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ON FORMAL PRODUCTS AND THE SEIDEL SPECTRUM OF GRAPHS

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Abstract. In [**2**] using the formal product and the so-called formal generating functions, we proved some results concerning cospectral graphs. In this paper, we define the Seidel formal product and investigate some properties of the Seidel spectrum. In particular, for any two overgraphs G_{S_1} and G_{S_2} of G we give necessary and sufficient conditions under which G_{S_1} and G_{S_2} have the same Seidel spectrum.

In this paper we consider only simple graphs. The vertex set of a graph G is denoted by $V(G)$, and its order by $|G|$. The spectrum of such a graph is the collection $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of eigenvalues of its (0,1) adjacency matrix and it is denoted with $\sigma(G)$. The Seidel spectrum $\sigma^*(G)$ is the collection $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of eigenvalues of its Seidel $(-1,1,0)$ adjacency matrix.

In the sequel, for any graph G denote by $A = A(G) = [a_{ij}]$, $A^* = A^*(G) = [a_{ij}^*]$, $P_G(\lambda)$ and $P_G^*(\lambda)$, the adjacency matrix, the Seidel adjacency matrix, the characteristic polynomial and the Seidel characteristic polynomial, respectively. If G and H are two graphs which have the same Seidel spectrum, we shall say that G and H are Seidel cospectral.

Let S be any (possibly empty) subset of the vertex set $V(G)$. Denote by G_S the graph obtained from the graph G by adding a new vertex x ($x \notin V(G)$), which is adjacent exactly to the vertices in S. The family of overgraphs G_S of the graph G is denoted by $\mathcal{G}(G)$, and it is called the *overset* of G.

For a matrix M denote by $\{M\}$ the adjoint of M, and let $\mathbf{sum}\, M$ denote the sum of all elements in M.

Let G be an arbitrary graph of order n and let $\mathbf{A} = [\mathbf{A_{ij}}] = \{\lambda I - A\}$. Of course, we have $\mathbf{A_{ij}} = \mathbf{A_{ji}}$ $(i, j = 1, 2, \ldots, n)$. For any two subsets X, Y of

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the vertex set $V(G)$, let $\langle X,Y\rangle = \sum_{i\in X}\sum_{j\in Y}\mathbf{A_{ij}}$. In [**2**] the expression $\langle X,Y\rangle$ was defined as *the formal product* of the sets X and Y, associated with the graph G. For any two subsets $X,Y\subseteq V(G)$, define $\langle X,Y\rangle^* = \sum_{i\in X}\sum_{j\in Y}\mathbf{A^*_{ij}}$, where $\mathbf{A}^* = [\mathbf{A^*_{ij}}] = \{\lambda I - A^*\}$. In this paper $\langle X,Y\rangle^*$ is called *the Seidel formal product* of the sets X and Y, associated with the graph G. Since $\mathbf{A^*_{ij}} = \mathbf{A^*_{ji}}$ $(i,j=1,2,\dots,n)$ we obtain that $\langle X,Y\rangle^* = \langle Y,X\rangle^*$ for any two subsets $X,Y\subseteq V(G)$. If $X\cap Y=\emptyset$, then the union of X and Y is denoted by $X+Y$. Let $X,Y,Z\subseteq V(G)$ be any three subsets of $V(G)$ such that $X\cap Y=\emptyset$. Then we find the relation

$$\langle X+Y,Z\rangle^* = \langle X,Z\rangle^* + \langle Y,Z\rangle^* .$$

Let $S\subseteq V(G)$ and G_S be the corresponding overgraph of G. For any set $S\subseteq V(G)$ let $T=V(G)\setminus S$. In [**2**] we proved the next result.

THEOREM 1 [**2**]. *For any graph G and any set $S\subseteq V(G)$, we have*

$$P_{G_S}(\lambda) = \lambda P_G(\lambda) - \langle S,S\rangle . \quad \square \tag{1}$$

Similar results were proved by E. Heilbronner (see [**1**, p. 59]) and by A. Schwenk (see [**3**]).

