

ON THE NON-COMMUTATIVE
NEUTRIX PRODUCT OF x_+^λ AND $x_+^{-\lambda-r}$

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Abstract. The non-commutative neutrix product of the distributions x_+^λ and $x_+^{-\lambda-r}$ is evaluated for $\lambda \neq 0, \pm 1, \pm 2, \dots$

In the following, we let $\rho(x)$ be an infinitely differentiable function having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$, (iii) $\rho(x) = \rho(-x)$,
(ii) $\rho(x) \geq 0$, (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let \mathcal{D} be the space of infinitely differentiable functions with compact support and let \mathcal{D}' be the space of distributions defined on \mathcal{D} . Then if f is an arbitrary distribution in \mathcal{D}' , we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. Let f and g be distributions in \mathcal{D}' for which on the interval (a, b) , f is the k -th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$

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is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^k \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2. Let f and g be distributions in \mathcal{D}' and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\text{N-}\lim_{n \rightarrow \infty} \langle f(x)g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in \mathcal{D} with support contained in the interval (a, b) , where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It was proved in [4] that if the product fg exists by Definition 1, then the product $f \circ g$ exists by Definition 2 and the two are equal.

The following theorem was proved in [5].

THEOREM 1. *The neutrix product $x_+^\lambda \circ x_+^{-\lambda-1}$ exists and*

$$x_+^\lambda \circ x_+^{-\lambda-1} = x_+^{-1} - [\gamma + \frac{1}{2} \psi(-\lambda) + \frac{1}{2} \psi(\lambda + 1) + 2c(\rho)] \delta(x)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$, where γ denotes Euler's constant and

$$\psi(\mu) = \frac{\Gamma'(\mu)}{\Gamma(\mu)}.$$

Before proving our main result we need the following definition of the Beta function given in [6].

Definition 3. The Beta function $B(\lambda, \mu)$ is defined for all λ, μ by

$$B(\lambda, \mu) = \text{N-}\lim_{n \rightarrow \infty} \int_{1/n}^{1-1/n} t^{\lambda-1} (1-t)^{\mu-1} dt.$$

It was proved that if $\lambda, \mu \neq 0, -1, -2, \dots$, then the above definition is in agreement with the standard definition of the Beta function.

In particular, it was proved in [6] that

$$B(-r, \lambda) = \frac{(-1)^r \Gamma(\lambda)}{r! \Gamma(\lambda - r)} [\phi(r) - \gamma - \psi(\lambda - r)]$$

for $r = 0, 1, 2, \dots$ and $\lambda, \mu \neq 0, -1, -2, \dots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r = 1, 2, \dots, \\ 0, & r = 0. \end{cases} \quad (1)$$

We now generalize theorem 1 in which the distribution x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^r$$

for $r = 1, 2, \dots$ and not as in Gel'fand and Shilov [7].

THEOREM 2. *The neutrix product $x_+^\lambda \circ x_+^{-\lambda-r}$ exists and*

$$x_+^\lambda \circ x_+^{-\lambda-r} = x_+^{-r} + a_r(\lambda) \delta^{(r-1)}(x), \quad (2)$$

where

$$a_r(\lambda) = \frac{(-1)^r [\gamma + 2c(\rho) + \frac{1}{2} \psi(\lambda + 1) + \frac{1}{2} \psi(-\lambda - r + 1) - \phi(r - 1)]}{(r-1)!} + \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1},$$

for $r = 1, 2, \dots$ and $\lambda \neq 0, \pm 1, \pm 2, \dots$.

