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TOPOLOGICAL ORDER COMPLEXES AND RESOLUTIONS OF DISCRIMINANT SETS

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ABSTRACT. If elements of a partially ordered set run over a topological space, then the corresponding order complex admits a natural topology, providing that similar interior points of simplices with close vertices are close to one another. Such *topological order complexes* appear naturally in the *conical resolutions* of many singular algebraic varieties, especially of *discriminant varieties*, i.e. the spaces of singular geometric objects. These resolutions generalize the *simplicial resolutions* to the case of non-normal varieties. Using these order complexes we study the cohomology rings of many spaces of nonsingular geometrical objects, including the spaces of nondegenerate linear operators in $\mathbf{R}^n$, $\mathbf{C}^n$ or $\mathbf{H}^n$, of homogeneous functions $\mathbf{R}^2 \to \mathbf{R}^1$ without roots of high multiplicity in $\mathbf{RP}^1$, of nonsingular hypersurfaces of a fixed degree in $\mathbf{CP}^n$, and of Hermitian matrices with simple spectra.

1. Introduction

We describe a method of computing the homology groups of many algebraic varieties, especially of *discriminant varieties*, and hence, by the Alexander duality, the cohomology groups of complements of such objects in $\mathbf{R}^n$ or $\mathbf{C}^n$.

This method is based on the notion of a *topological order complex*, associated with a partially ordered set, whose elements run over some topological space. If this space is discrete, then we get a standard order complex and the method of simplicial resolutions, see e.g. [18].

The first five examples of such complexes are listed in the following subsections 1.1–1.5.

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1.1. **Order complex of Grassmannians.** Let $\mathbf{K}$ be one of fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, and $G_i(\mathbf{K}^n)$ the Grassmannian manifold of all i-dimensional subspaces in $\mathbf{K}^n$. The disjoint union of all manifolds $G_i(\mathbf{K}^n)$, $i = 1, \dots, n$, is a poset (partially ordered set) by incidence of planes. The corresponding topological order complex $\Theta(\mathbf{K}^n)$ is defined as follows. Consider the join $G_1(\mathbf{K}^n) * \dots * G_n(\mathbf{K}^n)$, i.e., roughly speaking, the union of all simplices, whose vertices correspond to points of different Grassmannians. Such a simplex is *coherent* if the planes corresponding to all its vertices form a flag. The desired complex $\Theta(\mathbf{K}^n)$ is the union of all coherent simplices. This is a cone with vertex $\{\mathbf{K}^n\} \in G_n(\mathbf{K}^n)$. Let us denote by $\partial\Theta(\mathbf{K}^n)$ its *link*, i.e. the union of coherent simplices not containing this vertex $\{\mathbf{K}^n\}$.

THEOREM 1 (see [43], [40], [32]). *There is a PL-homeomorphism*

$$\partial\Theta(\mathbf{K}^n) \simeq S^M, \quad M = (\dim_{\mathbf{R}} \mathbf{K}) n(n-1)/2 + n - 2. \tag{1.1}$$

REMARK. Probably, this assertion is assumed in Remark 1.4 of [13]. I thank M. M. Kapranov for this reference.

These spaces $\Theta(\mathbf{K}^n)$ have a rich geometrical structure: filtrations, stratifications, etc. Their topological features are closely related to the topology of the corresponding linear groups $GL(n, \mathbf{K})$, see § 4.1.

1.2. **Order complex of configurations in S^1.** Let $\overline{B(S^1, i)}$ be the *configuration space* of unordered collections of i points of the circle S^1 (some of which can coincide).[1] It can be considered as the quotient space of the Cartesian power $(S^1)^i$ by the obvious action of the permutation group $S(i)$ and admits the corresponding quotient topology. The space $\sqcup_{i \geq 1} \overline{B(S^1, i)}$ is a poset by inclusion of configurations. It is convenient to consider $\overline{B(S^1, i)}$ as the space of all ideals of codimension i in the space of smooth functions $S^1 \to \mathbf{R}^1$, then this order relation coincides with the incidence of corresponding ideals. Denote by $\tilde{\Delta}_m(S^1)$ the union of all coherent simplices in the join $\overline{B(S^1, 1)} * \dots * \overline{B(S^1, m)}$.

THEOREM 2. *The space $\tilde{\Delta}_m(S^1)$ is homotopy equivalent to S^{2m-1}.*

The essential part of the proof is a theorem of C. Caratheodory (see § 4.2 below), claiming that the union of all m-vertex simplices spanning m-tuples of points of a generically embedded circle $S^1 \hookrightarrow \mathbf{R}^{2m}$ is homeomorphic to S^{2m-1}.

The geometrical features of the space $\tilde{\Delta}_m(S^1)$ are closely related to the topology of spaces of homogeneous polynomials $\mathbf{R}^2 \to \mathbf{R}^1$ of degree d without roots of multiplicity $\geq k$ in $\mathbf{RP}^1$, where $m = [d/k]$.

1.3. **Order complex of multigrassmannians.** Let $A = (a_1 \geq \dots \geq a_k)$ be a monotone sequence of natural numbers, $a_k \geq 2$, and $n \geq \sum a_i$. Denote by $\Gamma_A(n)$ the space of all unordered collections of k pairwise Hermitian-orthogonal complex subspaces of dimensions $a_1, \dots, a_k$ in $\mathbf{C}^n$. If all numbers a_i are different, then it coincides with the flag manifold $F(a_1, a_1 + a_2, \dots, a_1 + \dots + a_k)$, otherwise it is the

[1]The more standard notation for this space is $SP^i(S^1)$. We use a different notation, because in our applications for any manifold M some such spaces $\overline{B(M, i)}$ should be considered, which only occasionally coincide with $SP^i(M)$ if M is one-dimensional

quotient space of such a manifold by the action of the corresponding permutation group. In any case, this is a smooth compact complex manifold. The disjoint union of all such manifolds with a given n is a topological poset: a point $\gamma \in \Gamma_A(n)$ is subordinate to $\gamma' \in \Gamma'_{A'}(n)$ if any of k planes defining γ lies in some of k' planes defining γ'. The corresponding topological order complex is a cone with the vertex $\{\mathbf{C}^n\} \in \Gamma_{(n)}(n)$.

THEOREM 3 (see [52]). *The reduced rational homology group of the link of this order complex is trivial in all dimensions of the same parities as* n. *For instance, for* $n = 2, 3, 4$ *and* 5 *the Poincaré polynomials of these groups are equal to* t, $t^2(1+t^2)$, $t^3(1+t^4)(1+t^2+t^4)$ *and* $t^4(1+t^2+t^4+t^6)(1+t^2+t^4+t^6+t^8+t^{10})$ *respectively.*

In [52] a recursive method of calculating all these homology groups is proposed, however any compact formula for their dimensions is unknown for me.

The geometrical features of this complex are closely related to the topology of the space of Hermitian matrices without multiple eigenvalues in $\mathbf{C}^n$, see § 5.2 below and [52].

1.4. **Order complex of multiconfigurations.** Consider a multiindex $A = (a_1 \geq \dots \geq a_k)$, where all numbers a_i are natural and greater than 1. Given a topological space M (say, $M = S^1$), a *multiconfiguration of type* A in M is any collection of $a_1 + \dots + a_k$ distinct points in M divided into groups of cardinalities $a_1, \dots, a_k$. Denote by $V(M, A)$ the set of all A-configurations in M. It is convenient to consider any such configuration as a subspace (even a subring) in the space of continuous (or smooth if M is a manifold) functions $M \to \mathbf{R}^1$: namely, as the space of all functions taking equal values at the points of any group. The codimension of this subspace is equal to $\sum_{i=1}^k (a_i - 1)$, therefore this number is called the *complexity* of the multiindex A and of any multiconfiguration of type A. Let $\overline{V(M, A)}$ be the closure of $V(M, A)$ in the corresponding Grassmannian topology.