Using the same method as in the proof of Theorem 1, one can easily see that the Seidel characteristic polynomial of G_S reads

$$P^*_{G_S}(\lambda) = \lambda P^*_G(\lambda) - \langle S,S\rangle^* - \langle T,T\rangle^* + 2\langle S,T\rangle^* . \tag{2}$$

Let $S^\bullet = V(G)$ and denote the corresponding overgraph of G by $G^\bullet$. Since $\langle S^\bullet,S^\bullet\rangle^* = \langle S+T,S+T\rangle^*$, using (2) we obtain that

$$P^*_{G_S}(\lambda) = P^*_{G^\bullet}(\lambda) + 4\,\langle S,T\rangle^* . \tag{3}$$

Further, let S be any subset of the vertex set $V(G)$. To switch G with respect to S means:

- to remove all edges connecting S with $T=V(G)\setminus S$; and
- to introduce an edge between all nonadjacent vertices x,y such that one of them belongs to S and the other to T.

Two graphs G and H are switching (Seidel switching) equivalent if one of them is obtained from the other by switching. It is known that switching equivalent graphs have the same Seidel spectrum. We notice that, if $S\subseteq V(G)$ then the corresponding graphs G_S and G_T are switching equivalent.

On the other hand, since $\langle S,T\rangle^* = \langle T,S\rangle^*$, using relation (3) we obtain that G_S and G_T are Seidel cospectral graphs for any $S\subseteq V(G)$. Also, by (3) we obtain the following statement.

COROLLARY 1. *Let G_{S_1} and G_{S_2} be two arbitrary overgraphs of G. Then G_{S_1} and G_{S_2} are Seidel cospectral if and only if $\langle S_1, T_1\rangle^* = \langle S_2, T_2\rangle^*$.* □

For any adjacency matrix A, let $A^k = [a_{ij}^{(k)}]$. In [**2**] was proved that the formal product $\langle S, S\rangle$ and the characteristic polynomial $P_{G_S}(\lambda)$ $(S \subseteq V(G))$ can be expressed by the entries of A^k for all values of k. In this paper, we shall show that $\langle S, S\rangle^*$ and $P^*_{G_S}(\lambda)$ can be expressed by the entries of $(A^*)^k = [(a^*_{ij})^{(k)}]$, where, as usual, A^* is the Seidel adjacency matrix of G.

We first recall some results and definitions concerning canonical graphs which are given in [**4**] and [**5**].

Let G be an arbitrary connected graph of order n. We say that two vertices $x, y \in V(G)$ are equivalent in G and write $x \sim y$ if x is nonadjacent to y, and x and y have exactly the same neighbors in G. Relation $\sim$ is obviously an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by $\tilde{G}$, and is called the canonical graph of G.

We say that G is canonical if $G = \tilde{G}$ or equivalently $|G| = |\tilde{G}|$, i.e., if G has no two equivalent vertices. Let $\tilde{G}$ be the canonical graph of G, $|\tilde{G}| = k$, and $N_1, N_2, \ldots, N_k$ be the corresponding sets of equivalent vertices in G. Then we write $G = \tilde{G}(N_1, N_2, \ldots, N_k)$, or simply $G = \tilde{G}(n_1, n_2, \ldots, n_k)$, where $|N_i| = n_i$ $(i = 1, 2, \ldots, k)$, understanding that $\tilde{G}$ is a labelled graph.

It was proved in [**4**] that the characteristic polynomial $P_G(\lambda)$ of the graph G takes the form

$$P_G(\lambda) = n_1 \cdot n_2 \cdot \ldots \cdot n_k \, \lambda^{n-k} \begin{vmatrix} \dfrac{\lambda}{n_1} & -\tilde{a}_{12} & \ldots & -\tilde{a}_{1k} \\ -\tilde{a}_{21} & \dfrac{\lambda}{n_2} & \ldots & -\tilde{a}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{a}_{k1} & -\tilde{a}_{k2} & \ldots & \dfrac{\lambda}{n_k} \end{vmatrix}, \tag{4}$$

where $[\tilde{a}_{ij}]$ is the adjacency matrix of the canonical graph $\tilde{G}$.

