langle Y, J_1Y, J_2Y, J_3Y\rangle$. If X and Y are orthonormal vectors which generate the plane Π, then

$$\mathrm{Pr}_{\langle Y,J_1Y,J_2Y,J_3Y\rangle}X = \epsilon(Y)(\langle X,J_1Y\rangle J_1Y - \langle X,J_2Y\rangle J_2Y - \langle X,J_3Y\rangle J_3Y)$$

and (3.2) becomes

$$K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = 1 + 3\|\mathrm{Pr}_{\langle Y,J_1Y,J_2Y,J_3Y\rangle}X\|^2/\epsilon(X). \tag{3.6}$$

Hence, the paraquaternionic sectional curvature of a paraquaternionic projective space is 4. □

COROLLARY 3.3. *The paraquaternionic projective space for $n = 1$, $P_1(\mathbb{B})$, is of constant sectional curvature 4. It is diffeomorphic to S^4_2 and we have a fiber bundle of Hopf type, $\pi : S^7_4 \longrightarrow S^4_2$ with totally geodesic fibre S^3_1.* □

COROLLARY 3.4. *The curvature tensor of $(P_n(\mathbb{B}), g_0)$ is*

$$\begin{aligned}R(X,Y)Z = c\{&g_0(Y,Z)X - g_0(X,Z)Y \\ &+ \sum_r \epsilon_r g_0(J_rY,Z)J_rX - \sum_r \epsilon_r g_0(J_rX,Z)J_rY \\ &+ 2\sum_r \epsilon_r g_0(X,J_rY)J_rZ\},\end{aligned} \tag{3.8}$$

where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$ and c is a constant.

Proof. Using the submersion $\pi : S^{4n+3}_{2n+1} \longrightarrow P_n(\mathbb{B})$, from (3.2) it follows that

$$g_0(R(X,Y)Y,X) = c\{Q(X,Y) + 3\sum_r \epsilon_r g_0(X,J_rY)^2\}.$$

By polarization, this formula implies (3.8). □

Remark 1. For the paraquaternionic projective space corresponding to the pseudosphere of radius r, sectional curvature of some plane Π with respect to pseudoorthogonal base $\{X, Y\}$ is

$$K_{P_n(\mathbb{B})}(d\pi X, d\pi Y) = \frac{1}{r^2}(1 + 3\|\mathrm{Pr}_{\langle Y,J_1Y,J_2Y,J_3Y\rangle}X\|^2/\epsilon(X)).$$

It is well known that the sectional curvature of $P_n(\mathbb{B})$ is unbounded for $n > 1$ (see [**O**, p. 229]).

Remark 2. If we consider the metric $-\langle\cdot,\cdot\rangle$ on $\mathbb{B}^{n+1}$, then the paraquaternionic projective space $P_n(\mathbb{B})$ is of constant negative paraquaternionic sectional curvature -4 with respect to the induced metric $-g_0$. This is equivalent with the fact that the pseudosphere S^{4n+3}_{2n+1} and the pseudohyperbolic space H^{4n+3}_{2n+2} are anti-isometric.

Remark 3. Using the algebra of paracomplex numbers, $\mathbb{A} = \mathbb{R}\langle 1, i\rangle$, instead of the paraquaternionic numbers $\mathbb{B}$ we have the submersion

$$\pi : S^{2n+1}_n \longrightarrow P_n(\mathbb{A}) = S^{2n+1}_n(r)/S^1_1$$

with totally geodesic fibres $S^1_1 = \{u \in \mathbb{A} \mid \langle u, u\rangle = 1\}$. Its sectional curvature is

$$K_{P_n(\mathbb{A})}(d\pi X, d\pi Y) = \pm r^{-2}(1 + 3\|\mathrm{Pr}_{\langle Y, J_2 Y\rangle} X\|^2/\epsilon(X)).$$

In [**GM**] and [**E**], the projective model $P_n(B)$, diffeomorphic to TS^n, was studied using the appropriate submersion.

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