

## CONSTRUCTING KRIPKE MODELS OF CERTAIN FRAGMENTS OF HEYTING'S ARITHMETIC

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**Abstract.** We present nontrivial methods of constructing Kripke models for the fragments of  $HA$  obtained by restricting the induction schema to instances with  $\Pi_1$ - and  $\Pi_2$ -induction formulae respectively. The model construction for  $\Pi_1$ -induction was applied in [W96a] and [W97] to investigate the provably recursive functions of this theory. The construction of  $\Pi_2$ -induction models is a modification of Smoryński's collection operation introduced in [S73].

### 1. Introduction

A Kripke structure  $\mathcal{K}$  for a first-order language  $L$  is a pair

$$\mathcal{K} = ((K, \leq), (A_\alpha)_{\alpha \in K})$$

such that  $(K, \leq)$  is a (nonempty) reflexive partial order (the frame of  $\mathcal{K}$ ) and for each  $\alpha \in K$ ,  $A_\alpha$  is a classical  $L$ -structure  $A_\alpha = (A_\alpha, =_\alpha, (R_\alpha)_{R \in L}, (f_\alpha)_{f \in L})$  (not necessarily normal, i.e.,  $=_\alpha$  need not be true equality on  $A_\alpha$ ), with the proviso that the following 'monotonicity conditions' be fulfilled:

Whenever  $\alpha \leq \beta$ , then

1.  $A_\alpha$  is a subset of  $A_\beta$ ;
2. for every relation symbol  $R$  of  $L$  (including equality  $=$ ):  $R_\alpha \subseteq R_\beta$ ;
3. for every  $n$ -ary function symbol  $f$  of  $L$ :  $f_\alpha$  is the function  $f_\beta$  restricted to  $A_\alpha^n$ .

The forcing relation  $\Vdash_{\mathcal{K}}$ ,  $\Vdash$  for short, is defined as usual.  $\mathcal{K} \models \phi$  means that for all  $\alpha \in K$ ,  $\alpha \Vdash \phi$ . We also use  $\models$  for the classical satisfaction relation, sometimes

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writing  $\alpha \models \phi$  instead of  $A_\alpha \models \phi$ . We consider here the usual arithmetical language  $L_{ar}$  given by  $0, S, +, \cdot$  and  $<$ .  $i\Delta_0$  is the intuitionistic  $L_{ar}$ -theory axiomatized by the usual axioms for  $PA^-$  (cf. [K91]) together with the axiom schema of induction for  $\Delta_0$ -formulae.  $I\Delta_0$  is the classical version of  $i\Delta_0$ , i.e.,  $i\Delta_0$  augmented by classical logic.  $\Sigma_1$  is the class of  $L_{ar}$ -formulae of the form  $\exists \bar{y} \varphi(\bar{x}, \bar{y})$ ,  $\Pi_1$  the class of  $L_{ar}$ -formulae  $\forall \bar{y} \varphi(\bar{x}, \bar{y})$  with, in both cases,  $\varphi$  in  $\Delta_0$ . Given a formula class  $\Gamma$ ,  $i\Gamma$  is  $i\Delta_0$  plus induction over all formulae in  $\Gamma$  ( $I\Gamma$  being the corresponding classical theory).  $HA$  is  $i\Delta_0$  together with the axiom schema of induction for arbitrary formulae of  $L_{ar}$ , its classical counterpart is  $PA$ . Note that in  $i\Delta_0$ ,  $\Delta_0$ -formulae are decidable.

As a consequence, we have the following

1.1 LEMMA. *Let  $K = ((K, \leq), (A_\alpha)_{\alpha \in K})$  be any Kripke structure for  $L_{ar}$ . The following are equivalent:*

1.  $K \models i\Delta_0$ , i.e., for each  $\alpha \in K$ ,  $\alpha \Vdash i\Delta_0$ .
2. For each  $\alpha \in K$ ,  $A_\alpha$  is a classical model of  $I\Delta_0$  (i.e.,  $\alpha \models I\Delta_0$ ) and whenever  $\alpha \leq \beta$  in  $K$ ,  $A_\alpha \prec_{\Delta_0} A_\beta$ .