Using the same method as in [**4**] for obtaining relation (4), one can see that the Seidel characteristic polynomial $P^*_G(\lambda)$ of the graph G reads

$$P^*_G(\lambda) = (\lambda+1)^{n-k} \begin{vmatrix} \lambda+1-n_1 & -n_1\,\tilde{a}^*_{12} & \ldots & -n_1\,\tilde{a}^*_{1k} \\ -n_2\,\tilde{a}^*_{21} & \lambda+1-n_2 & \ldots & -n_2\,\tilde{a}^*_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -n_k\,\tilde{a}^*_{k1} & -n_k\,\tilde{a}^*_{k2} & \ldots & \lambda+1-n_k \end{vmatrix}, \tag{5}$$

where $[\tilde{a}^*_{ij}]$ is the Seidel adjacency matrix of the canonical graph $\tilde{G}$.

Let G be any (not necessarily canonical) graph of order n. Let $G_{x_1,x_2,\ldots,x_m}$ be the overgraph of G obtained by adding new vertices $x_1, x_2, \ldots, x_m$ equivalent to a vertex i of G, say $i = 1$, so that the vertices $x_1, x_2, \ldots, x_m, 1$ are mutually non-adjacent, and have the same neighbors in G. According to (5), applying the same method as in [**4**] for deriving relation (4), one can see that the Seidel characteristic polynomial of $G_{x_1,x_2,\ldots,x_m}$ reads

$$(6)\quad P^*_{G_{x_1,x_2,\ldots,x_m}}(\lambda) = (\lambda+1)^m \begin{vmatrix} \lambda - m & -(m+1)a^*_{12} & \ldots & -(m+1)a^*_{1n} \\ -a^*_{21} & \lambda & \ldots & -a^*_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a^*_{n1} & -a^*_{n2} & \ldots & \lambda \end{vmatrix},$$

where $[a^*_{ij}]$ is the Seidel adjacency matrix of the graph G.

Let S be any subset of $V(G)$ and let G_{2S} be the overgraph of G obtained by adding two new non-adjacent vertices x, y which are both adjacent exactly to the vertices from S. Note that $G_{2S} \in \mathcal{G}(G_S)$, and G_{2S} is obtained from G_S by adding a new vertex y which is equivalent to $x \in V(G_S)$. Therefore, using (2) we have the following relation

$$P^*_{G_{2S}}(\lambda) = \lambda P^*_{G_S}(\lambda) - \langle S, S\rangle^* - \langle T, T\rangle^* + 2\,\langle S, T\rangle^*,$$

where $\langle X, Y\rangle^*$ is the Seidel formal product associated with G_S.

PROPOSITION 1. *The Seidel characteristic polynomial $P^*_{G_{2S}}(\lambda)$ of the graph G_{2S} reads*

$$P^*_{G_{2S}}(\lambda) = (\lambda+1)\left[(\lambda-1)P^*_G(\lambda) - 2\,\langle S, S\rangle^* - 2\,\langle T, T\rangle^* + 4\,\langle S, T\rangle^*\right],$$

where $\langle X, Y\rangle^*$ is the Seidel formal product associated with the graph G.

Proof. Without loss of generality we may assume that $S = \{1, 2, \ldots, k\} \subseteq V(G)$ $(0 \le k \le n)$. Using relation (6), the Seidel characteristic polynomial $P^*_{G_{2S}}(\lambda)$ takes the form

$$P^*_{G_{2S}}(\lambda) = (\lambda+1) \begin{vmatrix} \lambda & \ldots & -a^*_{1k} & \ldots & -a^*_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -a^*_{k1} & \ldots & \lambda & \ldots & -a^*_{kn} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a^*_{n1} & \ldots & -a^*_{nk} & \ldots & \lambda & -1 \\ 2 & \ldots & 2 & \ldots & -2 & \lambda - 1 \end{vmatrix}.$$

Applying the same method as in the proof of Theorem 1, one can easily obtain the required statement. □

Let S be any subset of $V(G)$ and let G_{mS} be the overgraph of G obtained by adding m new mutually non-adjacent vertices $x_1, x_2, \ldots, x_m$, all adjacent exactly to the vertices in S.