EXAMPLE. Suppose that $M = S^1$, $k = 1$ and $a_1 = 2$. Then the space $V(M, A)$ is the configuration space $B(S^1, 2)$, i.e. an open Möbius band, and $\overline{V(M, A)}$ is the space $\overline{B(S^1, 2)}$, i.e. a closed Möbius band. However, the "function-theoretical" interpretation of these spaces differs from the one given in § 1.2. E.g., we consider the exceptional point (a, a) not as the ideal generated by the function $(x - a)^2$ but as the space of functions $S^1 \to \mathbf{R}^1$ such that $f'(a) = 0$.

For any natural d, the disjoint union of spaces $\overline{V(M, A)}$ over all admissible multiindices A of complexity $\leq d$ is a poset by a natural subordination of multiconfigurations (this subordination can be interpreted as the inverse inclusion of corresponding functional subspaces).

The topological and geometrical study of corresponding topological order complexes $\Omega(S^1, d)$ is known as the theory of finite-order knot invariants (and other cohomology classes of spaces of knots in $\mathbf{R}^n$, $n \geq 3$). Namely, the homology group of the quotient space $\Omega(S^1, d)/\Omega(S^1, d-1)$ is the first[2] step of the calculation of

[2]and, accordingly to M. Kontsevich, in the case of rational coefficients also the last

all such invariants and classes of order d modulo similar classes of order $d-1$, see § 5.1.

The knot theory is the simplest example of a theory *of second order*. Indeed, a smooth map $f: S^1 \to \mathbf{R}^n$ is not a knot if it satisfies some condition at some ≤ 2 points of S^1, namely the condition $f(a) = f(b)$ or $f'(a) = 0$.

In a similar way, we can consider theories of order 3, e.g. the classification of curves without triple (self)intersections. The corresponding order complex is defined in exactly the same way, but the elements a_i of an admissible multiindex A should be greater than or equal to 3.

1.5. **Order complex of singular sets of projective algebraic hypersurfaces.** For any natural d and n denote by $P(d,n)$ the space of homogeneous polynomials $\mathbf{C}^{n+1} \to \mathbf{C}^1$ of degree d, and consider all possible singular sets of hypersurfaces in $\mathbf{CP}^n$ defined by such polynomials. Any such set ξ defines a linear subspace in $P(d,n)$, consisting of polynomials, whose singular set contains ξ. Let us denote by $I(d,n,i)$ the set of all linear subspaces of codimension i defined in this way, and by $\overline{I(d,n,i)}$ its closure in the Grassmannian manifold of all subspaces of this codimension.

The disjoint union of all sets $\overline{I(d,n,i)}$ with arbitrary values of i is a poset; this is a subposet of the one considered in § 1.1 with $\mathbf{C}^n$ replaced by $P(d,n)$. The corresponding order complex $\Lambda(d,n)$ is a cone with the vertex {zero polynomial}. It plays an important role in the calculation of cohomology groups of spaces of nonsingular algebraic hypersurfaces in $\mathbf{CP}^n$, see § 4.3 below.

If $d = 2$, then it coincides with the complex $\Theta(\mathbf{C}^{n+1})$ considered in § 1.1, and if $n = 1$ then with the obvious "complexification" $\tilde{\Delta}_{[d/2]}(\mathbf{CP}^1)$ of the complex $\tilde{\Delta}_{[d/2]}(\mathbf{RP}^1)$ considered in § 1.2.

THEOREM 4 (see [49]). *For $(d,n) = (3,2)$ or $(3,3)$ the rational homology group of the link $\partial\Lambda(d,n)$ of our order complex $\Lambda(d,n)$ is trivial in all positive dimensions. The Poincaré polynomial of the rational homology group of the link $\partial\Lambda(4,2)$, reduced modulo a point, is equal to $t^{14}(1+t^3)(1+t^5)$.*

PROBLEM. Is the first assertion of this theorem true for any pair (d,n) with $d = 3$?

The explicit calculations in these complexes provide a plenty of nice and natural problems on configuration spaces and their homology groups.

EXAMPLE. Let be $n = 2, d = 4$. There are exactly the following singular sets of curves of order 4 in $\mathbf{CP}^2$ (in angular parentheses we indicate the dimension of the linear space of all polynomials of degree 4 having singular points at some set of the corresponding type):

1. Any point in $\mathbf{CP}^1$ $\langle 12 \rangle$
2. Any couple of points in $\mathbf{CP}^2$ $\langle 9 \rangle$
3. Any triple of points on the same line $\langle 7 \rangle$
4. Any triple of points not on the same line $\langle 6 \rangle$
5. Any line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ $\langle 6 \rangle$

6. Any three points on the same line plus one point not on this line $\langle 4 \rangle$
7. Any generic quadruple of points (i.e. none three of them lie on the same line; the corresponding polynomials are products of two quadrics) $\langle 3 \rangle$
8. Any line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ plus a point outside it $\langle 3 \rangle$
9. Any five points, four of which are in general position, and the fifth is the intersection point of some two lines passing through two non-intersecting pairs from this quadruple (the corresponding polynomials split into products of two linear functions and one quadric, however this quadric isn't defined uniquely by our points) $\langle 2 \rangle$
10. Six points, which are intersection points of some four lines in general position $\langle 1 \rangle$
11. Any non-singular quadric in $\mathbf{CP}^2$ $\langle 1 \rangle$
12. Any pair of lines $\langle 1 \rangle$
13. Entire space $\mathbf{CP}^2$, defined by the zero polynomial $\langle 0 \rangle$

In some sense, all the nonzero homology groups of $\partial\Lambda(4,2)$, mentioned in the last statement of Theorem 4, are provided by the stratum No. 10.

Let me formulate also a funny fact, which is essential for these calculations. Let $B(\mathbf{CP}^n, m)$ be the configuration space of all unordered collections of m distinct points in $\mathbf{CP}^n$. Let $\pm\mathbf{C}$ be the *sign local system* on it, i.e. the local system with fiber $\mathbf{C}$ such that the loops in $B(\mathbf{CP}^n, m)$, providing odd permutations of m points, act as multiplication by -1 in the fibre. Then the cohomology group of $B(\mathbf{CP}^n, m)$ with coefficients in this local system coincides with the usual cohomology group of the Grassmannian manifold $G_m(\mathbf{C}^{n+1})$:

$$H^i(B(\mathbf{CP}^n, m), \pm\mathbf{C}) \simeq H^{i+m-m^2}(G_m(\mathbf{CP}^{n+1}), \mathbf{C}).$$

In particular, this group is trivial in all dimensions if $m > n+1$.

For some other important examples and results on topological order complexes see [55] and bibliography there.

1.6. **Discriminants and their complements.** The general notion of a discriminant is as follows. Consider any function space $\mathcal{F}$, finitedimensional or not, and some class of singularities S which the functions from $\mathcal{F}$ can take at the points of the source manifold. The corresponding *discriminant variety* $\Sigma(S) \subset \mathcal{F}$ is the space of all functions that have such singular points. For example, let $\mathcal{F}$ be the space of (real or complex) polynomials of the form

$$x^d + a_1 x^{d-1} + \cdots + a_d, \tag{1.2}$$

and $S = \{\text{a multiple root}\}$. Then (in the complex case) $\Sigma(S)$ is the zero level set of the usual discriminant polynomial depending on the coefficients a_i; this is a motivation for the word "discriminant" in the general situation.

