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INTEGRATION OF THE PERTURBATED JACOBI PROBLEM

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Abstract. It is shown that recently obtained family of integrable potential perturbations of the Jacobi problem for the geodesics on the ellipsoid can be integrated by separation of variables in the elliptic coordinates. The way of reduction to the Liouville case is demonstrated and the complete integral of the Hamilton–Jacobi equation is given.

1. Introduction

In the theory of the Hamiltonian systems, the completely integrable case has a very important position. According to Lionville's theorem (see [**1, 2**]), the motion associated to the system has regular behavior. Unfortunately the integration procedure given by this theorem is ineffective. There are several methods of effectivisation of this procedure, classical and modern, as presented in the encyclopedic reviews [**1, 2**].

The integrability of the Hamiltonian system describing a particle moving under inertia on an ellipsoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1 \tag{1}$$

was proven by Jacobi by separation of variables in the elliptic coordinates (see, for example, [**1, 2**]).

A family of integrable potential perturbations of the above system was recently obtained (see [**5**]) using Kozlov's method (see [**3, 4**]). It was found in the form of Laurent polynomials:

$$V(x,y,z)=\sum_{r\geq n,m,p\geq s} a_{m,n,p}(a,b,c)x^m y^n z^p, \quad s,r\in Z.$$

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The basic potentials V_{m_0}, $m_0 = -2t$, $t \in N$, were given (see Theorem 1 in [**5**]) by the formulae

$$a_{m_0+2k,2s,-m_0-2-2(k+s)} = \binom{s+k-1}{k} \frac{c^{k+s}(c-a)^s(c-b)^k \prod_{i=k+1}^{k+s}(m_0+2i)}{b^k a^s (b-a)^{k+s}(-2)^s s!} \tag{2}$$

where $k \leq s$, and $m_0 + 2(k+s) < 0$.

In this paper we want to show that these systems are integrable by the method of separation of variables in the elliptic coordinate system. On the example of V_{-4} we demonstrate how such system can be reduced to the Liouville case (see [**6, 7**]) and give the complete integral of the Hamilton–Jacobi equation. (The exapmle $V_{-2} = 1/x^2$ has been studied by Kozlov in [**3**].)

2. The elliptic coordinates and the separation of the variables

If $0 < a < b < c$ are distinct positive real numbers, Jacobi's elliptic coordinates (in R^3) $\lambda_1, \lambda_2, \lambda_3$ can be defined by the function which the triple $(x, y, z) \in R^3$ maps the roots $(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_3 < \lambda_2 < \lambda_1$, of the equation

$$f(\lambda) = \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1.$$

Conversely (see [**1**]):

$$\begin{aligned} x^2 &= \frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(a-b)(a-c)}, \\ y^2 &= \frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{(b-a)(b-c)}, \\ z^2 &= \frac{(c-\lambda_1)(c-\lambda_2)(c-\lambda_3)}{(c-a)(c-b)} \end{aligned} \tag{3}$$

In these coordinates the constrain (1) is given by $\lambda_3 = 0$. The Hamiltonians of the analysed systems are of the form:

$$H_{m_0} = T + \tilde{V}_{m_0},$$

where

$$T = \frac{2}{\lambda_1 - \lambda_2}\left[\frac{(a-\lambda_1)(\lambda_1-b)(\lambda_1-c)}{\lambda_1}p_1^2 + \frac{(a-\lambda_2)(b-\lambda_2)(\lambda_2-c)}{\lambda_2}p_2^2\right]$$

$$p_j = (-1)^{j+1}(\lambda_1-\lambda_2)\frac{\lambda_j \dot{\lambda}_j}{4(a-\lambda_j)(b-\lambda_j)(c-\lambda_j)}, \quad j = 1, 2,$$

and $\tilde{V}_{m_0}(\lambda_1, \lambda_2)$ we got from $V_{m_0}(x, y, z)$ by expressing x^2, y^2, z^2 in (2) by (3).