COROLLARY 2. *The Seidel characteristic polynomial $P^*_{G_{mS}}(\lambda)$ $(m \in N)$ of the graph G_{mS} reads*

$$P^*_{G_{mS}}(\lambda) = (\lambda+1)^{m-1}\big[(\lambda-m+1)\,P^*_G(\lambda) - m\,\langle S,S\rangle^* - m\,\langle T,T\rangle^* + 2m\,\langle S,T\rangle^*\big]. \quad \square$$

By (2) we find that $\langle S,S\rangle^* + \langle T,T\rangle^* - 2\,\langle S,T\rangle^* = \lambda\,P^*_G(\lambda) - P^*_{G_S}(\lambda)$. Using Corollary 2 we have the following result.

COROLLARY 3. *Let $S \subseteq V(G)$. Then*

$$P^*_{G_{mS}}(\lambda) = (\lambda+1)^{m-1}\left[m\,P^*_{G_S}(\lambda) - (m-1)(\lambda+1)\,P^*_G(\lambda)\right],$$

for any $m \in N$. $\square$

COROLLARY 4. *Let G_{S_1} and G_{S_2} be any two overgraphs of G $(S_1, S_2 \subseteq V(G))$. If G_{S_1} and G_{S_2} are Seidel cospectral then G_{mS_1} and G_{mS_2} are also Seidel cospectral for every $m \in N$.* $\square$

Now, we shall need some well-known notions and results from the spectral theory of graphs (see [**1**]).

THEOREM 2 [**1**]. *Let A be the adjacency matrix of a multi-digraph G with vertices $1, 2, \ldots, n$, and $A^k = [a_{ij}^{(k)}]$; further, let $N_k(i,j)$ denote the number of walks of length k starting at vertex i and terminating at vertex j. Then*

$$N_k(i,j) = a_{ij}^{(k)} \qquad (k = 0, 1, 2, \ldots). \quad \square$$

THEOREM 3 [**1**]. *Let G be a graph with complement $\overline{G}$ and let $H_G(t) = \sum_{k=0}^{+\infty} N_k t^k$ be the generating function of the numbers N_k of walks of length k in the graph G. Then*

$$H_G(t) = \frac{1}{t}\left[(-1)^n\,\frac{P_{\overline{G}}(-1-1/t)}{P_G(1/t)} - 1\right], \tag{7}$$

where $N_k = \mathbf{sum}\,A^k$ $(k = 0, 1, 2, \ldots)$. $\square$

THEOREM 4 [**1**]. *If $P_G(\lambda)$ is the characteristic polynomial of a graph G and $P^*_G(\lambda)$ is the characteristic polynomial of the Seidel adjacency matrix $A^*(G)$ of G, then*

$$P_G(\lambda) = \frac{(-1)^n}{2^n}\,\frac{P^*_G(-2\lambda-1)}{1 + \dfrac{1}{2\lambda}\,H_G\!\left(\dfrac{1}{\lambda}\right)}. \tag{8} \quad \square$$

According to (7) and (8), by a straightforward calculation we obtain the relation

$$(9)\qquad P^*_G(-2\lambda-1) = 2^{n-1}\left(P_{\overline{G}}(-\lambda-1) + (-1)^n P_G(\lambda)\right).$$

Since $P^*_{G_S}(\lambda) = P^*_{G_T}(\lambda)$ for any $S \subseteq V(G)$, then using (9) the next result follows.