*Under these conditions, we have for each  $\alpha \in K$ , each  $\Delta_0$ -formula  $\phi(\bar{x})$  and each  $\bar{a} \in A_\alpha$ :*

$$\alpha \Vdash \phi(\bar{a}) \iff \alpha \models \phi(\bar{a}). \quad \square$$

For a proof, see [W96a]. If atomic formulae (and hence quantifier-free formulae, cf. [M84]) are decidable in  $K$  (as will be the case for  $K \models i\Delta_0$ ), we may assume without loss of generality that every  $A_\alpha$  is a normal structure and that, whenever  $\alpha \leq \beta$  in  $K$ ,  $A_\alpha$  is a substructure of  $A_\beta$  (such  $K$  will be called normal Kripke structures):

1.2 LEMMA. *Let  $K = ((K, \leq), (A_\alpha)_{\alpha \in K})$  be a Kripke structure for a language  $L$  such that  $K \models \forall \bar{x}(P\bar{x} \vee \neg P\bar{x})$  for each atomic formula  $P\bar{x}$  of  $L$ . Then there is a Kripke structure  $K^+ = ((K, \leq), (B_\alpha)_{\alpha \in K})$  for  $L$  such that every  $B_\alpha$  is a normal structure and whenever  $\alpha \leq \beta$  in  $K$ ,  $B_\alpha$  is a substructure of  $B_\beta$ , and such that  $K \models \phi \iff K^+ \models \phi$  for each  $L$ -sentence  $\phi$ .  $\square$*

This fact was already pointed out by Smoryński in [S73]; however, there are two pitfalls in proving it. Smoryński notes that one cannot just divide out  $=_\alpha$  locally, since the equivalence classes may grow when passing from  $\alpha$  to some  $\beta \geq \alpha$ , and so one will not obtain actual inclusion of domains. His proposal is to divide out globally as follows (note that he does not consider function symbols):

Let  $D := \bigcup_{\alpha \in K} A_\alpha$ . For  $c, d \in D$  define  $c \sim d : \iff \exists \alpha \in K \ c =_\alpha \ d$ . Put  $B_\alpha := \{[c] : c \in A_\alpha\}$ , where  $[c]$  is the equivalence class of  $c$  under  $\sim$ , let  $R_{B_\alpha}([c_1], \dots, [c_n]) : \iff R_{A_\alpha}(d_1, \dots, d_n)$  for some elements  $d_1, \dots, d_n \in A_\alpha$  such that  $d_i \in [c_i]$ .

But there is a problem here: ‘By accident’, one and the same object may exist in worlds of incomparable nodes of the frame, and with incompatible properties in the respective worlds. Thus, let  $\alpha, \beta$  and  $\gamma$  be nodes such that  $\alpha < \beta$ ,  $\alpha < \gamma$ , but  $\beta \not\leq \gamma$  and  $\gamma \not\leq \beta$ . Assume that  $A_\alpha$  does not contain the object  $c$ , whereas  $a \in A_\alpha$ .

Then it is possible that  $a =_\beta c$  and  $a \neq_\gamma c$ . Clearly, this makes it impossible to identify  $a$  and  $c$ .

Instead, it is necessary to proceed as follows: In a first step, divide out the equality relations  $=_\alpha$  locally. In a second step, for  $\alpha \leq \beta$ , identify two equivalence classes  $[a]_{=_\alpha}$  and  $[b]_{=_\beta}$  if  $[a]_{=_\alpha} \subseteq [b]_{=_\beta}$ , then take the transitive closure. The easy but tedious verification is left to the reader.

Given some set of axioms  $T$ , a Kripke structure  $K = ((K, \leq), (A_\alpha)_{\alpha \in K})$  for the language of  $T$  is called  $T$ -normal or locally  $T$  if for each  $\alpha \in K$ , the structure  $A_\alpha$  is a classical model of  $T$ ,  $A_\alpha \models T$ . The interplay of sets of axioms being forced at  $\alpha$  and those being classically true in  $A_\alpha$  is intriguing, see e.g. [W96].

## 2. Smoryński's collection operation

In his [S73], Smoryński introduced a powerful *collection operation* on Kripke structures which he used to prove a large number of results on intuitionistic logic and arithmetic. The idea behind his operation  $(\Sigma)'$  is this:

Let some family  $\mathcal{F} = \{K^i : i \in I\}$  of Kripke structures

$$K^i = ((K^i, \leq^i), (A_\alpha^i)_{\alpha \in K^i})$$

for some version  $L$  of the arithmetical language (containing a closed term for each natural number  $n$ ) be given. We may assume without loss of generality that for  $i \neq j$  in  $I$ ,  $K^i \cap K^j = \emptyset$ .