Many important topological spaces can be described as the complements of suitably defined discriminants. For instance, so are: spaces of polynomials without roots of multiplicity $\geq k$ $(k \geq 2)$; spaces of nondegenerate endomorphisms of $\mathbf{R}^n$,

$\mathbf{C}^n$ or $\mathbf{H}^n$ (homotopy equivalent to classical Lie groups $O(n)$, $U(n)$ and $Sp(n)$ respectively); spaces of Morse or generalized Morse functions on a manifold M or, more generally, spaces of smooth maps $M \to \mathbf{R}^n$ without complicated singularities; loop spaces ΩY and, more generally, spaces of maps $X \to Y$ where X is an m-dimensional cell complex and Y an $(m-1)$-connected one; spaces of based rational maps $\mathbf{C}P^1 \to \mathbf{C}P^n$ of fixed degree; spaces of nonsingular algebraic hypersurfaces of a given degree in $\mathbf{RP}^n$ or $\mathbf{CP}^n$; spaces of knots and links in n-dimensional manifolds, $n \geq 3$; spaces of generic plane curves; spaces of symmetric, Hermitian or hyper-Hermitian matrices in $\mathbf{R}^n$, $\mathbf{C}^n$ or $\mathbf{H}^n$; and many others, see e.g. [40], [39].

The regular topological study of such spaces was started by V. I. Arnold in [3]: he considered the space of complex polynomials without multiple roots in $\mathbf{C}^1$, i.e. the classifying space of the braid group. A seminal idea was proposed in this work: instead of studying the spaces of non-singular objects (which usually are open manifolds without any transparent geometrical structure) it is convenient to consider the complementary discriminant spaces, which have natural stratifications (corresponding to the classification of singular points), and try to express the topological features of the latter spaces in terms of these stratification.

In 1985, solving the Arnold's problem on the stable cohomology ring of complements of bifurcation diagrams, I found that such homology groups should be calculated by a generalization of *simplicial resolutions* of discriminants, see [41]. Numerous subsequent exercises (see e.g. [40], [39], [42], [45], [47], [50]) usually gave the strongest results on the homology groups in the particular problems, in which the considered discriminant varieties are normal (or at least their singular sets "of infinite multiplicity" are nonessential for the topological type). This method is briefly described in the next sections 2, 3.

However, there are many important situations, when this assumption fails; some of them are listed in the Abstract and are mentioned in the conclusions of subsections 1.1–1.5.

In sections 4 and 5.2 below we describe *conical resolutions* of discriminant spaces, which generalize the simplicial resolutions and are based on the topological order complexes like the ones considered in the previous subsections.

2. Order complexes of discrete posets and simplicial resolutions of subspace arrangements

In this section we consider a model application of simplicial resolutions: to the theory of plane arrangements, cf. [19], [21], [54], [44]. There are two different constructions of simplicial resolutions in this theory, see [54], [44]; I will follow here the one from [54].

Definition. Let $(A, \geq)$ be a discrete partially ordered set. The corresponding *order complex* $\Delta(A)$ (see e.g. [14]) is the simplicial complex, whose vertices are the elements of A, and simplices span all strictly monotone finite sequences $\{a_1 < \dots < a_m\}$, $a_i \in A$.

Consider any *affine plane arrangement* $\mathcal{L}$, i.e. a finite collection of affine subspaces $L_1, \dots, L_k$ of arbitrary dimensions in $\mathbf{R}^N$. Set $L = \cup L_i$, and, for any set of

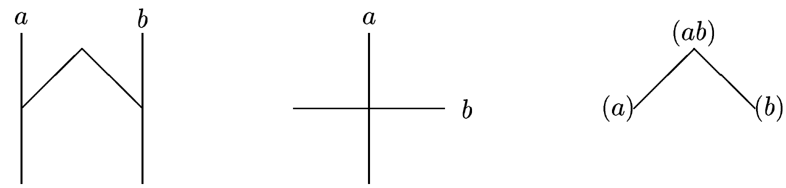

FIGURE 1. Resolution of a cross

indices $I \subset \{1, \ldots, k\}$, $L_I \equiv \cap_{i \in I} L_i$. Then all possible nonempty planes L_I form a partially ordered set (by inverse inclusion). Denote by $\Delta(\mathcal{L})$ the corresponding order complex. The simplicial resolution of the variety L can be constructed as a subset of the Cartesian product $\Delta(\mathcal{L}) \times \mathbf{R}^N$. Namely, for any plane of the form L_I we define the corresponding order subcomplex $\Delta(L_I) \subset \Delta(\mathcal{L})$ as the union of simplices, all whose vertices are subordinate to $\{L_I\}$, i.e. correspond to planes L_J *containing* L_I. This is a cone with vertex $\{L_I\}$. Let us denote by $\partial\Delta(L_I)$ its *link*, i.e. the union of all simplices in $\Delta(L_I)$ not containing the vertex $\{L_I\}$.

The resolution space $L' \subset \Delta(\mathcal{L}) \times \mathbf{R}^N$ is defined as the union of all spaces of the form $\Delta(L_I) \times L_I$ over all geometrically different planes L_I. The obvious projection $\Delta(\mathcal{L}) \times \mathbf{R}^N \to \mathbf{R}^N$ induces a map $L' \to L$. This map is proper and all its fibers are contractible complexes of the form $\Delta(L_I)$. It follows easily that this map is a homotopy equivalence, and moreover, its extension to the map of one-point compactifications, $\bar{L}' \to \bar{L}$, also is a homotopy equivalence.

EXAMPLE. Let L be the union of two crossing lines a and b in $\mathbf{R}^2$, see the middle part of Fig. 1. The corresponding order complex $\Delta(\mathcal{L})$ consists of two segments (see the right-hand part of Fig. 1) joining the vertices (a) and (b) (corresponding to these lines) to the vertex (ab) (corresponding to the intersection point). The resolution space L' consists of three complexes: the line $(a) \times a$, the line $(b) \times b$, and the complex $\Delta(\mathcal{L}) \times (a \cap b)$, see the left part of the picture.

In the general case, this resolution space L' has a natural increasing filtration $F_1 \subset F_2 \subset \cdots \subset F_{N-1} = L'$: its term F_m is the union of all spaces $\Delta(L_I) \times L_I$ over all planes L_I of *codimension* $\leq m$ in $\mathbf{R}^N$. The difference $F_m \setminus F_{m-1}$ is the disjoint union of spaces $(\Delta(L_I) \setminus \partial\Delta(L_I)) \times L_I$ over all planes L_I of dimension exactly $N - m$. Also we get a filtration $\bar{F}_0 \subset \bar{F}_1 \subset \cdots \subset \bar{F}_{N-1} = \bar{L}'$ of the one-point compactification $\bar{L}'$ of the space L': its term $\bar{F}_0$ is the added pont, and any space $\bar{F}_i$, $i > 0$, is the closure of the corresponding subspace $F_i \subset L'$.

The results of [54] imply in particular that this filtration homotopically splits: there is homotopy equivalence

$$\bar{L}' \sim \bar{F}_1 \vee (\bar{F}_2/\bar{F}_1) \vee \ldots \vee (\bar{F}_{N-1}/\bar{F}_{N-2}). \tag{2.1}$$

(An equivalent result was obtained in [44].)

This formula implies the Goresky–MacPherson formula for the cohomology of the complementary space $\mathbf{R}^N \setminus L$ (see [21]), and also the fact that the stable homotopy type of this space is determined by the dimensions of all planes L_I.