Proof. We first of all suppose that $-1 < \lambda < 0$ and put

$$\begin{aligned} (x_+^{-\lambda-r})_n &= x_+^{-\lambda-r} * \delta_n(x) \\ &= \begin{cases} \frac{(-1)^{r-1} \Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \int_{-1/n}^{1/n} (x-t)^{-\lambda-1} \delta_n^{(r-1)}(t) dt, & x > 1/n, \\ \frac{(-1)^{r-1} \Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \int_{-1/n}^x (x-t)^{-\lambda-1} \delta_n^{(r-1)}(t) dt, & -1/n \leq x \leq 1/n, \\ 0, & x < -1/n. \end{cases} \end{aligned}$$

Then

$$\begin{aligned}
& \frac{(-1)^{r-1}\Gamma(\lambda+r)}{\Gamma(\lambda+1)} \int_{-1}^1 x_+^\lambda (x_+^{-\lambda-r})_n x^i dx \\
&= \int_0^{1/n} x^{\lambda+i} \int_{-1/n}^x (x-t)^{-\lambda-1} \delta_n^{(r-1)}(t) dt dx \\
&\quad + \int_{1/n}^1 x^{\lambda+i} \int_{-1/n}^{1/n} (x-t)^{-\lambda-1} \delta_n^{(r-1)}(t) dt dx \\
&\quad + \int_0^{1/n} \delta_n^{(r-1)}(t) \int_t^1 x^{\lambda+i} (x-t)^{-\lambda-1} dx dt \\
&\quad + \int_{-1/n}^0 \delta_n^{(r-1)}(t) \int_0^1 x^{\lambda+i} (x-t)^{-\lambda-1} dx dt \\
&= n^{r-i-1} \int_0^1 \rho^{(r-1)}(v) \int_v^n u^{\lambda+i} (u-v)^{-\lambda-1} du dv \\
&\quad - (-1)^r n^{r-i-1} \int_0^1 \rho^{(r-1)}(v) \int_0^n u^{\lambda+i} (u+v)^{-\lambda-1} du dv,
\end{aligned} \tag{3}$$

where the substitutions $nt = v$ and $nx = u$ have been made in the first integral and $nt = -v$ and $nx = u$ in the second integral.

We have

$$\begin{aligned}
& \int_v^n u^{\lambda+i} (u-v)^{-\lambda-1} du - (-1)^r \int_0^n u^{\lambda+i} (u+v)^{-\lambda-1} du = \\
&= \int_v^n u^{\lambda+i} [(u-v)^{-\lambda-1} - (-1)^r (u+v)^{-\lambda-1}] du - (-1)^r \int_0^v u^{\lambda+i} (u+v)^{-\lambda-1} du
\end{aligned}$$

and it follows for the cases $i = 0, 1, \dots, r-2$ that

$$\begin{aligned}
& \text{N-}\lim_{n \rightarrow \infty} n^{r-i-1} \left[\int_v^n u^{\lambda+i} (u-v)^{-\lambda-1} du - (-1)^r \int_0^n u^{\lambda+i} (u+v)^{-\lambda-1} du \right] = \\
&= \text{N-}\lim_{n \rightarrow \infty} n^{r-i-1} \int_v^n u^{\lambda+i} [(u-v)^{-\lambda-1} - (-1)^r (u+v)^{-\lambda-1}] du \\
&= \text{N-}\lim_{n \rightarrow \infty} n^{r-i-1} \sum_{j=0}^{\infty} \binom{-\lambda-1}{j} [(-1)^j - (-1)^r] v^j \int_v^n u^{i-j-1} du \\
&= \frac{2(-1)^r}{r-i-1} \binom{-\lambda-1}{r-1} v^{r-1} = -\frac{2\Gamma(\lambda+r)}{(r-i-1)(r-1)!\Gamma(\lambda+1)} v^{r-1}.
\end{aligned}$$

It follows that

$$\text{N-}\lim_{n \rightarrow \infty} \int_{-1}^1 x_+^\lambda (x_+^{-\lambda-r})_n x^i dx = -\frac{1}{r-i-1} \tag{4}$$