We obtain a new Kripke structure by forming the disjoint union  $\Sigma\mathcal{F} = K = ((K, \leq), (A_\alpha)_{\alpha \in K})$  of  $\mathcal{F}$ , putting

- $K = \bigcup_{i \in I} K^i$ ;
- for  $\alpha, \beta \in K$ ,  $\alpha \leq \beta : \iff$  for some  $i \in I$ ,  $\alpha \leq^i \beta$ ;
- for  $\alpha \in K$ ,  $A_\alpha := A_\alpha^i$ , where  $i \in I$  is such that  $\alpha \in K^i$ .

Observing that for each  $\alpha \in K^i$ ,  $K_\alpha^i = K_\alpha$ , we see that if each  $K^i \models T$  for some theory  $T$ , then  $K = \Sigma\mathcal{F} \models T$ . The operation of disjoint union becomes interesting only in conjunction with a second operation  $K \mapsto K'$  which consists in attaching a (new) root to  $K$ :

Given a *normal* Kripke structure  $K$ ,  $K'$  is obtained from  $K$  by adding a new node  $\alpha_0$  to  $K$  (i.e.,  $K' := K \uplus \{\alpha_0\}$ ), stipulating that  $\alpha_0$  be minimal in  $K'$  (i.e.,  $\leq' := \leq \cup \{(\alpha_0, \beta) : \beta \in K'\}$ ) and letting  $A_{\alpha_0}$  be the standard model of arithmetic  $\mathbb{N}$ .

Note that the restriction to normal models, or at least an assumption that all worlds are normal in their 'standard part' given by the numerals, is important here: The interpretations  $f_\alpha$  of function symbols  $f$  are actual functions, compatible with  $=_\alpha$  but not multi-valued. Thus e.g. in the standard model  $A_{\alpha_0}$ , we certainly have  $S_\alpha(0) = 1$  (true equality!), whereas at some node  $\beta$  from  $K$ , only  $S_\beta(0) =_\beta 1$  is required, and we may actually have  $S_\beta(0) = c$  for some  $c \in A_\beta$  such that  $c \neq 1$  (but of course  $c =_\beta 1$ ). This would collide with the requirement that  $S_{\alpha_0}$  be the

restriction of  $S_\beta$  to  $A_{\alpha_0}$ . But this cannot happen if we exclude non-normal models from the start (which, by Lemma 1.2, is no real restriction).

It is now easy to see that  $\mathcal{K}'$  is again a Kripke structure; more difficult is the question which theories  $T$  are preserved under this operation, i.e., for which theories  $T$  is  $\mathcal{K}'$  a model of  $T$ , provided that  $\mathcal{K} \models T$ ?

Smoryński has shown that  $HA$  and some of its extensions are preserved under the  $'$  operation. We briefly go through his proof, pointing out the punch line and in fact showing that *every* fragment  $i\Gamma$  of  $HA$  is preserved.

**2.1 THEOREM (Smoryński).** *Let  $\mathcal{K} \models i\Gamma$ . Then  $\mathcal{K}' \models i\Gamma$  too.*

*Proof.* Clearly it is sufficient to show that  $\alpha_0 \Vdash' i\Gamma$  (writing  $\Vdash'$  for  $\Vdash_{\mathcal{K}'}$ ). The crucial axioms to check are the induction axioms. So let  $\varphi(x, \bar{z}) \in \Gamma$ ; we must show that

$$\alpha_0 \Vdash' \forall \bar{z} [\varphi(0, \bar{z}) \wedge \forall x (\varphi(x, \bar{z}) \rightarrow \varphi(Sx, \bar{z})) \rightarrow \forall x \varphi(x, \bar{z})].$$

Assume the contrary. Then for some  $\beta \geq' \alpha_0$  and  $\bar{b} \in A_\beta$ ,

$$\beta \not\Vdash' \varphi(0, \bar{b}) \wedge \forall x (\varphi(x, \bar{b}) \rightarrow \varphi(Sx, \bar{b})) \rightarrow \forall x \varphi(x, \bar{b}).$$

The assumption that  $\beta > \alpha_0$  leads to a contradiction, since at such  $\beta$ ,  $\Vdash$  and  $\Vdash'$  coincide, and so  $\beta \Vdash' i\Gamma$ .