3. Resolutions of swallowtails

The previous discrete construction has many continuous generalizations. Namely, we can consider the unions of infinitely many planes parametrized by some topological spaces. Such an union can be resolved in almost the same way as before, provided that only a finite number of planes can meet at the same point: the only additional difficulty here appears when collections of intersecting planes tend to degenerated mutual dispositions, where the dimensions of their intersection sets jump. Here is a simplest example.

Consider the space P_d of *real* polynomials of the form (1.2) and consider the subspace $\Sigma_k \subset P_d$ of polynomials having at least one root of multiplicity $\geq k$. In the simplest topologically non-trivial case, when $d = 4$ and $k = 2$, the discriminant Σ_k is ambient diffeomorphic to the direct product of the line $\mathbf{R}^1$ and the *swallowtail*, i.e. the variety shown in the lower part of Fig. 2.

Remark. The factor $\mathbf{R}^1$ here is the group of translations in the argument line. This group acts freely on the discriminant. All forthcoming constructions are invariant under this action, and we shall indicate them in our three-dimensional picture, keeping in mind that everything should be multiplied by $\mathbf{R}^1$.

The points of cuspidal edges in this picture correspond to the functions with a root of multiplicity 3, the very singular point corresponds to the function x^4, and the self-intersection points to functions with two different roots of multiplicity 2.

The homology groups of the one-point compactification $\bar{\Sigma}_k$ of the space Σ_k for arbitrary d and k were calculated in [6], we give here another calculation demonstrating our general method. Namely, we construct simplicial resolutions of Σ_k; for the case $d = 4, k = 2$ they are shown in the upper part of Fig. 2.

The main remark here is as follows: the space Σ_k is swept out by a one-parametric family of $(d-k)$-dimensional affine planes: the parameter runs over $\mathbf{R}^1$, and the plane corresponding to the point $x \in \mathbf{R}^1$ consists of all polynomials having a k-root at exactly this point x. The first step of our construction is the *tautological normalization* of the discriminant, obtained from its definition by the "elimination of quantifiers". Indeed, the discriminant Σ_k is defined as the space of all polynomials f such that $\exists$ a point $x \in \mathbf{R}^1$ such that f has a k-fold root at x. Instead, we can consider the space of all pairs (f, x) such that f has a k-root at x. The space of all such pairs is topologically trivial: it is a fiber bundle over $\mathbf{R}^1$, whose fibers are $(d-k)$-dimensional affine planes.

Unfortunately, such planes $\mathbf{R}^{d-k}$, corresponding to different points x, have nonempty intersections in P_d; in order to count them and to get a space homotopy equivalent to Σ_k we should add something to the previous tautological resolution. The best tool for this are the order complexes of all subsets in $\mathbf{R}^1$ which can be defined as sets of k-fold points of certain polynomials of the form (1.2).

3.1. The naive simplicial resolution.

The naive "geometrical" construction of this resolution is as follows (see the upper right part of Fig. 2). We fix a *generic* embedding $I : \mathbf{R}^1 \to \mathbf{R}^N$ of the argument line into the space of a large dimension. The genericity condition here claims that for any $[d/k]$ distinct points of the line

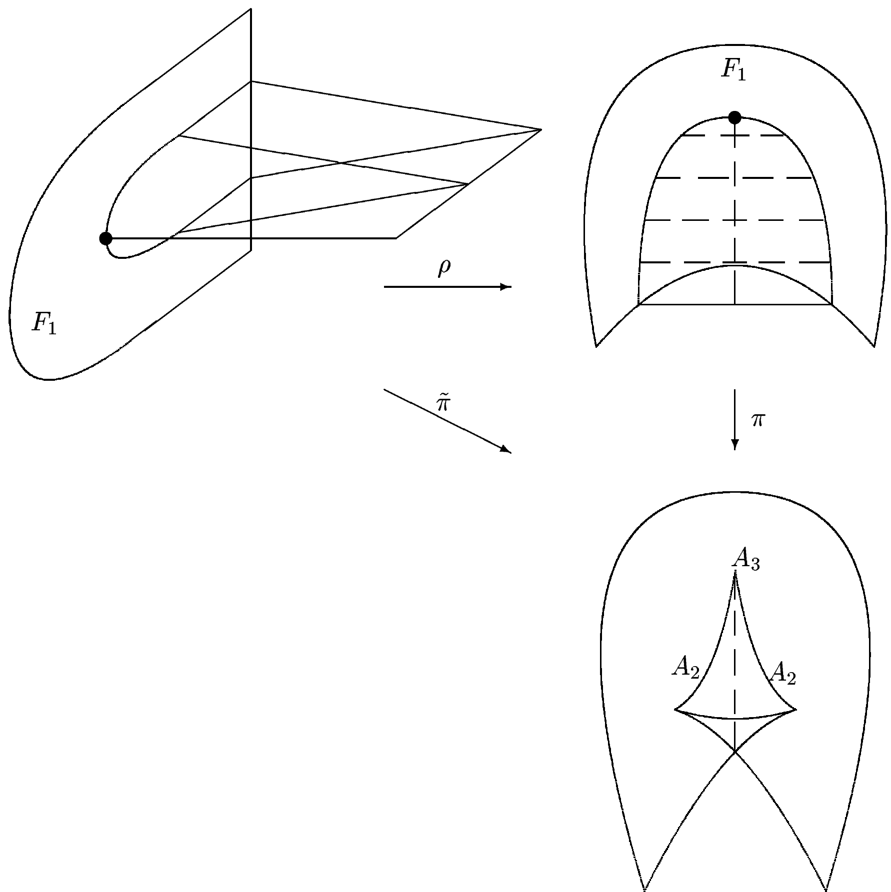

FIGURE 2. A swallowtail and its resolutions

their images cannot lie in the same $([d/k]-2)$-dimensional affine subspace. Our resolution is a subset of the direct product of this space $\mathbf{R}^N$ and the space P_d of all polynomials of the form (1.2). Namely, for any discriminant polynomial f we take *all* its roots of multiplicity $\geq k$, $z_1, \dots, z_t$, and consider the simplex in $\mathbf{R}^N \times P_d$, whose vertices are the points $(I(z_1), f), \dots, (I(z_t), f)$. The desired resolution space σ_k is the union of all such simplices. The obvious projection $\mathbf{R}^N \times P_d \to P_d$ defines a proper map of σ_k onto Σ_k; the extension of this map to a map of one-point compactifications of these spaces, $\pi : \bar{\sigma}_k \to \bar{\Sigma}_k$, is a homotopy equivalence.

On the other hand, the space σ_k (and hence also its one-point compactification $\bar{\sigma}_k$) has a natural increasing filtration: the term F_p of this filtration consists of all simplices of dimensions $\leq p-1$ participating in the previous construction; $F_0 = \{\text{the added point}\}$. The first element F_1 of this filtration in σ_k coincides with the above-described tautological normalization. The term $E^1_{p,q}$ of the corresponding homological spectral sequence is equal to $\bar{H}_{p+q}(F_p \setminus F_{p-1})$, where $\bar{H}_*$ denotes the *Borel–Moore homology group,* i.e. the homology group of the one-point compactification reduced modulo the added point. This space $F_p(\bar{\sigma}_k)\setminus F_{p-1}(\bar{\sigma}_k)$ has a natural structure of a fibre bundle, whose base is the configuration space $B(\mathbf{R}^1, p)$ of all subsets of cardinality p in the line, and the fibre over such a collection $(z_1, \dots, z_p)$ is the direct product of an open $(p-1)$-dimensional simplex and an affine space of dimension $d - p \cdot k$, consisting of all polynomials of the form (1.2), having k-fold roots at exactly these p points. Hence the space of this bundle is a cell, and $E^1_{p,q}$ is equal to $\mathbf{Z}$ if $q = d - p(k-1)$ and $p \leq d/k$, and is trivial for all other p and q. Obviously, this sequence degenerates at the term E^1 and gives us immediately the structure of groups $H_*(\bar{\sigma}_k) \equiv H_*(\bar{\Sigma}_k)$ and $H^*(P_d \setminus \Sigma_k)$.

