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# ON SUBADDITIVE PROCESSES ON DIRECT PRODUCT OF COUNTABLE AMENABLE GROUPS

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ABSTRACT. Direct product of two amenable groups is considered. If one of them is $\sum_{i=1}^{\infty} Z_2$, then $\sum_{i=1}^{\infty} Z_2 \times G$ is amenable. Subadditive processes on such a direct product are studied.

## 1. Introduction

Basic concepts and properties of amenable groups are given in [3]. We use the following facts about the class of amenable groups.

1) Subgroups of amenable groups are amenable.

2) Factors groups of amenable groups are amenable.

3) If $H$ is a normal subgroup of $G$ and both $H$ and $G/H$ are amenable, then so is $G$.

4) If $G = \bigcup_{n=1}^{\infty} G_n$ with $G_1 \subset G_2 \subset G_3 \subset \cdots$ with each $G_n$ amenable, then $G$ is amenable.

5) Each finite group and $\mathbf{Z}$ are amenable.

Let $G_1$ and $G_2$ be two countable amenable groups. Is the direct product $G_1 \times G_2$ of these groups also amenable or not? In the general case, this problem is rather difficult and may be there is no affirmative answer.

We consider a particular case, when one of these groups is $\sum_{i=1}^{\infty} Z_2$ (finitely generated countable commutative group) and the second group is an arbitrary countable amenable group. It is proved that the direct product of such amenable groups is also amenable. We also investigate subadditive processes on direct products of countable amenable groups. Basic definitions and theorems concerning subadditive processes are given in [1] and [2].

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Let $(X, \mathbf{F}, \mu)$ be a probability measure space, $G$ a countable amenable group and $T$ a measure preserving action of $G$ on $(X, \mathbf{F}, \mu)$, that is, $T$ is a homomorphism of $G$ into $\mathrm{Aut}(X, \mathbf{F}, \mu)$, where $\mathrm{Aut}(X, \mathbf{F}, \mu)$ is a group of measure preserving automorphism of $(X, \mathbf{F}, \mu)$. Let $\Psi(G)$ be the family of all finite subsets of $G$. Evidently, $\Psi(G)$ is a countable set.

Let us consider the transformation $\pi : \Psi(G) \to L^1_+(X)$, where $L^1_+(X)$ is the set of nonnegative integrable functions, that is, for arbitrary $H \in \Psi(G)$, $\pi(H) = q_H(x) \in L^1_+(X)$.

DEFINITION 1. A family of functions $\{q_H \in L^1_+(X) : H \in \Psi(G)\}$ is called a subadditive process, if the following conditions hold:

1) If $H = \bigcup_{i=1}^{n} B_i c_i$, where $B_i \in \Psi(G)$ and $c_i \in G$, $i = 1, \dots, n$, then

$$q_H(x) \leqslant \sum_{i=1}^{n} q_{B_i}(T^{c_i} x) \quad \text{for a.e. } x \in X$$

2) For each $g \in G$ and for a.e. $x \in X$, $q_{Hg}(x) = q_H(T_g x)$.

DEFINITION 2. A sequence of finite sets $F = \{F_n \in \Psi(G)\}_{n \geqslant 1}$ is called a Følner sequence, if $\lim\limits_{n \to \infty} \frac{|F_n g \Delta F_n|}{|F_n|} = 0$, for all $g \in G$.

## 2. Main results

In what follows let $\sum$ stand for $\sum\limits_{i=1}^{\infty}$.

THEOREM 1. *Let $G$ be a countable amenable group. The direct product $\sum Z_2 \times G$ is also an amenable group.*

PROOF. Let us consider $K_n = \sum_{i=1}^{n} Z_2$ with usual coordinately group action. Then $K_n$ is an amenable group. Assume $G_n = K_n \times G$. It is evident that $\tilde{K}_n = K_n \times \{e\}$, where $e \in G$ is the identity of $G$, is a normal subgroup of $G_n$ and $G_n / \tilde{K}_n = \{0\} \times G$ so, $G_n$ is an amenable group for any $n$. As $\mathrm{G}_1 \subset G_2 \subset G_3 \subset \cdots$ and $\sum Z_2 \times G = \bigcup_{n=1}^{\infty} G_n$, then $\sum Z_2 \times G$ is an amenable group.