Hence  $\beta = \alpha_0$  and  $\bar{b} \in A_{\alpha_0} = \mathbb{N}$ . We thus have (now suppressing parameters from  $A_{\alpha_0}$ ):

$$\alpha_0 \not\Vdash' \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x),$$

i.e., for some  $\beta \geq' \alpha_0$ ,

$$\beta \Vdash' \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(Sx))$$

but  $\beta \not\Vdash' \forall x \varphi(x)$ . Again  $\beta > \alpha_0$  is impossible since then  $\beta \Vdash' i\Gamma$ . So we have

(i)  $\alpha_0 \Vdash' \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(Sx))$  and

(ii)  $\alpha_0 \not\Vdash' \forall x \varphi(x)$ .

By (i) and the fact that for  $\beta > \alpha_0$  we have  $\beta \Vdash' i\Gamma$  we conclude that for all  $\beta > \alpha_0$ ,  $\beta \Vdash' \forall x \varphi(x)$ . Hence we may infer from (ii) that for some  $n \in A_{\alpha_0} = \mathbb{N}$ ,  $\alpha_0 \not\Vdash' \varphi(n)$ . Here's the punch line: Let  $m$  be the least number  $n$  such that  $\alpha_0 \not\Vdash' \varphi(n)$ . Such a minimal counterexample exists since we are in the standard model  $\mathbb{N}$ . (If  $A_{\alpha_0}$  were nonstandard, we would have to know that the forcing relation at  $\alpha_0$  is suitably definable in  $A_{\alpha_0}$ .) Now since  $\alpha_0 \Vdash' \varphi(0)$ , we have  $m \neq 0$ , so  $m = Sk$  for some  $k \in \mathbb{N}$ . By minimality of  $m$ ,  $\alpha_0 \Vdash' \varphi(k)$ , so by  $\alpha_0 \Vdash' \forall x (\varphi(x) \rightarrow \varphi(Sx))$  we obtain  $\alpha_0 \Vdash' \varphi(m)$ , contradiction.  $\square$

**2.2 COROLLARY.** *Every theory of the form  $i\Gamma$  has the explicit definability property (ED) and the disjunction property (DP), i.e., whenever  $i\Gamma$  proves a sentence*

$\exists x \varphi(x)$ , then for some  $n \in \mathbb{N}$ ,  $i\Gamma \vdash \varphi(n)$ , and whenever  $i\Gamma$  proves a sentence  $\psi \vee \chi$ , then either  $i\Gamma \vdash \psi$  or  $i\Gamma \vdash \chi$ .

*Proof.* We consider only (ED). Suppose that  $i\Gamma \not\vdash \varphi(n)$  for each  $n \in \mathbb{N}$ . By the completeness theorem, for each  $n$  there is a Kripke model  $K_n \models i\Gamma$  with  $K_n \not\models \varphi(n)$ . Now consider  $(\sum_{n \in \mathbb{N}} K_n)' =: K'$  which is a model of  $i\Gamma$  by theorem 2.1. Obviously,  $\alpha_0 \not\models' \exists x \varphi(x)$ , since this would imply  $\alpha_0 \Vdash' \varphi(n)$  for some  $n \in A_{\alpha_0} = \mathbb{N}$ , which is impossible by  $K_n \not\models \varphi(n)$ . By the soundness theorem,  $i\Gamma \not\vdash \exists x \varphi(x)$ .  $\square$

Smoryński's idea obviously yields a construction method for Kripke models that can be summarized as follows:

**2.3 THEOREM.** *Let  $(K, \leq)$  be any conversely wellfounded tree. Attach arbitrary models of  $I\Gamma$  to terminal nodes of  $(K, \leq)$  and the standard model  $\mathbb{N}$  of arithmetic to each internal node of  $(K, \leq)$ . The resulting Kripke structure is then a model of  $i\Gamma$ .*

### 3. Constructing models of $\Pi_1$ -induction

Sam Buss has shown in [B93] that there is an easy way to construct Kripke models of  $i\Sigma_1$  over arbitrary frames: *Every  $I\Sigma_1$ -normal Kripke structure is a Kripke model of  $i\Sigma_1$ .* But he also illustrates, by way of counterexample, that it is not as trivial to construct models of  $i\Pi_1$ . However, models of  $i\Pi_1$  over conversely wellfounded frames are not too difficult to build:

