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ON QUARTER-SYMMETRIC METRIC CONNECTIONS ON A HYPERBOLIC KAEHLERIAN SPACE

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ABSTRACT. We consider the problem of two kinds of quarter-symmetric metric connections on a hyperbolic Kaehlerian space, their curvature conditions and invariants.

1. Introduction

In 1924, Friedman and Schouten introduced the idea of semi-symmetric linear connection in differentiable manifolds. In 1932, Hayden introduced the idea of metric connection with torsion in Riemannian manifolds. In 1970, Yano studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Gołąb introduced the idea of a quarter-symmetric linear connection in differentiable manifolds. In 1980, Mishra and Pandey introduced a quarter-symmetric F-connection in Riemannian manifolds with F-structures; especially, they considered the case of Kaehlerian structure. In 1982, Yano and Imai studied some curvature conditions for quarter-symmetric metric connections in Riemannian, Hermitian and Kaehlerian manifolds.

Our aim is to transpond some of previous results into hyperbolic Kaehlerian spaces and to give curvature and derivational conditions for two naturaly introduced quarter-symmetric connections in such kind of manifolds.

2. On a hyperbolic Kaehlerian manifold

A hyperbolic Kaehlerian manifold (space) is an even-dimensional pseudo-Riemannian manifold endowed with an F-structure satisfying

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(2.1) $$F_k{}^i F_j{}^k = \delta^i_j,$$

(2.2) $$F_{ij} = F_i{}^k g_{kj} = -F_{ji},$$

(2.3) $$\overset{\circ}{\nabla}_k F_i{}^j = 0,$$

where $\overset{\circ}{\nabla}$ denotes the Levi–Civita covariant differentiation in the underlying pseudo-Riemannian manifold. By the condition (2.2), the isomorphism F sends any tangent vector into an orthogonal one; as the structure has n linearly independent eigenvectors, they are null or isotropic vector fields. We can even use them as a base of the tangent space, if we find it convenient. Also, there are tangent vectors of negative scalar square. If we need it, we can form a base of the tangent space with a half of positive-squared basic vectors and the other half of negative-squared. Moreover, negative-squared vectors may be images of positive-squared ones under the structure.

Therefore, a hyperbolic Kaehlerian space itself is a product of two equally-dimensioned Riemannian spaces; one of them has positive definite metrics and the other one has negative definite metrics. The structure sends any of these two subspaces into the other. The geometry of hyperbolic Kaehlerian spaces is rather different from the geometry of Kaehlerian spaces.

3. A quarter-symmetric metric connection

According to Yano and Imai [4], if ∇ is a quarter-symmetric connection ∇ with torsion tensor $T^i_{jk} = p_j A^i_k - p_k A^i_j$, ($p_j$ being components of a 1-form and A^i_k components of any $(1,1)$ tensor) and if it is a metric one $\nabla_k g_{ij} = 0$, then its components have the following form:

$$\Lambda^i_{jk} = \{{}^{\,i}_{jk}\} - p_k U_j{}^i + p_j V^i{}_k - p^i V_{jk},$$

where $\{{}^{\,i}_{jk}\}$ denotes the coefficients of Levi–Civita connection and

$$U_{ij} = (A_{ij} - A_{ji})/2; \quad V_{ij} = (A_{ij} + A_{ji})/2$$

and, consequently $A_{ij} = U_{ij} + V_{ij}$.

On a hyperbolic Kaehlerian space, there are two fundamental tensors; one of them is symmetric and the other one is skew-symmetric. It is natural to construct a quarter-symmetric metric connection over them. Then $V_{ij} = g_{ij}$, $U_{ij} = F_{ij}$ and

(3.1) $$\Lambda^i_{jk} = \{{}^{\,i}_{jk}\} - p_k F^i_j + p_j \delta^i_k - p^i g_{jk}.$$

These are components of *natural quarter-symmetric metric connection* in a hyperbolic Kaehlerian space.