for $i = 0, 1, \dots, r - 2$, since it is easily proved by induction that

$$\int_0^1 v^r \rho^{(r)}(v) dv = \frac{1}{2}(-1)^r r!$$

When $i = r - 1$, we have on making the substitution $u = v/y$

$$\begin{aligned} \int_v^n u^{\lambda+r-1} (u-v)^{-\lambda-1} du &= v^{r-1} \int_{v/n}^1 y^{-r} (1-y)^{-\lambda-1} dy \\ &= v^{r-1} \int_{v/n}^1 y^{-r} \left[(1-y)^{-\lambda-1} - \sum_{j=0}^{r-1} \binom{-\lambda-1}{j} (-y)^j \right] dy \\ &\quad + v^{r-1} \sum_{j=0}^{r-1} (-1)^j \binom{-\lambda-1}{j} \int_{v/n}^1 y^{j-r} dy \\ &= v^{r-1} \int_{v/n}^1 y^{-r} \left[(1-y)^{-\lambda-1} - \sum_{j=0}^{r-1} \binom{-\lambda-1}{j} (-y)^j \right] dy \\ &\quad + v^{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{-\lambda-1}{j} \frac{1 - (n/v)^{r-j-1}}{j-r+1} \\ &\quad - \binom{-\lambda-1}{r-1} (-v)^{r-1} (\ln v - \ln n). \end{aligned}$$

It follows that

$$\text{N-}\lim_{n \rightarrow \infty} \int_v^n u^{\lambda+r-1} (u-v)^{-\lambda-1} du = v^{r-1} B(-r+1, -\lambda) - \binom{-\lambda-1}{r-1} (-v)^{r-1} \ln v$$

and so

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} \int_0^1 \rho^{(r-1)}(v) \int_v^n u^{\lambda+r-1} (u-v)^{-\lambda-1} du dv &= \tag{5} \\ &= \frac{1}{2}(-1)^{r-1} (r-1)! B(-r+1, -\lambda) - \binom{-\lambda-1}{r-1} (r-1)! \left[\frac{1}{2} \phi(r-1) + c(\rho) \right] \\ &= \frac{1}{2}(-1)^{r-1} (r-1)! B(-r+1, -\lambda) + \frac{(-1)^r \Gamma(\lambda+r)}{\Gamma(\lambda+1)} \left[\frac{1}{2} \phi(r-1) + c(\rho) \right] \end{aligned}$$

since it is easily proved by induction that

$$\int_0^1 v^r \ln v \rho^{(r)}(v) dv = \frac{1}{2}(-1)^r r! \phi(r) + (-1)^r r! c(\rho).$$

Further, making the substitution $u = v(y^{-1} - 1)$, we have

$$\begin{aligned}
\int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} du &= v^{r-1} \int_{v/(n+v)}^1 y^{-r}(1-y)^{\lambda+r-1} dy \\
&= v^{r-1} \int_{v/(n+v)}^1 y^{-r} \left[(1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1} \binom{\lambda+r-1}{j} (-y)^j \right] dy + \\
&\quad + v^{r-1} \sum_{j=0}^{r-1} (-1)^j \binom{\lambda+r-1}{j} \int_{v/(n+v)}^1 y^{j-r} dy \\
&= v^{r-1} \int_{v/(n+v)}^1 y^{-r} \left[(1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1} \binom{\lambda+r-1}{j} (-y)^j \right] dy + \\
&\quad + v^{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{\lambda+r-1}{j} \frac{1 - (n/v+1)^{r-j-1}}{j-r+1} +
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{N-}\lim_{n \rightarrow \infty} \int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} du &= v^{r-1} B(-r+1, \lambda+r) + \\
&\quad + v^{r-1} \sum_{j=0}^{r-2} \binom{\lambda+r-1}{j} \frac{(-1)^j}{r-j-1} + \binom{\lambda+r-1}{r-1} (-1)^r v^{r-1} \ln v
\end{aligned}$$

and so

$$\begin{aligned}
\text{N-}\lim_{n \rightarrow \infty} \int_0^1 \rho^{(r-1)}(v) \int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} du dv &= \frac{1}{2} (-1)^{r-1} (r-1)! B(-r+1, \lambda+r) \\
&\quad + \frac{1}{2} (-1)^{r-1} (r-1)! \sum_{j=0}^{r-2} \binom{\lambda+r-1}{j} \frac{(-1)^j}{r-j-1} \\
&\quad - \binom{\lambda+r-1}{r-1} (r-1)! \left[\frac{1}{2} \phi(r-1) + c(\rho) \right] \\
&= \frac{1}{2} (-1)^{r-1} (r-1)! B(-r+1, \lambda+r) \\
&\quad + \frac{1}{2} (-1)^{r-1} (r-1)! \sum_{j=0}^{r-2} \binom{\lambda+r-1}{j} \frac{(-1)^j}{r-j-1} \\
&\quad - \frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)} \left[\frac{1}{2} \phi(r-1) + c(\rho) \right]. \tag{6}
\end{aligned}$$