**3.1 LEMMA.** *Let  $K = ((K, \leq), (A_\alpha)_{\alpha \in K})$  be a Kripke structure such that for each  $\beta \in K$  there is a terminal node  $\alpha \geq \beta$  in  $K$  (this is always the case when  $(K, \leq)$  is conversely wellfounded). Suppose further that for all terminal nodes  $\alpha \in K$ ,  $A_\alpha \models I\Sigma_1$  and that for all internal nodes  $\beta \in K$ ,  $A_\beta \models I\Delta_0$ . Assume that whenever  $\alpha \leq \beta$  in  $K$ ,  $A_\beta$  is a  $\Delta_0$ -elementary extension of  $A_\alpha$ . Then  $K \models i\Pi_1$ .*

*Proof.* We proceed by brute force (see [W96a] for an alternative proof). Take any  $\alpha \in K$  and let  $\psi(x, \bar{y}, \bar{z})$  be a  $\Delta_0$ -formula. We have to prove that  $\alpha$  forces

$$\forall \bar{z} (\forall \bar{y} \psi(0, \bar{y}, \bar{z}) \wedge \forall x (\forall \bar{y} \psi(x, \bar{y}, \bar{z}) \rightarrow \forall \bar{y} \psi(Sx, \bar{y}, \bar{z})) \rightarrow \forall x \forall \bar{y} \psi(x, \bar{y}, \bar{z})).$$

So let  $\beta \geq \alpha$ ,  $\bar{b} \in A_\beta$  and suppose that

$$\beta \Vdash \forall \bar{y} \psi(0, \bar{y}, \bar{b}) \wedge \forall x (\forall \bar{y} \psi(x, \bar{y}, \bar{b}) \rightarrow \forall \bar{y} \psi(Sx, \bar{y}, \bar{b})).$$

Thus every terminal node above  $\beta$  classically satisfies this last sentence. Since all terminal nodes are classical models of  $I\Sigma_1$  ( $\equiv I\Pi_1$ ), every such node also satisfies  $\forall x \forall \bar{y} \psi(x, \bar{y}, \bar{b})$ . This sentence is  $\Pi_1$  and thus downwards preserved in  $\Delta_0$ -elementary extensions. In particular, it is classically true in every  $\gamma \geq \beta$ . But we want to show that  $\beta \Vdash \forall x \forall \bar{y} \psi(x, \bar{y}, \bar{b})$ . However, this just means that for all  $\gamma \geq \beta$  and all  $c, \bar{d} \in A_\gamma$ ,  $\gamma \Vdash \psi(c, \bar{d}, \bar{b})$ , i.e., by 1.1 that  $\gamma \models \forall x \forall \bar{y} \psi(x, \bar{y}, \bar{b})$ , which, as we have just seen, is indeed the case.  $\square$

3.2 *Remark.* This result shows that Buss' counterexample is not just accidentally not conversely wellfounded: Every  $PA$ -normal Kripke structure on conversely wellfounded frames validates  $i\Pi_1$ .

#### 4. Some general Kripke model theory

In the next section, we will use a modification of Smoryński's  $(\Sigma)'$  operation to construct models of  $i\Pi_2$  from *arbitrary* classical models of  $IS_2$ . The present section develops a little theorem in general Kripke model theory as a preparation for that result.

4.1 *Definition.* A formula  $\varphi$  of some language  $L$  is positive if it is built up from atomic formulae using only  $\wedge$ ,  $\vee$  and  $\exists$ .

4.2 *Remark.* For all Kripke structures  $K$ ,  $\alpha \in K$ ,  $\bar{a} \in A_\alpha$  and every positive formula  $\varphi(\bar{x})$  we have:  $\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a})$ . (In fact this property characterizes the positive formulae: If always  $\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a})$ , then  $\varphi$  is intuitionistically equivalent to a positive formula; cf. [M83].) If in the respective Kripke model  $\Delta_0$ -formulae are decidable, we may, by 1.1, relax the definition of positivity by replacing 'atomic' with ' $\Delta_0$ ', and the property will continue to hold.

4.3 *Definition.* Let  $\Gamma$  be the smallest class of formulae containing the positive formulae, closed under application of  $\forall$  and  $\wedge$  and under the following rule: If  $\varphi$  is positive and  $\psi \in \Gamma$ , then  $\varphi \rightarrow \psi$  is in  $\Gamma$ .

4.4 *Remark.* Under classical logic,  $\Gamma$  is the class of  $\forall\exists$ -formulae. This is also true in intuitionistic theories within which atomic formulae are decidable.