The inserted simplices, participating in this construction (i.e. spanning the points $I(z_1), \dots, I(z_t)$), can be considered as supports of certain order complexes similar to subcomplexes $\Delta(L_I)$ from § 2. Namely, for any t-element subset of $\mathbf{R}^1$ let us consider the poset of all its nonempty subsets. The corresponding order complex is naturally isomorphic to the first barycentric subdivision of a $(t-1)$-dimensional simplex. For instance, any inserted segment in the upper right part of Fig. 2 should be considered as the union of two segments, joining its boundary points (corresponding to some two points $a, b \in \mathbf{R}^1$) to the middle point (corresponding to the two-element subset $(a, b) \subset \mathbf{R}^1$).

3.2. **The formal simplicial resolution.** Now we describe a more formal construction of the same resolution, not depending on any embedding $\mathbf{R}^1 \to \mathbf{R}^N$. First we construct a slightly different (but homotopy equivalent) resolution space as a subset of the space $\left(B(\mathbf{R}^1, 1) * \overline{B(\mathbf{R}^1, 2)} * \dots * \overline{B(\mathbf{R}^1, [d/k])}\right) \times P_d$, where $\overline{B(\mathbf{R}^1, i)}$ is the space of unordered collections of i points in $\mathbf{R}^1$ (some of which can coincide). It is convenient to consider this space as that of all ideals of codimension i in the ring of smooth functions $\mathbf{R}^1 \to \mathbf{R}^1$.

The natural topology in the space of such ideals it the "Hilbert-scheme topology". (In multidimensional generalizations, when we resolve discriminants in spaces of functions on manifolds M of higher dimensions, we should consider the configuration space $B(M, i)$ as a subset of the infinitedimensional Grassmannian manifold of all subspaces of codimension i in $C^\infty(M, i)$ and define $\overline{B(M, i)}$ as its closure in this manifold.)

The disjoint union of these spaces $B(\mathbf{R}^1, 1), \dots, \overline{B(\mathbf{R}^1, [d/k])}$ is a partially ordered set (by inclusion of configurations, or, which is the same, by the inverse inclusion of corresponding ideals). Consider their join $B(\mathbf{R}^1, 1) * \dots * \overline{B(\mathbf{R}^1, [d/k])}$, set $m = [d/k]$, and define the topological order complex $\tilde{\Delta}_m(\mathbf{R}^1)$ as the union of all coherent simplices in this join. The desired resolution space is a subset of the product $\tilde{\Delta}_m(\mathbf{R}^1) \times P_d$. Namely, for any polynomial $f \in \Sigma_k$ with exactly i roots of multiplicity $\geq k$ we consider the set $s(f)$ of all its $(\geq k)$-fold roots as a point in $B(\mathbf{R}^1, i)$ and take the subcomplex $\tilde{\Delta}(f) \subset \tilde{\Delta}_m(\mathbf{R}^1)$, consisting of coherent simplices, all whose vertices are subsets of $s(f)$. Finally, the resolution space $\tilde{\sigma}_k \subset \tilde{\Delta}_m(\mathbf{R}^1) \times P_d$ is the *closure* of the union of all simplices of the form $\tilde{\Delta}(f) \times \{f\}$.

Again, all fibers of the obvious projection $\tilde{\pi} : \tilde{\sigma}_k \to \Sigma_k$ are compact cones, and this projection defines a homotopy equivalence of one-point compactifications of these spaces.

However, topologically this resolution $\tilde{\pi}$ does not coincide with the resolution constructed in § 3.1. In the case $d = 4, k = 2$ it is shown in the upper left part of Fig. 2. The half-line in the right-hand part of this left part symbolizes the configuration space $\overline{B(\mathbf{R}^1, 2)}$ (which is in fact diffeomorphic to the direct product of such a half-line and $\mathbf{R}^1$, see Remark in the beginning of this section). This configuration space is a manifold with boundary: its boundary points correspond to configurations of two equal points. In the construction of the corresponding order complex $\tilde{\Delta}_2$, any non-boundary point (a, b) of this configuration space is joined by segments with two points (a) and (b) of the space $B(\mathbf{R}^1, 1) \equiv \mathbf{R}^1$ (so that the union of these two segments can be considered as a segment joining the points (a) and (b)), and the boundary point (a, a) is joined by a single segment with the point $(a) \in B(\mathbf{R}^1, 1)$. Any such segment (for any $a \in \mathbf{R}^1$) is a contractible space. Contracting any such segment into a single point, we get a space homotopy equivalent to the previous one (and homeomorphic to the one indicated in the upper right part of Fig. 2).

For general d and k this factorization is defined as follows. For any point $\alpha \in \overline{B(\mathbf{R}^1, i)}$ we define its *geometrization* α' as the point of some $B(\mathbf{R}^1, n)$, $n \leq i$, obtained from the configuration α by taking all its points without multiplicities. Given a point $\beta \in B(\mathbf{R}^1, n)$, we say that a coherent simplex in $\tilde{\Delta}_m(\mathbf{R}^1)$ is a *ghost simplex of* β if the geometrizations of all its vertices coincide with β. Define the *ghost subspace* $\Gamma(\beta) \subset \tilde{\Delta}_m(\mathbf{R}^1)$ as the union of all ghost simplices of β. This is a compact cone with the vertex $\{\beta\}$. Then we obtain the quotient space $\Delta_m(\mathbf{R}^1)$ of $\tilde{\Delta}_m(\mathbf{R}^1)$, contracting any ghost simplex to one point and extending this contraction by linearity to all coherent simplices containing such ghost simplices as their faces.

REMARK. The obtained quotient space is in obvious set-theoretical bijection with the subset of $\tilde{\Delta}_m(\mathbf{R}^1)$ consisting of coherent simplices, all whose vertices are geometrical, i.e. belong to subspaces $B(\mathbf{R}^1, n) \subset \overline{B(\mathbf{R}^1, n)}$. However as topological spaces they are different.

The canonical projection $\tilde{\Delta}_m(\mathbf{R}^1) \to \Delta_m(\mathbf{R}^1)$ is a proper stratified map of semialgebraic sets, whose restriction on the pre-image of any stratum of $\Delta_m(\mathbf{R}^1)$ in $\tilde{\Delta}_m(\mathbf{R}^1)$ is a fibre bundle with compact contractible fibres. In particular, it defines a homotopy equivalence of one-point compactifications of these spaces.

Further, this factorization map $\tilde{\Delta}_m(\mathbf{R}^1) \to \Delta_m(\mathbf{R}^1)$ defines in the obvious way the map $\tilde{\Delta}_m(\mathbf{R}^1) \times P_d \to \Delta_m(\mathbf{R}^1) \times P_d$. Its restriction to the space $\tilde{\sigma}_k$ maps this space onto a certain subspace $\sigma'_k \subset \Delta_m(\mathbf{R}^1) \times P_d$. This map also is proper and stratified with contractible fibers.