COROLLARY 5. *Let* $S \subseteq V(G)$. *Then*

$$P_{G_S}(\lambda) - P_{G_T}(\lambda) = (-1)^n \left(P_{\overline{G_S}}(-\lambda-1) - P_{\overline{G_T}}(-\lambda-1)\right). \quad \square$$

In [**2**] using the so-called generalized adjacency matrices, we proved that for any $S \subseteq V(G)$ the formal product

$$(10)\qquad \langle S,S\rangle = \frac{P_G(\lambda)}{\lambda}\, \mathfrak{F}_S\Big(\frac{1}{\lambda}\Big),$$

where $\mathfrak{F}_S(t) = \sum_{k=0}^{+\infty} d^{(k)}\, t^k$ $\left(|t| < \lambda_1^{-1};\ \lambda_1 \in \sigma(G)\right)$, and $d^{(k)} = \sum_{i\in S}\sum_{j\in S} a_{ij}^{(k)}$ for any non-negative integer k. We note that $d^{(k)}$ is the number of walks of length k with endpoints in S.

The function $\mathfrak{F}_S(t)$ is called the"formal generalized function" associated with the graph G_S. Similarly, the function

$$\mathfrak{F}^*_{S,T}(t) = \sum_{k=0}^{+\infty} e_*^{(k)}\, t^k \qquad \left(|t| < (\lambda_1^*)^{-1};\ \lambda_1^* \in \sigma^*(G)\right),$$

will be called the "Seidel formal generating function" associated with the graph G_S, where $T = V(G) \smallsetminus S$ and $e_*^{(k)} = \sum_{i\in S}\sum_{j\in T} (a_{ij}^*)^{(k)}$ $(k = 0,1,2,\dots)$.

We shall prove that for any $S \subseteq V(G)$, the Seidel formal product

$$(11)\qquad \langle S,T\rangle^* = \frac{P^*_G(\lambda)}{\lambda}\, \mathfrak{F}^*_{S,T}\Big(\frac{1}{\lambda}\Big).$$

The last relation may be proved by using some "Seidel generalized matrices", in a way similar to that used to prove (10). However, in this paper we shall give an alternative proof of relation (11), as follows.

First, let

$$H^*_G(t) = \sum_{k=0}^{+\infty} N^*_k\, t^k \qquad \left(|t| < (\lambda_1^*)^{-1};\ \lambda_1^* \in \sigma^*(G)\right),$$

where $N_k^* = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^*)^{(k)}$ $(k = 0,1,2,\ldots)$. In this paper, $H_G^*(t)$ is called the "Seidel generating function". If we set

$$(12)\qquad [H_G^*(t)] = \sum_{k=0}^{+\infty}(A^*)^k t^k \qquad (|t| < (\lambda_1^*)^{-1}),$$

then, it is clear that $H_G^*(t) = \mathbf{sum}\,[H_G^*(t)]$.

Proposition 2. *Let X, Y be any two sets of the vertex set $V(G)$. Then*

$$\langle X,Y\rangle^* = \frac{P_G^*(\lambda)}{\lambda}\,\mathfrak{F}_{X,Y}^*\Big(\frac{1}{\lambda}\Big),$$

where $\mathfrak{F}_{X,Y}^*(t) = \sum_{k=0}^{+\infty} c_*^{(k)} t^k$ *and* $c_*^{(k)} = \sum_{i\in X}\sum_{j\in Y}(a_{ij}^*)^{(k)}$ $(k = 0,1,2,\ldots)$.

Proof. Using (12) we find that

$$[H_G^*(t)] = (I - t\,A^*)^{-1} = \frac{\{I - t\,A^*\}}{|I - t\,A^*|}.$$

If we set $[\mathbf{B}_{\mathbf{ij}}^*] = \{I - t\,A^*\}$, then from the previous relation we obtain

$$\mathbf{B}_{\mathbf{ij}}^* = t^n\,P_G^*\Big(\frac{1}{t}\Big)\sum_{k=0}^{+\infty}(a_{ij}^*)^{(k)}t^k \qquad (1\le i,j\le n).$$