4.5 LEMMA. *Let  $\varphi(\bar{z}) \in \Gamma$ , let  $K$  be a Kripke structure,  $\alpha \in K$ ,  $\bar{a} \in A_\alpha$  and suppose that for all  $\beta > \alpha$  we have  $\beta \Vdash \varphi(\bar{a})$ . Then:*

$$\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a}).$$

*Proof.* We proceed by induction on the definition of  $\varphi \in \Gamma$ .

If  $\varphi(\bar{z})$  is positive, then the claim follows from our remark above. Using the induction hypothesis, the case of conjunction is trivial.

So suppose that  $\varphi(\bar{z})$  is of the form  $\forall y \psi(y, \bar{z})$  with  $\psi(y, \bar{z}) \in \Gamma$ . By assumption for all  $\beta > \alpha$ ,  $\beta \Vdash \forall y \psi(y, \bar{a})$ . (We will from now on suppress mention of the parameters  $\bar{a}$  and just write  $\forall y \psi(y)$ .) First suppose that  $\alpha \Vdash \forall y \psi(y)$ . In particular, for all  $c \in A_\alpha$ ,  $\alpha \Vdash \psi(c)$ , and so by persistence of forcing, for all  $c \in A_\alpha$  and all  $\beta \geq \alpha$   $\beta \Vdash \psi(c)$ . By the induction hypothesis then for all  $c \in A_\alpha$   $\alpha \models \psi(c)$ , i.e.,  $\alpha \models \forall y \psi(y)$ .

Now suppose that  $\alpha \not\Vdash \forall y \psi(y)$ . By definition, for some  $\beta \geq \alpha$  and some  $c \in A_\beta$ ,  $\beta \not\Vdash \psi(c)$ . But since all  $\beta > \alpha$  force  $\forall y \psi(y)$ , we must have  $\alpha \not\Vdash \psi(c)$  for some  $c \in A_\alpha$ . But for all  $\beta > \alpha$   $\beta \Vdash \psi(c)$  and so by induction hypothesis  $\alpha \models \psi(c)$  and thus  $\alpha \not\models \forall y \psi(y)$ .

For the case of implication, suppose  $\varphi$  is of the form  $\psi \rightarrow \chi$ , where  $\psi$  is positive and  $\chi \in \Gamma$ . (We are again suppressing parameters from  $A_\alpha$ .) Assume that for all  $\beta > \alpha$ ,  $\beta \Vdash \psi \rightarrow \chi$ .

First let  $\alpha \Vdash \psi \rightarrow \chi$ . We consider two cases. For the first case, suppose that  $\alpha \Vdash \psi$ . Then for all  $\beta \geq \alpha$   $\beta \Vdash \psi$  and thus for all  $\beta \geq \alpha$   $\beta \Vdash \chi$ . By induction hypothesis then  $\alpha \models \chi$  and of course  $\alpha \models \psi \rightarrow \chi$ . In the second case  $\alpha \not\Vdash \psi$ . Since  $\psi$  is positive, we then have  $\alpha \not\models \psi$  and so vacuously  $\alpha \models \psi \rightarrow \chi$ .

For the other direction, assume that  $\alpha \not\Vdash \psi \rightarrow \chi$ . Then for some  $\beta \geq \alpha$   $\beta \Vdash \psi$  and  $\beta \not\Vdash \chi$ . Since by assumption for all  $\beta > \alpha$   $\beta \Vdash \psi \rightarrow \chi$ , we must in fact have  $\alpha \Vdash \psi$  and  $\alpha \not\Vdash \chi$ . Since  $\alpha \Vdash \psi$ , for all  $\beta > \alpha$   $\beta \Vdash \chi$ . Hence by induction hypothesis  $\alpha \not\models \chi$ .  $\alpha \Vdash \psi$  and  $\psi$  is positive, so  $\alpha \models \psi$ . Together we get  $\alpha \not\models \psi \rightarrow \chi$ .  $\square$

4.6 *Question.* Does the property exhibited in Lemma 4.5 characterize the class  $\Gamma$ ?

4.7 *Remark.* If  $\Delta_0$ -formulae are decidable in the model under consideration, the theorem remains true if we use the relaxed definition of positivity indicated in remark 4.2 in the definition of  $\Gamma$  (cf. lemma 1.1), so that we are then talking about  $\Pi_2$ -formulae.