4. Curvature tensor and invariant

We can calculate the components of the curvature tensor of the connection (3.1). Then

(4.1)
$$\begin{aligned}R^i_{jkl} = K^i_{jkl} &+ F^i_j(\overset{\circ}{\nabla}_k p_l - \overset{\circ}{\nabla}_l p_k) + \delta^i_k(\overset{\circ}{\nabla}_l p_j - p_j p_l + q_j p_l + \frac{1}{2}p_s p^s g_{jl})\\ &- \delta^i_l(\overset{\circ}{\nabla}_k p_j - p_j p_k + p_k q_j + \frac{1}{2}p_s p^s g_{jk}) + g_{jl}(\overset{\circ}{\nabla}_k p^i - p^i p_k + p_k q^i + \frac{1}{2}p_s p^s \delta^i_k)\\ &- g_{jk}(\overset{\circ}{\nabla}_l p^i - p^i p_l + p_l q^i + \frac{1}{2}p_s p^s \delta^i_l) - p_l p_j F^i_k + p^i p_k F_{jl} + p_k p_j F^i_l - p^i p_l F_{jk}.\end{aligned}$$

By q_j are denoted the components of the image of 1-form p_j by the structure. K^i_{jkl} are components of the curvature tensor of the Levi–Civita connection. Then, lowering the upper index in (4.1) and supposing that p_j is a gradient, we obtain

$$\begin{aligned}R_{ijkl} = K_{ijkl} &- g_{il}p_{kj} + g_{ik}p_{lj} - g_{kj}p_{li} + g_{lj}p_{ki}\\ &+ p_j p_l F_{ik} + p_i p_k F_{jl} - p_j p_k F_{il} - p_i p_l F_{jk},\end{aligned} \tag{4.2}$$

where p_{kj} stands for $\overset{\circ}{\nabla}_k p_j - p_k p_j + p_k q_j + \frac{1}{2}p_s p^s g_{kj}$. Transvecting (4.2) by g^{il}, we get

$$R_{jk} = K_{jk} - (n-2)p_{kj} - g_{kj}p^i_i - p_j q_k + p_k q_j - p_s p^s F_{kj} \tag{4.3}$$

for Ricci tensors of natural quarter-symmetric metric connection (R_{jk}) and Levi-Civita connection (K_{jk}) respectively. Then, for the curvature scalars of these two connections, we have $R - K = -2(n-1)p^i_i$ and, consequently $p^i_i = \dfrac{K-R}{2(n-1)}$. So

$$(n-2)p_{kj} = K_{jk} - R_{jk} - \frac{K-R}{2(n-1)}g_{kj} - p_j q_k + p_k q_j - p_s p^s F_{jk}. \tag{4.4}$$

The Ricci tensor R_{jk} may not be symmetric. Actually, we can get

$$R_{jk}F^{jk} = 2(n-2)p_s p^s, \qquad R_{kj}F^{jk} = -2(n-2)p_s p^s.$$

¿From the original expression for p_{kj}, we can see that the only member which may not be symmetric is $p_k q_j$. The expression (4.4) also has members of such a kind. From this fact, we can get

$$p_j q_k = \frac{1}{(n-2)(n-4)}[R_{jk} - (n-3)R_{kj}] + \frac{1}{n-4}p_s p^s F_{jk} \tag{4.5}$$

and, consequently

$$p_j p_l = \frac{1}{(n-2)(n-4)}[R_{jk} - (n-3)R_{kj}]F^k_l - \frac{1}{n-4}p_s p^s g_{lj}. \tag{4.6}$$

Using (4.5), we obtain from (4.4)

$$p_{kj} = \frac{1}{n-2}\Big[K_{jk} - R_{jk} - g_{jk}\frac{K-R}{2(n-1)}\Big] - \frac{1}{(n-2)^2}(R_{kj} - R_{jk}) - \frac{1}{n-4}p_s p^s F_{jk}. \tag{4.7}$$

By (4.7), we obtain from (4.2)