Theorem 1 is thus proved. □

COROLLARY 1. *Let $G$ be a countable amenable group. Then the direct product $(\sum Z_2)^d \times G$ is also an amenable group for arbitrary positive integer $d$.*

Let $G_1 = \sum Z_2$ and $G$ be two amenable groups. We shall consider a simpler case. Let $(X_1, \mathbf{F}_1, \mu_1)$ and $(X_2, \mathbf{F}_2, \mu_2)$ be two probability spaces, $T'$ a measure preserving action of $G_1$ on $(X_1, \mathbf{F}_1, \mu_1)$ and $T''$ a measure preserving action of $G$ on $(X_2, \mathbf{F}_2, \mu_2)$, that is, $T' : G_1 \to \mathrm{Aut}(X_1, \mathbf{F}_1, \mu_1)$, $T'' : G \to \mathrm{Aut}(X_2, \mathbf{F}_2, \mu_2)$ are homomorphisms. Then $T' \times T''$ is a measure preserving action of $\sum Z_2 \times G$ on $(X_1 \times X_2, \mathbf{F}_1 \otimes \mathbf{F}_2, \mu_1 \times \mu_2)$, where $T_{(g_1, g_2)} = (T'_{g_1} \times T''_{g_2})$ for $g_1 \in G_1$, $g_2 \in G$. Let $\Psi(G_1 \times G)$ be a family of finite subsets of $G_1 \times G$ of the form $F' \times F''$, where $F'$ is a sequence of finite subsets of $G_1$ and $F''$ is a sequence of finite subsets of $G$, that is, $\Psi(G_1 \times G) = \Psi(G_1) \times \Psi(G)$.

DEFINITION 3. A sequence $\tilde{F} = \{F'_n \times F''_n\}_{n \geqslant 1}$ is called a quasi-direct product of Følner sequences, if $\{F'_n\}_{n \geqslant 1}$ and $\{F''_n\}_{n \geqslant 1}$ are Følner sequences.

Let us consider $\pi' : \Psi(G_1) \to L^1_+(X_1, F_1, \mu_1)$ and $\pi'' : \Psi(G) \to L^1_+(X_2, F_2, \mu_2)$. Assume that $\pi : \Psi(G_1 \times G) \to L^1_+(X_1, F_1, \mu_1) \times L^1_+(X_2, F_2, \mu_2)$ such that $\pi(g_1, g_2) = (\pi'(g_1) \times \pi''(g_2))$ for $g_1 \in G_1$, $g_2 \in G$.

DEFINITION 4. A family of functions

$$\{q_{H_1 \times H_2} \in L^1_+(X_1) \times L^1_+(X_2) : H_1 \in \Psi(G_1), H_2 \in \Psi(G)\}$$

is called a quasi-subadditive process if both $\{q_{H_1} \in L^1_+(X_1) : H_1 \in \Psi(G_1)\}$ and $\{q_{H_2} \in L^1_+(X_2) : H_2 \in \Psi(G)\}$ are subadditive processes.

THEOREM 2. *Let $G_1$ and $G$ be two countable amenable groups. If $\{q_{H_1 \times H_2} \in L^1_+(X_1) \times L^1_+(X_2) : H_1 \in \Psi(G_1),\ H_2 \in \Psi(G)\}$ is a quasi-subadditive process, then for arbitrary quasi-Følner sequence $\tilde{F} = \{F'_n \times F''_n\}_{n \geqslant 1}$, there exists a measurable function $\tilde{q} = \tilde{q}_1.\tilde{q}_2$ such that $\tilde{q}_i \in L^1_+(X_i)$ for $i = 1, 2$; $q_1(T'_{g_1}(x_1)) = \tilde{q}_1(x_1)$, $q_2(T''_{g_2}(x_2)) = \tilde{q}_2(x_2)$ for a.e. $x_1 \in X_1$, $x_2 \in X_2$: $\lim_{n\to\infty} \frac{1}{|F'_n \times F''_n|} q_{F'_n \times F''_n} = \tilde{q}$ a.e. and*

$$\lim_{n\to\infty} \frac{1}{|F'_n \times F''_n|} \int_{X_1 \times X_2} q_{F'_n \times F''_n}\, d\mu_1\, d\mu_2 = \inf_n \frac{1}{|F'_n \times F''_n|} \int_{X_1 \times X_2} q_{F'_n \times F''_n}\, d\mu_1\, d\mu_2 = \int_{X_1} \tilde{q}_1\, d\mu_1 \int_{X_2} \tilde{q}_2\, d\mu_2.$$

PROOF. By Definition 3 and Definition 4, we can obtain a subadditive sequence $\{q_{F'_n \times F''_n}\}_{n \geqslant 1}$ converging to $\tilde{q} = \tilde{q}_1.\tilde{q}_2$, i.e.,

$$\lim_{n\to\infty} \frac{1}{|F'_n \times F''_n|} q_{F'_n \times F''_n} = \inf_n \left\{ \frac{1}{|F'_n \times F''_n|} q_{F'_n \times F''_n} \right\}_{n \geqslant 1} = \tilde{q}.$$

$\tilde{q} = \tilde{q}_1.\tilde{q}_2$ is a measurable function. Also $0 \leqslant \tilde{q} \leqslant \frac{1}{|F'_n \times F''_n|} q_{F'_n \times F''_n} \in L^1_+(X_1) \times L^1_+(X_2)$.