PROPOSITION 1. *This map $\rho : \tilde{\sigma}_k \to \sigma'_k$ factorizes the resolution map $\tilde{\pi} : \tilde{\sigma}_k \to \Sigma_k$, i.e. there exists a proper map $\pi' : \sigma'_k \to \Sigma_k$ such that $\tilde{\pi} \equiv \pi \circ \rho$ (namely, π is the restriction of the projection $\Delta_m(\mathbf{R}^1) \times P_d \to P_d$). The extensions of all*

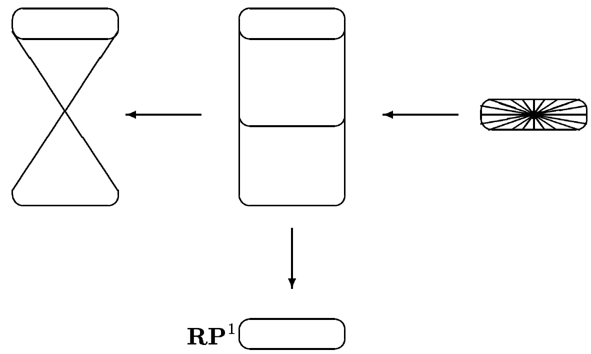

FIGURE 3. Resolution of a cone

these maps to one-point compactifications define the homotopy equivalences of all three spaces $\overline{\tilde{\sigma}_k}$, $\overline{\sigma'_k}$ *and* $\overline{\Sigma_k}$. *The resolution space* σ'_k *thus constructed is naturally homeomorphic to the space* σ_k *constructed in* § *3.1.* □

4. Resolutions of non-normal discriminants

In this and the next sections we discuss the following generalizations of the above constructions.

First, we consider non-normal discriminant varieties, whose tautological resolutions have non-discrete fibers.

Next, we consider theories of degrees ≥ 2. The discriminant spaces, considered in §§ 3, 4, are defined in the terms of *monosingularities,* i.e. the restrictions are formulated in the terms of the behavior of the function or map at some single point of the source manifold. A theory of degree 2 appears when the simultaneous behavior at two different points should be taken into account. The simplest such theory is that of knots and links, where the essential singularities are the self-intersection points, i.e. the conditions of the form $f(x) = f(y)$.

A problem, including difficulties of both these kinds, is provided by the theory of Hermitian matrices with (non)simple spectra.

An example of 3-d order theory appears if we study the plane curves with(out) triple intersections, see 5.3.

4.1. Resolutions of determinants. Let $\mathbf{K}$ be any of fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. Consider the *determinant* variety $Det(\mathbf{K}^n)$ of all degenerate operators $\mathbf{K}^n \to \mathbf{K}^n$.

Its tautological resolution is again defined by elimination of quantifiers. Namely, by definition an operator $A \in End(\mathbf{K}^n)$ belongs to $Det(\mathbf{K}^n)$ if $\exists$ a point $x \in \mathbf{KP}^{n-1}$ such that $\{x\} \in \ker A$. Define the resolution space $det_1(\mathbf{K}^n)$ as the space of all pairs $(x, A) \in \mathbf{KP}^{n-1} \times End(\mathbf{K}^n)$ such that $\{x\} \in \ker A$. This is a very simple space: tautologically, it admits the structure of a $(n^2 - n)$-dimensional $\mathbf{K}$-vector bundle over $\mathbf{KP}^{n-1}$. There is obvious projection $det_1(\mathbf{K}^n) \to Det(\mathbf{K}^n)$, which is a homeomorphism in a neighborhood of operators having 1-dimensional kernels, but its pre-image over an operator with $\dim\ker = l$ is equal to $\mathbf{KP}^{l-1}$.

Say, let $\mathbf{K} = \mathbf{R}$, $n = 2$. The space $End(\mathbf{R}^2)$ of all operators $\mathbf{R}^2 \to \mathbf{R}^2$ is 4-dimensional, and $Det(\mathbf{R}^2)$ is a 3-dimensional conical subvariety in it, see Fig. 3. There is a single point in $Det(\mathbf{R}^2)$, over which the tautological resolution is not a homeomorphism: namely, the zero operator. Its preimage is the line $\mathbf{RP}^1$. In order to get a space homotopy equivalent to $Det(\mathbf{R}^2)$ we need to insert a disc, whose boundary coincides with this preimage, see Fig. 3. It is useful to consider this disk as the space $\Theta(\mathbf{R}^2)$, see § 1.1.

For other $\mathbf{K}$ and n the conical resolution of $Det(\mathbf{K}^n)$ is constructed as a subset of the direct product $\Theta(\mathbf{K}^n) \times Det(\mathbf{K}^n)$. To any plane $L \subset \mathbf{K}^n$ there corresponds a subspace $\Theta(L) \subset \Theta(\mathbf{K}^n)$, namely, the union of all coherent simplices, all whose vertices correspond to planes lying in L. This is a cone with vertex $\{L\}$. Also define $\kappa(L) \subset End(\mathbf{K}^n)$ as the linear space of all operators $\mathbf{K}^n \to \mathbf{K}^n$, whose kernels contain L, and define the conical resolution $\delta(\mathbf{K}^n) \subset \Theta(\mathbf{K}^n) \times Det(\mathbf{K}^n)$ as the union of all products $\Theta(L) \times \kappa(L)$ over all planes L of all dimensions $1, \dots, n$. It is easy to see that the obvious projection $\delta(\mathbf{K}^n) \to Det(\mathbf{K}^n)$ induces a homotopy equivalence of one-point compactifications of these spaces (indeed, this projection is proper and all its fibers are contractible cones of the form $\Theta(L)$). On the other hand, the space $\delta(\mathbf{K}^n)$ has a nice filtration: its term F_i is the union of products $\Theta(L) \times \kappa(L)$ over planes L of dimensions $\leq i$. The term $F_i \setminus F_{i-1}$ of this filtration is the space of a fiber bundle over $G_i(\mathbf{K}^n)$. Its fiber over a point $\{L\}$ is the space $(\Theta(L) \setminus \partial\Theta(L)) \times \kappa(L)$, which by Theorem 1 is homeomorphic to an Euclidean space. Thus the Borel–Moore homology group of this term can be reduced to that of the base. The spectral sequence, generated by this filtration and converging to the Borel–Moore homology of $Det(\mathbf{K}^n)$ (or, equivalently, to the cohomology of the complementary space $GL(\mathbf{K}^n)$), degenerates in the first term and gives us, in particular, the homological *Miller splitting*

$$H_m(GL(\mathbf{C}^n)) = \oplus_{k=0}^{n} H_{m-k^2}(G_k(\mathbf{C}^n))$$

and similar splittings for $\mathbf{R}$ and $\mathbf{H}$, see [43], [40].

4.2. **Homogeneous polynomials in $\mathbf{R}^2$ without multiple zeros.** Another example of a non-normal discriminant variety is as follows. Consider the space HP_d of all homogeneous polynomials $\mathbf{R}^2 \to \mathbf{R}^1$ of degree d, and define the discriminant subset $\Sigma_k \subset HP_d$ as the set of polynomials vanishing with multiplicity $\geq k$ on some line in $\mathbf{R}^2$. Its resolution can be constructed similarly to § 3.2, but not to § 3.1. This is due to the fact that the tautological resolution of Σ_k (consisting of pairs $\{(a \in \mathbf{RP}^1, f \in HP_d) | f$ has a k-fold zero along the corresponding line $\{a\}\}$) has a non-discrete preimage over the point {identical zero} $\in \Sigma_k$.