(4.8)

$$
\begin{aligned}
&K_{ijkl} - \frac{1}{n-2}\Big[g_{il}K_{jk} - g_{ik}K_{jl} + g_{jk}K_{il} - g_{jl}K_{ik} - \frac{K}{n-1}(g_{il}g_{kj} - g_{ik}g_{jl})\Big]\\
&= R_{ijkl} - \frac{1}{n-2}\big[g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik} - \frac{R}{n-1}(g_{il}g_{jk} - g_{ik}g_{lj})\big]\\
&\quad - \frac{1}{(n-2)^2}\big[g_{il}(R_{kj}-R_{jk}) - g_{ik}(R_{lj}-R_{jl}) + g_{kj}(R_{li}-R_{il}) - g_{lj}(R_{ki}-R_{ik})\big]\\
&\quad - \frac{1}{(n-2)(n-4)}\big\{[R_{js} - (n-3)R_{sj}]F_l{}^sF_{ik} + [R_{is} - (n-3)R_{si}]F_k{}^sF_{jl}\\
&\qquad - [R_{js} - (n-3)R_{sj}]F_k{}^sF_{il} - [R_{is} - (n-3)R_{si}]F_l{}^sF_{jk}\big\}\\
&\quad + \frac{2}{n-4}p_sp^s(F_{ik}g_{lj} + F_{jl}g_{ik} - F_{il}g_{jk} - F_{jk}g_{il}).
\end{aligned}
$$

There is Weyl's conformal curvature tensor on the left-hand side of (4.8). There is no member depending on p_i (the generator of the quarter-symmetric metric connection) in (4.8) except the last row. Now, suppose

$$F_{ik}g_{lj} + F_{jl}g_{ik} - F_{il}g_{jk} - F_{kj}g_{il} = 0.$$

Transvecting this relation by F^{ki}, we obtain $(n+2)g_{lj} = 0$. So, the only way for the tensor on the right-hand side of (4.10) to be an invariant of chosen natural quarter-symmetric metric connection is $p_sp^s = 0$. That means that there will be no invariant unless the generator of chosen natural quarter-symmetric metric connection is isotropic.

So, we proved

THEOREM 1. *The tensor*

(4.9)

$$
\begin{aligned}
&R_{ijkl} - \frac{1}{n-2}\Big[g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik} - \frac{R}{n-1}(g_{il}g_{jk} - g_{ik}g_{lj})\Big]\\
&\quad - \frac{1}{(n-2)^2}\big[g_{il}(R_{kj}-R_{jk}) - g_{ik}(R_{lj}-R_{jl}) + g_{kj}(R_{li}-R_{il}) - g_{lj}(R_{ki}-R_{ik})\big]\\
&\quad - \frac{1}{(n-2)(n-4)}\big\{[R_{js} - (n-3)R_{sj}]F_l{}^sF_{ik} + [R_{is} - (n-3)R_{si}]F_k{}^sF_{jl}\\
&\qquad - [R_{js} - (n-3)R_{sj}]F_k{}^sF_{il} - [R_{is} - (n-3)R_{si}]F_l{}^sF_{jk}\big\}
\end{aligned}
$$

is invariant for all natural quarter-symmetric metric connections whose generator is an isotropic gradient.

Also, we can prove

THEOREM 2. *The Ricci tensor of the natural quarter-symmetric metric connection cannot be symmetric, even if its generator is an isotropic gradient.*

PROOF. If the generator is not an isotropic 1-form, then there is obviously one skew-symmetric member in the sum (4.3). Even if $p_sp^s = 0$, R_{jk} will be symmetric if and only if, by (4.5), $p_jq_k = 0$, or, by (4.6), $p_jp_l = 0$. This means that both the generator and its image should vanish and the quarter-symmetric metric connection in this case is just the Levi–Civita connection.

If the generator is not isotropic and if the Ricci tensor is symmetric, then, from (4.4) we obtain

$$p_j q_k - p_k q_j = \frac{2}{n-4} p_s p^s F_{kj}$$

and from (4.5)

$$p_j q_k - p_k q_j = -\frac{2}{n-4} p_s p^s F_{kj},$$

what cannot be true. □

5. Curvature-type conditions

Yano and Imai in [4] asked whether the curvature tensor of analogue of natural quarter-symmetric metric connection may vanish on an elliptic Kaehlerian space. The answer was no, for such kind of quarter-symmetric connection. For hyperbolic Kaehlerian spaces, the answer is also no, but by other reasons.