By Fatou's Lemma, we have $\tilde{q}_i \in L^1_+(X_i)$ for $i = 1, 2$. Then $\tilde{q} \in L^1_+(X_1) \times L^1_+(X_2)$. It is easy to see that $\tilde{q} = \tilde{q}_1.\tilde{q}_2$ is $T_{(g_1, g_2)} = (T'_{g_1} \times T''_{g_2})$-invariant for $g_1 \in G_1$, $g_2 \in G$. By Lebesque's Bounded Convergence Theorem, we can write

$$\lim_{n\to\infty} \frac{1}{|F'_n|} \int_{X_1} q_{F'_n} d\mu_1 = \int_{X_1} \tilde{q}_1 d\mu_1, \quad \lim_{n\to\infty} \frac{1}{|F''_n|} \int_{X_2} q_{F''_n} d\mu_2 = \int_{X_2} \tilde{q}_2 d\mu_2.$$

Finally, we have

$$\lim_{n\to\infty} \frac{1}{|F'_n \times F''_n|} \int_{X_1 \times X_2} q_{F'_n \times F''_n} d\mu_1 d\mu_2 = \inf_n \frac{1}{|F'_n \times F''_n|} \int_{X_1 \times X_2} q_{F'_n \times F''_n} d\mu_1 d\mu_2 = \int_{X_1} \tilde{q}_1 d\mu_1 \int_{X_2} \tilde{q}_2 d\mu_2.$$

Theorem 2 is thus proved. □

Now we can consider subadditive processes for $\mathrm{G}_1 \times G$, if $G_1 = \sum Z_2$. Let $(X_1, \mathbf{F}_1, \mu_1)$ and $(X_2, \mathbf{F}_2, \mu_2)$ be probability spaces. Let $T$ be a measure preserving action of $\sum Z_2 \times G$ on $(X_1 \times X_2, \mathbf{F}_1 \otimes \mathbf{F}_2, \mu_1 \times \mu_2)$; that is, $T$ is a homomorphism of $\sum Z_2 \times G$ into $\mathrm{Aut}(X_1 \times X_2, \mathbf{F}_1 \otimes \mathbf{F}_2, \mu_1 \times \mu_2)$, where $G$ is a countable group.

Let $\Psi(\sum Z_2 \times G)$ be the family of all finite subsets of $\sum Z_2 \times G$. Consider the transformation $\pi : \Psi(\sum Z_2 \times G) \to L^1_+(X_1 \times X_2)$.

DEFINITION 5. A family of functions $\{q_H \in L^1_+(X_1 \times X_2) : H \in \Psi(\sum Z_2 \times G)\}$ is called a subadditive process, if the following conditions hold:

If $H = \bigcup_{i=1}^n B_i.c_i$, where $B_i \in \Psi(\sum Z_2 \times G)$ and $c_i \in \sum Z_2 \times G$, $i = 1, \dots, n$, then for a.e. $x \in X_1 \times X_2$, $q_H(x) \leqslant \sum_{i=1}^n q_{B_i}(T^{c_i}x)$.

2) For each $g \in \sum Z_2 \times G$ and for a.e. $x \in X_1 \times X_2$ $q_{Hg}(x) = q_H(T_g x)$.

We have the following theorem.

THEOREM 3. *Let $G$ be a countable amenable group. If $\{q_H \in L^1_+(X_1 \times X_2) : H \in \Psi(\sum Z_2 \times G)\}$ is a subadditive process, then for arbitrary Følner sequence $F = \{\tilde{F}_n\}_{n\geqslant 1}$, $\tilde{F}_n \in \Psi(\sum Z_2 \times G)$, there exists a measurable function $\tilde{q}$ such that $q_H \in L^1_+(X_1 \times X_2)$, $\tilde{q}(T(x_1,x_2)) = \tilde{q}(x_1,x_2)$ for a.e. $(x_1,x_2) \in X_1 \times X_2$; $\lim\limits_{n\to\infty} \frac{1}{|\tilde{F}_n|} q_{\tilde{F}_n} = \tilde{q}$ a.e., and*

$$\lim_{n\to\infty} \frac{1}{|\tilde{F}_n|} \int_{X_1\times X_2} q_{\tilde{F}_n}\, d(\mu_1\times\mu_2) = \inf_n \frac{1}{|\tilde{F}_n|} \int_{X_1\times X_2} q_{\tilde{F}_n}\, d(\mu_1\times\mu_2) = \int_{X_1\times X_2} \tilde{q}\, d(\mu_1\times\mu_2)$$

PROOF. Easily follows from Theorem 1. □

REMARK 1. Since $\sum Z_2 \times G$ is a amenable, there is a Følner sequence $F = \{\tilde{F}_n\}_{n\geqslant 1}$, $\tilde{F}_n \in \Psi(\sum Z_2 \times G)$. However, it is difficult to construct a sequence such as mentioned above.

REMARK 2. In the same way, it is possible to prove Theorem 3 for $(\sum Z_2)^d \times G$.

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## References

[1] J. F. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Royal Statist. Soc. B **30** (1968), 499–510.

[2] U. Krengel, *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.

[3] D. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Anal. Math. **48** (1987), 1–141.

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