If we set $t = 1/\lambda$ and substitute t in the last relation, we can easily see that

$$\frac{1}{\lambda^{n-1}}\,\mathbf{A}_{\mathbf{ij}}^* = \frac{1}{\lambda^n}\,P_G^*(\lambda)\sum_{k=0}^{+\infty}(a_{ij}^*)^{(k)}\,\frac{1}{\lambda^k},$$

where $[\mathbf{A}_{\mathbf{ij}}^*] = \{\lambda I - A^*\}$. Consequently, for any two sets $X, Y \subseteq V(G)$ the following relation is obtained

$$\begin{aligned}\langle X,Y\rangle^* &= \sum_{i\in X}\sum_{j\in Y}\mathbf{A}_{\mathbf{ij}}^* = \frac{P_G^*(\lambda)}{\lambda}\sum_{i\in X}\sum_{j\in Y}\Big[\sum_{k=0}^{+\infty}(a_{ij}^*)^{(k)}\,\frac{1}{\lambda^k}\Big]\\ &= \frac{P_G^*(\lambda)}{\lambda}\sum_{k=0}^{+\infty}\Big[\sum_{i\in X}\sum_{j\in Y}(a_{ij}^*)^{(k)}\Big]\frac{1}{\lambda^k}\\ &= \frac{P_G^*(\lambda)}{\lambda}\,\mathfrak{F}_{X,Y}^*\Big(\frac{1}{\lambda}\Big). \quad \square\end{aligned}$$

In particular, for $Y = X$ we denote the corresponding Seidel formal generating function $\mathfrak{F}^*_{X,X}(t)$ by $\mathfrak{F}^*_X(t)$. Therefore, according to (10), for any $S \subseteq V(G)$ we have

$$\langle S, S\rangle^* = \frac{P^*_G(\lambda)}{\lambda}\, \mathfrak{F}^*_S\Big(\frac{1}{\lambda}\Big)\,,$$

where $\mathfrak{F}^*_S(t) = \sum\limits_{k=0}^{+\infty} d^{(k)}_* \, t^k$ and $d^{(k)}_* = \sum\limits_{i\in S}\sum\limits_{j\in S} (a^*_{ij})^{(k)}$.

Corollary 6. *Let $X, Y \subseteq V(G)$. Then the formal product*

$$\langle X, Y\rangle = \frac{P_G(\lambda)}{\lambda}\, \mathfrak{F}_{X,Y}\Big(\frac{1}{\lambda}\Big)\,,$$

where $\mathfrak{F}_{X,Y}(t) = \sum\limits_{k=0}^{+\infty} c^{(k)} t^k$ and $c^{(k)} = \sum\limits_{i\in X}\sum\limits_{j\in Y} a^{(k)}_{ij}$ $(k = 0, 1, 2, \ldots)$. □

We note that $\mathfrak{F}_{X,Y}(t)$ is the generating function for the number of walks of length k with starting point in X and endpoint in Y.

As an immediate consequence of Proposition 2 and Corollary 1, we get:

Corollary 7. *Let G_{S_1} and G_{S_2} be two arbitrary overgraphs of G. Then G_{S_1} and G_{S_2} are Seidel cospectral if and only if $\mathfrak{F}^*_{S_1,T_1}(t) = \mathfrak{F}^*_{S_2,T_2}(t)$.* □

For any $S \subseteq V(G)$, we shall define a function

$$\mathfrak{F}^*_{[S]}(t) = \mathfrak{F}^*_S(t) + \mathfrak{F}^*_T(t) - 2\,\mathfrak{F}^*_{S,T}(t)\,,$$

where $T = V(G) \setminus S$. Now, using (2) and (11) we can easily see that the Seidel characteristic polynomial of G_S reads

$$P^*_{G_S}(\lambda) = P^*_G(\lambda)\,\Big[\lambda - \frac{1}{\lambda}\,\mathfrak{F}^*_{[S]}\Big(\frac{1}{\lambda}\Big)\Big]\,. \tag{13}$$

Let $S^\bullet = V(G)$ and denote the corresponding overgraph of G by $G^\bullet$. Using Proposition 2, we find that $\mathfrak{F}^*_{S^\bullet}(t) = H^*_G(t)$. Since $T^\bullet = V(G) \setminus S^\bullet = \emptyset$, it follows that $\langle T^\bullet, X\rangle^* = 0$ for any $X \subseteq V(G)$. Whence we obtain $\mathfrak{F}^*_{[S^\bullet]}(t) = \mathfrak{F}^*_{S^\bullet}(t)$. Using (13) and the last relation, we have

$$P^*_{G^\bullet}(\lambda) = \lambda\, P^*_G(\lambda) - \frac{P^*_G(\lambda)}{\lambda}\, H^*_G\Big(\frac{1}{\lambda}\Big)\,.$$

Finally, using the Seidel formal generating functions, we shall prove an elementary result.