THEOREM 3. *If a hyperbolic Kaehlerian space admits a natural quarter-symmetric connection, than its curvature tensor cannot vanish.*

PROOF. Suppose that the tensor R^i_{jkl} vanishes. Then its Ricci tensor also vanishes and, by (4.6), $p_j p_l$ also vanishes. Then the components of the generator also vanish. □

Further, one can ask whether or not the tensor R_{ijkl} satisfies the conditions which are most common for curvature-type tensors. We can easily see that the curvature tensor satisfies $R_{ijkl} = -R_{ijlk}$ and $R_{ijkl} = -R_{jikl}$. But, we have

THEOREM 4. *The curvature tensor of the natural quarter-symmetric metric connection on a hyperbolic Kaehlerian space never satisfies $R_{ijkl} = R_{klij}$.*

PROOF. If the components of the curvature tensor satisfied the above identity, then the Ricci tensor of such a connection would be symmetric. But it is not. □

The fourth common identity for most curvature-like tensors is the first Bianchi identity. Suppose that the curvature tensor of natural quarter-symmetric metric connection satisfies $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. Then, by (4.2)

$$g_{il}(p_{jk} - p_{kj}) - g_{ik}(p_{jl} - p_{lj}) + g_{ij}(p_{kl} - p_{lk}) + 2p_i p_k F_{jl} - 2p_i p_l F_{jk} + 2p_i p_j F_{lk} = 0.$$

Transvecting the equation above by g^{il}, we obtain

$$(n-2)(p_{jk} - p_{kj}) + 2p_k q_j - 2p_j q_k - 2p_s p^s F_{jk} = 0.$$

By (4.4)

$$(n-2)(p_{jk} - p_{kj}) = -R_{jk} + R_{kj} - 2p_j q_k + 2p_k q_j + 2p_s p^s F_{jk}.$$

Then

$$R_{kj} - R_{jk} = 4(p_j q_k - p_k q_j) \tag{5.1}$$

and

$$(n-2)(p_{jk}-p_{kj}) = 2(p_j q_k - p_k q_j + p_s p^s F_{jk}).$$

Then, by the definition of p_{kj}, we have

$$(n-4)(p_j q_k - p_k q_j) = -2 p_s p^s F_{jk}. \tag{5.2}$$

We have two possibilities here

(1) the structure has a bivectorial form
(2) the generator is isotropic and $p_j q_k = p_k q_j$.

The second possibility leads to contradiction with Theorem 2 in view of (5.1). The structure cannot have such a bivectorial form. We are going to prove it. Suppose that $F_{jk} = \alpha(p_j q_k - p_k q_j)$; then, as $F_{jk}F^{kj} = n$, we have $-\alpha^2(p_k p^k q_s q^s) = n$. But $q_k q^k = -p_s p^s$ and then $(p_k p^k)^2 = n/\alpha^2$. By (5.5), $\alpha = -\dfrac{2p_s p^s}{n-4}$ and finally

$$(p_k p^k)^4 = \frac{n(n-4)^2}{4}. \tag{5.3}$$

On the other hand, $q_j = p_k F_j{}^k$. If F_{kj} has such a bivectorial form, then

$$p_k F_j{}^k = \alpha(p_j q^k - p^k q_j) p_k = -\alpha p_k p^k q_j = \frac{2(p_s p^s)^2}{n-4} q_j$$

and

$$(p_s p^s)^2 = \frac{n-4}{2}. \tag{5.4}$$

contradicts to (5.3). So, we have proved

THEOREM 5. *The curvature tensor of the natural quarter-symmetric metric connection do not satisfy the first Bianchi identity, unless the generator is isotropic.*

6. A special quarter-symmetric metric connection

If $U_{ij} = 0$ and $V_{ij} = g_{ij}$, then $\Lambda^i_{jk} = \{{}^{\,i}_{jk}\} + p_j \delta^i_k - p^i g_{jk}$ and such a connection is in fact a semi-symmetric metric connection. This shows that the generalization from semi-symmetric metric connection to quarter-symmetric metric connection was adequate. If however, $U_{ij} = F_{ij}$ and $V_{ij} = 0$, then $\Lambda^i_{jk} = \{{}^{\,i}_{jk}\} - p_k F_j{}^i$. We call such a connection a *special quarter-symmetric metric connection.*