It follows from equation (1) that

$$\begin{aligned}
B(-r+1, \lambda+r) &= \frac{(-1)^{r-1} \Gamma(\lambda+r)}{(r-1)! \Gamma(\lambda+1)} [\phi(r-1) - \gamma - \psi(\lambda+1)], \\
B(-r+1, -\lambda) &= \frac{(-1)^{r-1} \Gamma(-\lambda)}{(r-1)! \Gamma(-\lambda-r+1)} [\phi(r-1) - \gamma - \psi(-\lambda-r+1)] \\
&= \frac{\Gamma(\lambda+r)}{(r-1)! \Gamma(\lambda+1)} [\phi(r-1) - \gamma - \psi(-\lambda-r+1)]
\end{aligned}$$

and so

$$\begin{aligned}
B(-r+1, -\lambda) - (-1)^r B(-r+1, \lambda+r) &= \\
&= \frac{\Gamma(\lambda+r)}{(r-1)! \Gamma(\lambda+1)} [2\phi(r-1) - 2\gamma - \psi(\lambda+1) - \psi(-\lambda-r+1)]. \quad (7)
\end{aligned}$$

It now follows from equations (3), (5), (6) and (7) that

$$\begin{aligned}
\text{N-}\lim_{n \rightarrow \infty} \frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)} \int_{-1}^1 x_+^\lambda (x_+^{-\lambda-r})_n x^{r-1} dx &= \\
&= \frac{1}{2} (r-1)! [B(-r+1, -\lambda) - (-1)^r B(-r+1, \lambda+r)] \\
&\quad - \frac{1}{2} (r-1)! \sum_{j=0}^{r-2} \binom{\lambda+r-1}{j} \frac{(-1)^{r-j}}{r-j-1} - \frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)} [\phi(r-1) + 2c(\rho)] \\
&= \frac{(-1)^{r-1} (r-1)! \Gamma(\lambda+r)}{\Gamma(\lambda+1)} b_r(\lambda), \quad (8)
\end{aligned}$$

where

$$\begin{aligned}
b_r(\lambda) &= \frac{(-1)^r [\gamma + 2c(\rho) + \frac{1}{2} \psi(\lambda+1) + \frac{1}{2} \psi(-\lambda-r+1)]}{(r-1)!} + \\
&\quad + \frac{\Gamma(\lambda+1)}{2\Gamma(\lambda+r)} \sum_{j=0}^{r-2} \binom{\lambda+r-1}{j} \frac{(-1)^j}{r-j-1}.
\end{aligned}$$

It was proved in [3] that

$$\begin{aligned}
\langle x_+^{-r}, \varphi(x) \rangle &= \\
&= \int_0^\infty x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i - \frac{\varphi^{(r-1)}(0)}{(r-1)!} x^{r-1} H(1-x) \right] dx + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0),
\end{aligned}$$

for all φ in \mathcal{D} . In particular, if the support of φ is contained in the interval $[-1, 1]$, we have

$$\begin{aligned}
\langle x_+^{-r}, \varphi(x) \rangle &= \quad (9) \\
&= \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} \int_1^\infty x^{-r+i} dx - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) \\
&= \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0).
\end{aligned}$$

Now let φ be an arbitrary function in \mathcal{D} with support contained in the interval $[-1, 1]$. By the mean value theorem

$$\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,$$

where $0 < \xi < 1$ and so

$$\begin{aligned} \langle x_+^\lambda (x_+^{-\lambda-r})_n, \varphi(x) \rangle &= \int_0^1 x^\lambda (x_+^{-\lambda-r})_n \varphi(x) dx \\ &= \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_0^1 x^{\lambda+i} (x_+^{-\lambda-r})_n dx + \frac{1}{r!} \int_0^1 x^\lambda [x^r (x_+^{-\lambda-r})_n] \varphi^{(r)}(\xi x) dx. \end{aligned}$$