The possible singular sets, which the discriminant polynomials can define in $\mathbf{RP}^1$, are any configurations of $1, 2, \dots, [d/k]$ points and entire $\mathbf{RP}^1$. Thus we need to construct the complex $\widetilde{H\Delta}_{[d/k]}(\mathbf{RP}^1)$ as the topological order complex of all these sets. By Theorem 2 (see § 1.2) the *link* of this complex (consisting of all coherent simplices not containing the vertex $\{\mathbf{RP}^1\}$) is homotopy equivalent to

$S^{2[d/k]-1}$. Similarly to § 3.2 we can consider the geometrization $H\Delta_{[d/k]}$ of this complex, contracting all "ghost simplices". This geometrization has the following interpretation (similar to the "naive" construction from § 3.1). Let us embed S^1 generically into the space of a large dimension $N \geq 2m$, $m \equiv [d/k]$, and consider the union of convex hulls of all m-tuples of points of this embedded circle. The "genericity" condition implies that all these convex hulls are simplices, which can intersect only on their common faces.

The original theorem of C. Caratheodory (applying to the particular embedding $t \mapsto (\sin t, \cos t, \dots, \sin(mt), \cos(mt)))$ claims that this union is homeomorphic to the sphere S^{2m-1}. Theorem 2 follows immediately from this one and the fact that the "geometrization" reduction of links of these order complexes is a homotopy equivalence.

4.3. **Spaces of nonsingular algebraic projective hypersurfaces.** For any natural d and n, we study the cohomology group of the space $\Pi(d,n)$ of all non-singular projective hypersurfaces in $\mathbf{CP}^n$ of degree d. The real version of this problem, especially its part concerning the 0-dimensional cohomology, is the well-known rigid isotopy classification problem, see e.g. [22].

Our algorithm of calculating these homology groups (see [49]) in its essential part repeats the one described in §§ 3.2, 4.2. We consider the linear space of all homogeneous polynomials $\mathbf{C}^{n+1} \to \mathbf{C}$ of degree d. The subset in it, defining non-singular varieties in $\mathbf{CP}^n$, is the complement of the conical *discriminant* hypersurface of singular polynomials, hence it is homeomorphic to the Cartesian product of the desired space $\Pi(d,n)$ of nonsingular hypersurfaces and the punctured line $\mathbf{C}^*$. The Alexander duality reduces its cohomology groups to the Borel–Moore homology groups of the discriminant cone. To calculate these groups, we consider the poset of all possible subsets in $\mathbf{CP}^n$ which can serve as singular sets of certain varieties of degree d (see § 1.5), take its closure in the Hilbert-scheme topology, and construct the conical resolution of the discriminant as a subset of the product of this order complex and the space of polynomials. The natural filtration in this space is defined by the (co)dimensions of linear spaces of polynomials, having singularities at a given subset in $\mathbf{CP}^n$. For instance, the last term $F_N \setminus F_{N-1}$ of this filtration corresponds to the greatest possible singular set, i.e. to entire $\mathbf{CP}^n$, and coincides with entire order complex less its link.

Here are some first results on the groups $H^*(\Pi(d,n))$ obtained in this way. These groups with $d=2$ are well-known: the space $\Pi(2,n)$ is homotopy equivalent to the $(n+1)$-st Lagrangian Grassmannian $U(n+1)/O(n+1)$, whose homology groups were calculated in [12].

THEOREM 5 (see [49]). *The Poincaré polynomial of the group* $H^*(\Pi(d,n), \mathbf{C})$ *with* $(d,n) = (3,2)$ (*respectively,* $(3,3)$, *respectively,* $(4,2)$) *is equal to* $(1+t^3)(1+t^5)$ (*respectively,* $(1+t^3)(1+t^5)(1+t^7)$, *respectively,* $(1+t^3)(1+t^5)(1+t^6)$).

Two first assertions of this theorem lead to the following conjecture.

CONJECTURE. *For any* $n \geq 2$ *the rational cohomology ring of the space* $\Pi(3,n)$ *is isomorphic to that of the projective linear group* $PGL(n+1, \mathbf{C})$.

A principal part of these calculations is the study of the order complex $\Lambda(d,n)$, in particular of topological properties of classes of singular sets, see § 1.5.

REMARK. It turns out that all these classes corresponding to one-dimensional singular sets cannot provide a nontrivial contribution to the group $H^*(\Pi(4,2),\mathbf{C})$. Namely, such a nontrivial contribution is provided by strata No. 1, 2, 4, 10 and 13. The most mysterious of them[3] is the stratum No. 10: it provides the six-dimensional generator of the ring $H^*(\Pi(4,2),\mathbf{C})$.

PROBLEM. To express this generator in the interior terms of the space $\Pi(4,2)$ (and not by means of the Alexander duality).

One other problem is as follows. In all the non-trivial cases I have calculated (i.e. the ones described in Theorem 5 and in the next Theorem 6) the corresponding spectral sequences degenerate at the first term: $E^1 \equiv E^\infty$ (although in the trivial case $d=2$ it is not more so). Is it true also for greater d and n?

THEOREM 6 (see [49]). *The rational cohomology ring of the space of homogeneous quadratic vector fields in $\mathbf{C}^3$, having no singularities outside the origin, is isomorphic to that of homogeneous polynomials $\mathbf{C}^3 \to \mathbf{C}^1$ of degree 3 defining a nonsingular curve in $\mathbf{CP}^2$; in particular, its Poincaré polynomial is equal to $(1+t)(1+t^3)(1+t^5)$. This isomorphism is induced by the gradient embedding, sending any polynomial to the set of its partial derivatives.*

5. THEORIES OF SECOND AND THIRD ORDER

5.1. **Knot theory and related combinatorial problems.** It was said very much on the finite-order knot invariants, see e.g. [11], [10], [24].

Their original construction was essentially (up to minor modifications) the same as in § 3.1. We consider the space of all smooth maps $S^1 \to \mathbf{R}^3$, define the discriminant subspace as the set of all maps having either self-intersections or singular points, define its tautological resolution by the elimination of quantifiers (i.e. as the space of all pairs of the form {a couple of points $(a,b) \in \overline{B(S^1,2)}$, a map $f : S^1 \to \mathbf{R}^3$} such that either $a \neq b$ and $f(a) = f(b)$, or $a = b$ and $f'(a) = 0$), and then insert some simplices to get a space "homotopy equivalent to"[4] the discriminant.

There are numerous combinatorial problems arising from this resolution. First of all, the singular strata of the discriminant (and hence of its resolution) are described in the terms of configurations of points pasted together by corresponding singular maps. Their types are codified by finite sequences of natural numbers $(a_1 \geq a_2 \geq \ldots \geq a_k; b)$: such a code denotes the set of maps gluing together some groups of $a_1, \ldots, a_k$ points and additionally having b singular points. Of course,

[3]We exclude the exceptional stratum No. 13, which accumulates all the difficulties and beauties of all other strata.

[4]the quotes here are due to the fact that we are in the infinite-dimensional situation, and the standard notion of the homotopy equivalence cannot be applied. However, some speculation with finite-dimensional approximations allows us to work with these spaces as with semialgebraic finite-dimensional varieties

there can be many nonequivalent configurations with the same code. Also, the *complexes of connected and two-connected graphs* appear naturally in the fibers of the "naive" resolution. Indeed, if we consider a map with exactly one m-fold self-intersection point, then its preimage in the tautological resolution consists of $\binom{m}{2}$ points, and following the "naive" construction we should span them by a $\left(\binom{m}{2}-1\right)$-dimensional simplex. This simplex belongs to the $(m-1)$-st term of the natural filtration of the resolution, while some of its faces belong to smaller terms.

