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A NOTE ON STABILITY OF MINIMAL SURFACES IN n-DIMENSIONAL HYPERBOLIC SPACE $H^n(c)$

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Abstract. We improve a result of Barbosa-do Carmo about stability of minimal surfaces in n-dimensional hyperbolic space $H^n(c)$.

1. Introduction. In [**1**] Barbosa and do Carmo obtain the following well-known result

THEOREM 1 [**1**]. *Let M be a minimal surface in an n-dimensional hyperbolic space $H^n(c)$. Assume that D is a simply connected compact domain with piecewise smooth boundary on M. Let A denote the second fundamental form of M. If*

$$\int_D (|c| + \frac{|A|^2}{2}) dv < \frac{4\pi}{3}, \tag{1}$$

then D is stable.

In this note, we improve the Theorem above as follows

THEOREM 2. *Let M be a minimal surface in an n-dimensional hyperbolic space $H^n(c)$. Assume that D is a simply connected compact domain with piecewise smooth boundary on M. Let A denote the second fundamental form of M. If*

$$\int_D (\frac{|c|}{5} + \frac{|A|^2}{2}) dv < \frac{4\pi}{3}, \tag{2}$$

then D is stable.

Remark. Obviously, our condition (2) is better than condition (1) of Barbosa-do Carmo's.

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2. Preliminaries. Let $H^n(c)$ be an n-dimensional simply connected space of constant negative curvature c; we also call it the hyperbolic space. Let M be a minimal surface in $H^n(c)$; we denote by K the Gauss curvature of M with respect to the induced metric ds^2_M. Let A be the second fundamental form of M.

We need the following lemmas to prove Theorem 2.

smallskip LEMMA 1. *If M be a minimal surface in $H^n(c)$, then*

$$|\nabla(|A|^2)|^2 \le 2|A|^2|\nabla A|^2. \tag{3}$$

Proof. Let M be a minimal surface $H^n(c)$. By an elementary observation one can see that at each point the dimension of the image of the second fundamental form A of M is at most 2. Thus we may choose $e_3, \dots, e_n$ so that $h^\alpha_{ij} = 0$ for all i, j and $\alpha \ge 5$, i.e., we may choose the basis $e_1, e_2, \dots, e_n$ so that the component h^α_{ij} of A satisfy

$$(h^3_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h^5_{ij}) = \cdots = (h^n_{ij}) = O, \tag{4}$$

for some functions λ and μ. Let $|A|^2 = \sum\limits_{\alpha,i,j} (h^\alpha_{ij})^2$ be the square length of the second fundamental form of M and $K_N = \sum\limits_{\alpha,\beta,i,j} R^2_{\alpha\beta ij}$ be the normal scalar curvature of M. By (4) and Ricci equation we easily check that $|A|^2 = 2(\lambda^2+\mu^2)$, $K_N = 16\lambda^2\mu^2$.

Noting $\sum\limits_k (h^\alpha_{11k})^2 = \sum\limits_k (h^\alpha_{12k})^2$, $3 \le \alpha \le n$, by (4), we have

$$\begin{aligned} |\nabla(|A|^2)|^2 &= 4\sum_k (\sum_{i,j,\alpha} h^\alpha_{ij} h^\alpha_{ijk})^2 \\ &= 16\sum_k (\lambda h^3_{11k} + \mu h^4_{12k})^2 \\ &\le 16\sum_k (\lambda^2+\mu^2)[(h^3_{11k})^2 + (h^4_{12k})^2] \\ &= 8|A|^2 \sum_k [(h^3_{11k})^2 + (h^4_{11k})^2]. \end{aligned} \tag{5}$$

On the other hand, we have

$$\begin{aligned} |\nabla A|^2 &= 2\sum_{i,k,\alpha} (h^\alpha_{iik})^2 = 4\sum_{k,\alpha} (h^\alpha_{11k})^2 \\ &\ge 4\sum_k [(h^3_{11k})^2 + (h^4_{11k})^2]. \end{aligned} \tag{6}$$

We get (3) from (5) and (6). *The proof of Lemma 1 is completed.*

LEMMA 2. *If M be a minimal surface in $H^n(c)$, then*

$$\frac{1}{2}\Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2$$
$$\geq |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4.$$

Proof. Denote the matrix (h^α_{ij}) by H_α, $3 \leq \alpha \leq n$. By Gauss-Codazzi-Ricci equations it was shown in [**4**] that

$$\frac{1}{2}\Delta(|A|^2) = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + \sum_{\alpha,i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})$$
$$+ \sum_{\alpha,\beta,i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}$$
$$= |\nabla A|^2 + \sum_{\alpha,\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta}(tr(H_\alpha H_\beta))^2 + 2c|A|^2.$$

By (4), it is easy to check the following formulas

$$\sum_{\alpha,\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 = -16\lambda^2\mu^2, \quad \sum_{\alpha,\beta}(tr(H_\alpha H_\beta))^2 = 4(\lambda^4 + \mu^4). \tag{8}$$

Substituting (8) into (7), we get

$$\frac{1}{2}\Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - 8(\lambda^2 + \mu^2)^2 + 4(\lambda^4 + \mu^4)$$
$$= |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2.$$

We completed the proof of Lemma 2.

The following proposition is crucial to prove our Theorem 2.

PROPOSITION 1. *Let M be a minimal surface in $H^n(c)$, ds^2_M be the induced metric. Then the Gauss curvature $\overline{K}$ of the conformal metric $\overline{ds^2} = \sigma ds^2_M$ satisfies $\overline{K} \leq 2$, where*

$$\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} > 0.$$

Proof. By Gauss equation $2K = 2c - |A|^2$,

$$\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} = \frac{4c}{5} - K. \tag{9}$$

Thus we can define a conformal metric $\overline{ds^2} = \sigma ds^2_M$ on M. As it is wellknown, the Gauss curvature $\overline{K}$ of $\overline{ds^2}$ satisfies (for example, see [**2**])

$$-\sigma\overline{K} = -K + \frac{1}{2}\frac{\Delta\sigma}{\sigma} - \frac{|\nabla\sigma|^2}{2\sigma^2}, \tag{10}$$

where Δ is the Laplacian operator of the metric ds_M^2.

By (9) and (10), we get

$$-\sigma\overline{K} = \sigma - \frac{4c}{5} + \frac{1}{2}\frac{\Delta\sigma}{\sigma} - \frac{|\nabla\sigma|^2}{2\sigma^2}. \tag{11}$$

By use of Lemma 1, Lemma 2 and (9),

$$\begin{aligned}\frac{1}{2}\Delta\sigma = \frac{1}{4}\Delta(|A|^2) &\geq \frac{1}{2}|\nabla A|^2 + c|A|^2 - \frac{3}{4}|A|^4\\ &\geq \frac{1}{2}\frac{|\nabla\sigma|^2}{\sigma} - 3\sigma^2 + \frac{4c\sigma}{5}.\end{aligned} \tag{12}$$

Combining (11) with (12), we obtain

$$\overline{K} \leq 2.$$

We completed the proof of Proposition 1.

3. The Proof of Theorem 2. By use of our Proposition 1, we can prove Theorem 2 in the same way as Barbosa and do Carmo did in [**1**] for Theorem 1. So we omit the proof of Theorem 2 here.

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