Since the sequence of continuous functions $\{x^r (x_+^{-\lambda-r})_n\}$ converges uniformly to the continuous function $x^{-\lambda}$ on the closed interval $[0, 1]$, it follows on using equations (4), (8) and (9) that

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} \langle x_+^\lambda (x_+^{-\lambda-r})_n, \varphi(x) \rangle &= \\ &= \text{N-}\lim_{n \rightarrow \infty} \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^1 x_+^{\lambda+i} (x_+^{-\lambda-r})_n dx + \lim_{n \rightarrow \infty} \frac{1}{r!} \int_{-1}^1 x_+^\lambda [x^r (x_+^{-\lambda-r})_n] \varphi^{(r)}(\xi x) dx \\ &= \frac{1}{r!} \int_0^1 \varphi^{(r)}(\xi x) dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + (-1)^{r-1} b_r(\lambda) \varphi^{(r-1)}(0) \\ &= \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) \\ &\quad + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) + (-1)^{r-1} b_r(\lambda) \varphi^{(r-1)}(0) \\ &= \langle x_+^{-r}, \varphi(x) \rangle + (-1)^{r-1} a_r(\lambda) \varphi^{(r-1)}(0), \end{aligned}$$

giving equation (1) on the interval $[-1, 1]$ when $-1 < \lambda < 0$. However, since $x_+^\lambda \cdot x_+^{-\lambda-r} = x_+^{-r}$ on any closed interval not containing the origin, equation (1) holds on the real line when $-1 < \lambda < 0$.

Now suppose that equation (2) holds for some r and $\lambda \neq 0, \pm 1, \pm 2, \dots$. This is true for $r = 1$. It is also true for $r + 1$ when $-1 < \lambda < 0$. Assume it is also true for $r + 1$ when $-k < \lambda < -k + 1$. Then if $-k < \lambda < -k + 1$, equation (2) holds by our assumption and differentiating this equation, we get

$$\lambda x_+^{\lambda-1} \circ x_+^{-\lambda-r} - (\lambda + r) x_+^\lambda \circ x_+^{-\lambda-r-1} = -r x_+^{-r-1} + a_r(\lambda) \delta^{(r)}(x).$$

It follows from our assumptions that

$$\lambda x_+^{\lambda-1} \circ x_+^{-\lambda-r} = \lambda x_+^{-r-1} + [(\lambda + r) a_{r+1}(\lambda) + a_r(\lambda)] \delta^{(r)}(x).$$

We have

$$\begin{aligned}
(\lambda + r)a_{r+1}(\lambda) + a_r(\lambda) &= \frac{(-1)^{r-1}(\lambda + r)}{r!} [\gamma + \frac{1}{2}\psi(-\lambda - r) + \frac{1}{2}\psi(\lambda + 1) + 2c(\rho)] \\
&+ \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-1} \binom{\lambda + r}{j} \frac{(-1)^j}{r-j} + \frac{(-1)^r(\lambda + r)\phi(r)}{r!} \\
&+ \frac{(-1)^r}{(r-1)!} [\gamma + \frac{1}{2}\psi(-\lambda - r + 1) + \frac{1}{2}\psi(\lambda + 1) + 2c(\rho)] \\
&+ \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r-j-1} - \frac{(-1)^r\phi(r-1)}{(r-1)!}.
\end{aligned}$$

Noting that

$$\begin{aligned}
(\lambda + r)\psi(-\lambda - r) &= 1 + (\lambda + r)\psi(-\lambda - r + 1), \\
\lambda\psi(\lambda + 1) &= 1 + \lambda\psi(\lambda), \\
\sum_{j=0}^{r-1} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r-j} &= \sum_{j=0}^{r-1} \binom{\lambda + r}{j} \frac{(-1)^j}{r-j} + \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r-j-1},
\end{aligned}$$

it follows that

$$(\lambda + r)a_{r+1}(\lambda) + a_r(\lambda) = \lambda a_{r+1}(\lambda - 1)$$

and we see that equation (2) holds when $-k - 1 < \lambda < -k$.