Let G be the complete graph K_n and S be any subset of $V(K_n)$. Denote by $K(m)$ the corresponding overgraph of K_n, where $|S| = m$ $(0 \le m \le n)$.

Proposition 3. *If $S \subseteq V(K_n)$ and $|S| = m$, then*

$$P^*_{K(m)}(\lambda) = (\lambda-1)^{n-2}\left[\lambda^3 + (n-2)\lambda^2 - (2n-1)\lambda + 4m^2 - 4mn + n\right].$$

Proof. Since K_n is a regular graph of degree $n-1$, we obtain that $N^*_k = (-1)^k n(n-1)^k$. Let $\alpha_k = (a^*_{11})^{(k)}$ and $\beta_k = (a^*_{12})^{(k)}$ $(k = 0,1,2,\ldots)$. It is clear that $(a^*_{ii})^{(k)} = \alpha_k$ $(i = 1,2,\ldots,n)$ and $(a^*_{ij})^{(k)} = \beta_k$ $(i \neq j)$. Therefore, $(-1)^k n(n-1)^k = n\alpha_k + (n^2-n)\beta_k$. Since $\alpha_k = -(n-1)\beta_{k-1}$, the expression for

$$\alpha_k = \frac{(-1)^k(n-1)^k + (n-1)}{n} \quad \text{and} \quad \beta_k = \frac{(-1)^k(n-1)^k - 1}{n},$$

can be obtained by solving the linear recursions

$$\beta_k = \beta_{k-1} + (-1)^k(n-1)^{k-1} \quad \text{and} \quad \alpha_k = -(n-1)\beta_{k-1},$$

with $\alpha_0 = 1$ and $\beta_0 = 0$.

Since $d^{(k)}_* = \sum_{i\in S}\sum_{j\in S}(a^*_{ij})^{(k)}$ and $|S| = m$, we have $d^{(k)}_* = m\alpha_k + (m^2-m)\beta_k$. Whence we get

$$d^{(k)}_* = \frac{(-1)^k m^2(n-1)^k - m^2 + mn}{n}. \tag{14}$$

Further, using (14) we find

$$\mathfrak{F}^*_S(t) = \frac{m^2}{n\big(1+(n-1)t\big)} - \frac{m^2}{n(1-t)} + \frac{m}{1-t}. \tag{15}$$

Similarly, we obtain

$$\mathfrak{F}^*_T(t) = \frac{(n-m)^2}{n\big(1+(n-1)t\big)} - \frac{(n-m)^2}{n(1-t)} + \frac{n-m}{1-t}. \tag{16}$$

Now, denote by $e^{(k)}_*$ the corresponding coefficients of the function $\mathfrak{F}^*_{S,T}(t)$. Since $e^{(k)}_* = m(n-m)\beta_k$, we can see that

$$\mathfrak{F}^*_{S,T}(t) = \frac{m(n-m)}{n\big(1+(n-1)t\big)} + \frac{m^2}{n(1-t)} - \frac{m}{1-t}.$$

Using (15), (16) and the last relation, by an easy calculation we obtain that the corresponding function $\mathfrak{F}^*_{[S]}(t)$ reads

$$\mathfrak{F}^*_{[S]}(t) = \frac{(4mn - 4m^2 - n)\,t + n}{\big(1+(n-1)t\big)(1-t)}.$$

Finally, if we set $t = 1/\lambda$ in the previous relation, then by using (13), having in mind that

$$P^*_{K_n}(\lambda) = (-1)^n P_{K_n}(-\lambda) = (\lambda - 1)^{n-1}\big(\lambda + (n-1)\big)\,,$$

we obtain the statement. $\square$

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