Namely, the faces of this simplex are naturally encoded by the graphs with m given vertices; a face belongs to a smaller term of the filtration if and only if it corresponds to a not connected graph. Such faces form a simplicial subcomplex of our simplex, thus the homological study of the discriminant leads to the calculation of the homology group of the complex of not connected graphs (or of the complex of connected graphs, related with it by the exact sequence of our simplex), see [44]. This homology group is concentrated in dimension $m-2$ and is isomorphic to $\mathbf{Z}^{(m-1)!}$.

However, we could construct our resolution following the scheme of § 3.2, i.e. inserting over any m-point configuration not the naive $\binom{m}{2}$-vertex simplex but the order complex of all proper subspaces in $C^\infty(S^1, \mathbf{R}^3)$, consisting of maps gluing together the points of some or other subconfigurations of this m-point one, cf. § 1.4. This order complex is $(m-2)$-dimensional from the very beginning. It can be naturally embedded into our simplex as a subcomplex of its barycentric subdivision. This embedding induces an isomorphism of homology groups of these complexes modulo their intersections with the lower terms of the canonical filtration of the corresponding resolution (while for the latter order complex this intersection is nothing but its link).

The complex of two-connected graphs allows us to define the *higher indices* (or *residues*), which a knot invariant defines at a singular knot. In the banal theory of finite-type invariants one considers only the singular knots in $\mathbf{R}^3$ having several (say, k) transverse double self-intersections and no more complicated singular points. Such a singular knot can be slightly moved in 2^k locally different ways to obtain non-singular knots. Given a knot invariant, i.e. a locally constant function on the space of knots, its k-th index at our singular knot is defined as the alternated sum of its values at all these 2^k resolutions; these indices play a key role in the calculus of knot invariants (under the name of weight functions). In a similar way we can define higher indices at more complicated singularities. However, they usually are not numerical. E.g., if our map $S^1 \to \mathbf{R}^3$ has exactly one generic m-fold selfintersection point, then this index takes values in the homology group of *the complex of two-connected graphs* with m vertices. This group was calculated in [17] and [37]. If our map has several such points, maybe of different multiplicities, then the values are taken in the tensor product of such homology groups.

REMARK. Essentially the same spectral sequence calculates all the cohomology groups of the space of knots in any (≥ 4)-dimensional manifold and probably provides interesting invariants of such manifolds. M. Kontsevich has proved (unpublished) that in the case of knots in $\mathbf{R}^n$, $n \geq 3$, this spectral sequence (with complex coefficients) degenerates at the first term: $E^1 \equiv E^\infty$. (His famous integral formula [24], [11] for the knot invariants in $\mathbf{R}^3$ is only a very particular version of this result). In the case of an arbitrary manifold the similar statement is not true, see [47].

5.2. **Spaces of Hermitian matrices with simple spectra.** V. I. Arnold [4], [9] has studied the topology of spaces of Hermitian operators in $\mathbf{C}^n$ with non-simple spectra in a relation with the theory of adiabatic connections and the quantum Hall effect (see also [28], [29] concerning physical motivations of this problem). The natural filtration of these spaces by the sets of operators with a fixed number of eigenvalues defines a spectral sequence, providing interesting combinatorial and homological information on these sets, see [9].

In [52] a different spectral sequence was constructed, also converging to the homology groups of these spaces and based on the above-described techniques.

This spectral sequence degenerates at the term E_1 and is hypothetically multiplicative. When n grows, it converges to a stable spectral sequence, calculating the cohomology group of the space of infinite Hermitian operators without multiple eigenvalues; all terms $E_r^{p,q}$ of this stable sequence are finitely generated.

The main object of this construction is the *topological order complex of all collections of pairwise Hermitian-orthogonal complex subspaces in* $\mathbf{C}^n$, see § 1.3.

Again, the resolution of the space of singular (i.e. having multiple eigenvalues) operators is a subspace of the Cartesian product of this order complex and the space of all Hermitian operators. The singular strata of this resolution are classified in almost the same way as these for the space of knots (the multiindex $A \equiv (a_1 \geq a_2 \geq \ldots \geq a_k)$ denotes the set of operators with an eigenspace of dimension a_1, an eigenspace of dimension a_2 orthogonal to the previous one (but maybe with the same eigenvalue), etc.).

Although the cohomology ring of the space of non-discriminant Hermitian operators is known (see [12], [9]), this construction provides a very natural filtration in this ring. This filtration is stable with respect to embeddings of spaces of such operators induced by the embeddings $\mathbf{C}^n \to \mathbf{C}^{n+1} \to \cdots$, thus defining a stable structure in the cohomology ring of the space of infinite-dimensional generic Hermitian operators.

E.g., the 2-dimensional cohomology group of our space is generated by the first Chern classes $c_1^{(i)}$ of all linear vector bundles formed by eigenvectors of operators (ordered by the increase of corresponding eigenvalues) subject to a single relation: the sum $\sum_i c_1^{(i)}$ of all these classes is equal to zero. For any m, the m-th term of our filtration of this 2-cohomology group is nothing other than the space of integral sequences $\sum_i \alpha_i c_1^{(i)}$, where $\{\alpha_i\}$ is the sequence of values at integral points of some polynomial of degree $\leq d$. (A basis in the space of such polynomials consists of polynomials $\beta_d \equiv i(i-1)\cdot \ldots \cdot (i-d+1)/d!$, $d = 1, 2, \ldots, m$.) An important problem is to give an explicit expression of all other terms $E^{p,q}$ of this sequence in terms of Chern classes of these linear bundles.

5.3. **Triple points free plane curves.** V. I. Arnold [7], [8] has introduced three local invariants of generic plane closed immersed curves. The most interesting of them, the *strangeness*, is in fact an invariant of triple points free immersed curves and can be defined as the linking number with the (suitably oriented) discriminant subvariety in $Imm(S^1, \mathbf{R}^2)$ consisting of all immersions having a triple point. The investigation of invariants of generic immersed plane curves was continued in [1], [53], [34], [36], [31], [27], and many other works; in particular A. B. Merkov [27] has proved that the (suitably defined) finite-order invariants of such curves provide a complete system of invariants.

About the same time, I studied *ornaments*, i.e. collections of closed plane curves without triple intersections (but maybe with singular points) and constructed some their invariants by the methods similar to the ones described in § 5.1, see [44], [45], [39], [40]. The recent work [50] contains a translation of these methods to the classification of triple points free immersed plane curves, and also to the neighboring theory of *doodles*, i.e. closed plane curves without triple points but maybe with singularities. (The last theory has been previously studied by M. Khovanov [23] and by A. B. Merkov [27], who in particular has discovered the first example of a non-trivial one-component doodle and proved that the finite-order invariants form a complete system of invariants also in this problem. A slightly different notion was considered in [20].)

These methods lead naturally to the study of *complexes of connected 3-hypergraphs* in the same way as the knot theory leads to the calculus of connected graphs, see [44], [16], [25]. (In fact, these problems are even more important here. The elements of the homology group of the complex of connected graphs with m vertices can be involved more or less only in the calculation of the $\geq (m-3)$-dimensional cohomology group of the space of knots; in particular such complexes with $m = 2$ and 3 only are important for the study of knot invariants. On the contrary, the homology groups of connected hypergraphs with arbitrarily many vertices play important role in the calculation of invariants of doodles and/or immersed triple-points free curves.)

E.g., the first nontrivial invariant of one-component doodles is of order 4. The 3-hypergraphs can be depicted by collections of triangles, therefore we get the calculus of triangular diagrams in the almost the same way as the calculus of chord diagrams appears in the knot theory. The chord diagram depicting the easiest (of order 2) knot invariant is the cross $\bigoplus$. Analogously, the triangular diagram depicting our 4-th order invariant of doodles is as follows: .

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