Equation (2) therefore holds by induction for negative $\lambda \neq -1, -2, \dots$ and $r = 1, 2, \dots$. A similar argument shows that equation (2) holds for positive $\lambda \neq 1, 2, \dots$. This completes the proof of the theorem.

COROLLARY 2.1 *The neutrix product $x_-^\lambda \circ x_-^{\lambda-r}$ exists and*

$$x_-^\lambda \circ x_-^{\lambda-r} = x_-^{-r} - (-1)^r a_r(\lambda) \delta^{(r-1)}(x) \quad (10)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$.

Proof. Equation (10) follows immediately on replacing x by $-x$ in equation (2).

In the next corollary, the distribution $(x + i0)^\lambda$ is defined by

$$(x + i0)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and

$$(x + i0)^{-r} = x^{-r} + \frac{(-1)^r i\pi}{(r-1)!} \delta^{(r-1)}(x), \quad (11)$$

for $r = 1, 2, \dots$, see [7].

COROLLARY 2.2 *The neutrix product $(x+i0)^\lambda \circ (x+i0)^{-\lambda-r}$ exists and*

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-r} = (x+i0)^{-r} \quad (12)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$

Proof. The neutrix product is distributive with respect to addition and so

$$\begin{aligned} (x+i0)^\lambda \circ (x+i0)^{-\lambda-r} &= \\ &= x_+^\lambda \circ x_+^{-\lambda-r} + (-1)^r x_-^\lambda \circ x_-^{-\lambda-r} + (-1)^r e^{-i\lambda\pi} x_+^\lambda \circ x_-^{-\lambda-r} + e^{i\lambda\pi} x_-^\lambda \circ x_+^{-\lambda-r}. \end{aligned} \quad (13)$$

Further, it was proved in [4] that

$$x_+^\lambda \circ x_-^{-\lambda-r} = (-1)^{r-1} x_-^\lambda \circ x_+^{-\lambda-r} = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!} \delta^{(r-1)}(x) \quad (14)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$. It follows from equations (2), (10), (11), (13) and (14) that

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-r} = x^{-r} + \frac{(-1)^r i\pi}{(r-1)!} \delta^{(r-1)}(x) = (x+i0)^{-r},$$

proving equation (12).

We finally note that the following results can be proved similarly.

$$\begin{aligned} |x|^\lambda \circ (\operatorname{sgn} x |x|^{-\lambda-2r+1}) &= x^{-2r+1}, \\ |x|^\lambda \circ (\operatorname{sgn} x |x|^{-\lambda-2r}) &= \operatorname{sgn} x |x|^{-2r} + \left[2a_{2r}(\lambda) + \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-1)!} \right] \delta^{(2r-1)}(x), \\ (\operatorname{sgn} x |x|^\lambda) \circ |x|^{-\lambda-2r+1} &= x^{-2r+1}, \\ (\operatorname{sgn} x |x|^\lambda) \circ |x|^{-\lambda-2r} &= \operatorname{sgn} x |x|^{-2r} + \left[2a_{2r}(\lambda) - \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-1)!} \right] \delta^{(2r-1)}(x), \\ |x|^\lambda \circ |x|^{-\lambda-2r+1} &= |x|^{-2r+1} + \left[2a_{2r-1}(\lambda) - \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-2)!} \right] \delta^{(2r-2)}(x), \\ |x|^\lambda \circ |x|^{-\lambda-2r} &= x^{-2r}, \\ (\operatorname{sgn} x |x|^\lambda) \circ (\operatorname{sgn} x |x|^{-\lambda-2r+1}) &= |x|^{-2r+1} + \left[2a_{2r-1}(\lambda) + \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-2)!} \right] \delta^{(2r-2)}(x), \\ (\operatorname{sgn} x |x|^\lambda) \circ (\operatorname{sgn} x |x|^{-\lambda-2r}) &= x^{-2r} \end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$

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