## 5. Constructing models of $\Pi_2$ -induction

In the variant indicated at the end of the previous section, we can use our result 4.5 to construct certain models of the intuitionistic version  $i\Pi_2$  of  $III_2$ :

5.1 THEOREM. *Let  $K$  be an  $I\Sigma_2$ -normal Kripke structure over a conversely well-founded frame. Then  $K$  is a model of  $i\Pi_2$ , i.e., for each  $\alpha \in K$ ,  $\bar{a} \in A_\alpha$  and every  $\Pi_2$ -formula  $\varphi(x, \bar{y})$  we have*

$$\alpha \Vdash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a})) \rightarrow \forall x \varphi(x, \bar{a}).$$

*Proof.* Note that in  $I\Sigma_2$ -normal Kripke structures all extensions are  $\Delta_0$ -elementary (since the MRDP theorem can be proved in  $I\Sigma_2$ , cf. [HP93]), and so  $K \models i\Delta_0$  by 1.1.

We proceed by bar induction on  $\alpha$ . For terminal  $\alpha$  there is nothing to show since  $III_2 \equiv I\Sigma_2$ .

So let  $\alpha$  be an internal node and suppose that

$$\alpha \not\Vdash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a})) \rightarrow \forall x \varphi(x, \bar{a}).$$

Then for some  $\beta \geq \alpha$  we have  $\beta \Vdash \varphi(0, \bar{a})$ ,  $\beta \Vdash \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a}))$  and  $\beta \not\Vdash \forall x \varphi(x, \bar{a})$ . But by induction hypothesis, we must have  $\beta = \alpha$ , so  $\alpha \Vdash \varphi(0, \bar{a})$ ,  $\alpha \Vdash \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a}))$  and  $\alpha \not\Vdash \forall x \varphi(x, \bar{a})$ .

By persistence of  $\Vdash$  we obtain for each  $\beta > \alpha$  that  $\beta \Vdash \varphi(0, \bar{a})$ ,  $\beta \Vdash \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a}))$  and hence by induction hypothesis that  $\beta \Vdash \forall x \varphi(x, \bar{a})$ . So it follows from  $\alpha \not\Vdash \forall x \varphi(x, \bar{a})$  that already for some  $b \in A_\alpha$ ,  $\alpha \not\Vdash \varphi(b, \bar{a})$ ; however, for each  $\beta > \alpha$  we have  $\beta \Vdash \varphi(b, \bar{a})$  and  $\varphi$  is  $\Pi_2$ , so by 4.5  $\alpha \not\models \varphi(b, \bar{a})$ . Now

let  $c$  be the least element  $e$  of  $A_\alpha$  such that  $\alpha \not\vdash \varphi(e, \bar{a})$ . (This is possible since  $\alpha \models III_2$ .)

Then the following obtain:

1.  $\alpha \not\vdash \varphi(c, \bar{a})$  (by 4.5, since  $\alpha \not\vdash \varphi(c, \bar{a})$  and for all  $\beta > \alpha$  we have  $\beta \Vdash \varphi(c, \bar{a})$ );
2. for each  $d <_\alpha c$ ,  $\alpha \Vdash \varphi(d, \bar{a})$  (again by 4.5, since for  $d <_\alpha c$   $\alpha \models \varphi(d, \bar{a})$  and for each  $\beta > \alpha$   $\beta \Vdash \varphi(d, \bar{a})$ ).

But now  $c$  is not 0, since  $\alpha \Vdash \varphi(0, \bar{a})$  and  $\alpha \not\vdash \varphi(c, \bar{a})$ . Hence  $c = Sd$  for some  $d \in A_\alpha$ . But by 2.  $\alpha \Vdash \varphi(d, \bar{a})$ , so by  $\alpha \Vdash \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(Sx, \bar{a}))$  we get  $\alpha \Vdash \varphi(Sd, \bar{a})$ , i.e.,  $\alpha \Vdash \varphi(c, \bar{a})$ , contradicting 1.  $\square$

**5.2 Remark.** It is clear by Theorem 5.1 that every *PA*-normal Kripke structure over a wellfounded frame is a model of  $iII_2$ . This result is in some sense optimal: Zambella and Visser show in a forthcoming paper that there are *PA*-normal Kripke structures over two-element frames, the respective *PA*-models an end-extension, which are not models of  $i\Sigma_2$ .

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