THEOREM. *The systems with Hamiltonians $H_{m_0}(\lambda_1, \lambda_2, p_1, p_2)$ are integrable by the method of separation of variables in the elliptic coordinate system.*

Proof. By the Levi–Civita criterion (see [**2, 7**]) one has to show that

$$\begin{aligned} &\frac{\partial H_{m_0}}{\partial p_1}\cdot\frac{\partial H_{m_0}}{\partial p_2}\cdot\frac{\partial^2 H_{m_0}}{\partial\lambda_1\partial\lambda_2}-\frac{\partial H_{m_0}}{\partial p_1}\cdot\frac{\partial H_{m_0}}{\partial\lambda_2}\cdot\frac{\partial^2 H_{m_0}}{\partial p_2\partial\lambda_1}\\ &-\frac{\partial H_{m_0}}{\partial\lambda_1}\cdot\frac{\partial H_{m_0}}{\partial p_2}\cdot\frac{\partial^2 H_{m_0}}{\partial p_1\partial\lambda_2}+\frac{\partial H_{m_0}}{\partial\lambda_1}\cdot\frac{\partial H_{m_0}}{\partial\lambda_2}\cdot\frac{\partial^2 H_{m_0}}{\partial p_1\partial p_2}=0 \end{aligned} \tag{4}$$

The system with Hamiltonian $H_0 = T$ is integrable by separation of variables in the elliptical coordinates. So (4) is equivalent to

$$(\lambda_1-\lambda_2)\frac{\partial^2\tilde{V}_{m_0}}{\partial\lambda_1\partial\lambda_2}+\frac{\partial\tilde{V}_{m_0}}{\partial\lambda_2}-\frac{\partial\tilde{V}_{m_0}}{\partial\lambda_1}=0.$$

This can be verifyed directly, using (2) and (3).

□

3. Explicit reduction to the Lioville case of separation of variables

We demonstrate explicit reduction of such systems to the Liouville case of separation of variables on the example

$$V_{-4}=x^{-4}z^2+\frac{c(c-a)}{b(b-a)}x^{-4}y^2.$$

Then we give complete integral of the Hamilton-Jacobi equation. Recall that separation of variables in the simpler case of $V_{-2} = 1/x^2$ was given by Kozlov in [**3**].

In elliptic coordinates we have

$$\tilde{V}_{-4}(\lambda_1,\lambda_2)=\frac{(a-b)^2(c-\lambda_1)(c-\lambda_2)-(c-a)^2(b-\lambda_1)(b-\lambda_2)}{(a-\lambda_1)^2(a-\lambda_2)^2}.$$

The following proposition is obtained by simple algebraic transformations.

PROPOSITION. *The potential $\tilde{V}_{-4}$ can be expressed in the form:*

$$\tilde{V}_{-4}(\lambda_1,\lambda_2)=\frac{1}{\lambda_1-\lambda_2}(\varphi(\lambda_1)-\varphi(\lambda_2)),$$

where

$$\varphi(\lambda)=\frac{\lambda}{a^2(a-\lambda)^2}((a-b)^2(c-\lambda)c-(c-a)^2(b-\lambda)b).$$

As a corollary we get (see [**6**]):

COROLLARY. *Complete integral of the Hamilton-Jacobi equation is*

$$\mathcal{K} = \int \sqrt{\frac{2\lambda_1}{(a-\lambda_1)(\lambda_1-b)(\lambda_1-c)}\,(h\lambda_1 - \varphi(\lambda_1) + \alpha)}\,d\lambda_1 + \int \sqrt{\frac{2\lambda_2}{(a-\lambda_2)(b-\lambda_2)(\lambda_2-c)}\,(-h\lambda_2 + \varphi(\lambda_2) - \alpha)}\,d\lambda_2,$$

where α, h are constants representing values of two integrals in involution.

In the case of polynomial potential perturbations (the case $m_0 = 0$ from [**5**]) there is nothing new comparing to the situation considered by Kozlov. The polynomials differ from the Kozlov's potential $V(x,y,z) = k(x^2+y^2+z^2)$ by the constants, according to the constrained condition (1).

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