For the curvature tensor of a special quarter-symmetric metric connection, we can easily get by a straightforward calculation

$$R_{ijkl} = K_{ijkl} + F_{ij}(\mathring{\nabla}_k p_l - \mathring{\nabla}_l p_k).$$

Then, we have

THEOREM 6. *If the generator of a special quarter-symmetric metric connection is a gradient, then the curvature tensor of this connection equals to the curvature tensor of Levi–Civita connection.*

The other question is whether or not the curvature tensor of a special quarter-symmetric connection can vanish. Of course, the generator may not be a gradient. Then, as Yano and Imai presumed in a similar case for elliptic Kaehlerian spaces,

$$K_{ijkl} = F_{ij}\alpha_{kl}, \tag{6.1}$$

where α_{kl} stands for $\overset{\circ}{\nabla}_l p_k - \overset{\circ}{\nabla}_k p_l$. At first, we want to know whether α_{kl} can be proportional to the structure tensor ($\alpha_{kl} = \alpha F_{kl}$). If so, then

$$\alpha F_{ij}F_{kl} = -\alpha F_{ij}F_{lk}, \tag{6.2}$$

$$\alpha F_{ij}F_{kl} = -\alpha F_{ji}F_{kl} \tag{6.3}$$

and, as the curvature tensor of the Levi–Civita connection satisfies the first Bianchi identity,

$$\alpha(F_{ij}F_{kl} + F_{ik}F_{lj} + F_{il}F_{jk}) = 0. \tag{6.4}$$

If $\alpha \neq 0$, we transvect (6.4) by F^{ji}. Then this yields $(n+2)F_{kl} = 0$, what is senseless. So, if we assume that the tensor α_{kl} is proportional to the structure tensor, then the proportionality coefficient vanishes and the generator is a gradient. In view of Theorem 6 then the underlying Riemannian space is flat.

Suppose, now, that $R_{ijkl} = 0$ and that α_{kl} is not proportional to F_{kl}. We want to find some possible relations between the structure tensor and the generator. As both the structure and α_{kl} are skew-symmetric tensors, from (6.1) we can see that analogues of (6.2) and (6.3) are always fulfilled. Then, as the curvature tensor of Levi–Civita connection is invariant under changing places of the first and the second pair of indices $F_{ij}\alpha_{kl} = F_{kl}\alpha_{ij}$.

Besides, the first Bianchi identity has the form:

$$F_{ij}\alpha_{kl} + F_{ik}\alpha_{lj} + F_{il}\alpha_{jk} = 0. \tag{6.5}$$

Transvecting (6.5) by F^{ji}, we obtain $n\alpha_{kl} + \delta^j_k\alpha_{lj} + \delta^j_l\alpha_{jk} = 0$, i.e., $n\alpha_{kl} = 0$ and the generator must be a gradient. We have proved

THEOREM 7. *The curvature tensor of a special quarter-symmetric metric connection on a hyperbolic Kaehlerian space can not vanish unless the generator is a gradient. If it is, the curvature tensor vanishes if and only if the underlying pseudo-Riemannian space is flat.*

Also, we can obviously state

COROLLARY. *The curvature tensor of a special quarter-symmetric metric connection can not satisfy the first Bianchi identity, unless it vanishes.*

REMARK. A special quarter-symmetric metric connection on a hyperbolic Kaehlerian space is an F-connection.

On the contrary, for the natural quarter-symmetric metric connection, which can even have a curvature-like tensor invariant, we have:

THEOREM 8. *A natural quarter-symmetric metric connection on a hyperbolic Kaehlerian space is not an F-connection.*

PROOF. Assume that $\nabla_k F_{ij} = 0$ for a connection given by (3.1). Then

$$\overset{\circ}{\nabla}_k F_{ij} - p_i F_{kj} - q_j g_{ki} - p_j F_{ik} + q_i g_{jk} = 0$$

or

$$p_i F_{kj} + q_j g_{ik} + p_j F_{ik} - q_i g_{jk} = 0. \tag{6.6}$$

Transvecting (6.6) by F^{jk}, we obtain $(n-2)p_j = 0$ and such a connection is just a Levi–Civita connection. □

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