PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 79(93) (2006), 109–114 DOI: 10.2298/PIM0693109S

COMMON SPECTRAL PROPERTIES OF LINEAR OPERATORS A AND B SUCH THAT $ABA = A^2$ AND $BAB = B^2$

Christoph Schmoeger

Communicated by Stevan Pilipović

Abstract. Let A and B be bounded linear operators on a Banach space such that $ABA = A^2$ and $BAB = B^2$. Then A and B have some spectral properties in common. This situation is studied in the present paper.

1. Terminology and motivation

Throughout this paper X denotes a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. For $A \in \mathcal{L}(X)$, let $N(A)$ denote the null space of A, and let $A(X)$ denote the range of A. We use

$$\sigma(A),\ \sigma_p(A),\ \sigma_{ap}(A),\ \sigma_r(A),\ \sigma_c(A) \text{ and } \rho(A)$$

to denote spectrum, the point spectrum, the approximate point spectrum, the residual spectrum, the continuous spectrum and the resolvent set of A, respectively. An operator $A \in \mathcal{L}(X)$ is *semi-Fredholm* if $A(X)$ is closed and either $\alpha(A) := \dim N(A)$ or $\beta(A) := \operatorname{codim} A(X)$ is finite. $A \in \mathcal{L}(X)$ is *Fredolm* if A is semi-Fredholm, $\alpha(A) < \infty$ and $\beta(A) < \infty$. The *Fredholm spectrum* $\sigma_F(A)$ of A is given by

$$\sigma_F(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not Fredholm}\}.$$

The dual space of X is denoted by X^* and the adjoint of $A \in \mathcal{L}(X)$ by A^*.

The following theorem motivates our investigation:

Theorem 1.1. *Let $P, Q \in \mathcal{L}(X)$ such that $P^2 = P$ and $Q^2 = Q$. If $A = PQ$ and $B = QP$, then*

(1) $ABA = A^2$ *and* $BAB = B^2$*;*
(2) $\sigma(A) \setminus \{0\} = \sigma(B) \setminus \{0\}$*;*
(3) $\sigma_p(A) \setminus \{0\} = \sigma_p(B) \setminus \{0\}$*;*
(4) $\sigma_{ap}(A) \setminus \{0\} = \sigma_{ap}(B) \setminus \{0\}$*;*

2000 *Mathematics Subject Classification*: Primary 47A10.
Key words and phrases: operator equation, spectrum.

(5) $\sigma_r(A) \setminus \{0\} = \sigma_r(B) \setminus \{0\}$;
(6) $\sigma_c(A) \setminus \{0\} = \sigma_c(B) \setminus \{0\}$;
(7) $\sigma_F(A) \setminus \{0\} = \sigma_F(B) \setminus \{0\}$.

PROOF. (1) $ABA = PQQPPQ = PQPQ = A^2$, $BAB = QPPQQP = QPQP = B^2$. (2) follows from [**2**, Proposition 5.3], (3), (4), (5) and (6) are shown in [**1**, Theorem 3] and (7) follows from [**1**, Theorem 6]. □

The main result of this paper reads as follows:

THEOREM 1.2. *Let $A, B \in \mathcal{L}(X)$ such that $ABA = A^2$ and $BAB = B^2$. Then*
(1) $\sigma_p(A) \setminus \{0\} = \sigma_p(AB) \setminus \{0\} = \sigma_p(BA) \setminus \{0\} = \sigma_p(B) \setminus \{0\}$;
(2) $\sigma_{ap}(A) \setminus \{0\} = \sigma_{ap}(AB) \setminus \{0\} = \sigma_{ap}(BA) \setminus \{0\} = \sigma_{ap}(B) \setminus \{0\}$;
(3) $\sigma_r(A) \setminus \{0\} = \sigma_r(AB) \setminus \{0\} = \sigma_r(BA) \setminus \{0\} = \sigma_r(B) \setminus \{0\}$;
(4) $\sigma_c(A) \setminus \{0\} = \sigma_c(AB) \setminus \{0\} = \sigma_c(BA) \setminus \{0\} = \sigma_c(B) \setminus \{0\}$;
(5) $\sigma(A) = \sigma(B) = \sigma(AB) = \sigma(BA)$;
(6) $\sigma_F(A) = \sigma_F(B) = \sigma_F(AB) = \sigma_F(BA)$.

A proof of Theorem 1.2 will be given in Section 2 of this paper.

For results concerning the operator equations $ABA = A^2$ and $BAB = B^2$ see [**4**], [**6**] and [**7**].

2. Proofs

Throughout we assume that $A, B \in \mathcal{L}(X)$ and that $ABA = A^2$ and $BAB = B^2$. It it easy to see that if $0 \in \rho(A)$ or $0 \in \rho(B)$, then $A = B = I$. So we always assume that $0 \in \sigma(A)$ and $0 \in \sigma(B)$.

PROPOSITION 2.1. $\sigma_p(A) \setminus \{0\} = \sigma_p(AB) \setminus \{0\} = \sigma_p(BA) \setminus \{0\} = \sigma_p(B) \setminus \{0\}$.

PROOF. It suffices to show that $\sigma_p(A) \setminus \{0\} \subseteq \sigma_p(AB) \setminus \{0\} \subseteq \sigma_p(B) \setminus \{0\}$. To this end let $\lambda \in \sigma_p(A) \setminus \{0\}$. Hence there is $x \in X \setminus \{0\}$ such that $Ax = \lambda x$. Then $BAx = \lambda Bx$ and $A^2x = \lambda^2 x$, thus $\lambda ABx = ABAx = A^2x = \lambda^2 x$; this gives

$$ABx = \lambda x, \tag{2.1}$$

hence $\lambda \in \sigma_p(AB) \setminus \{0\}$ and $B(Bx) = B^2x = BABx = \lambda Bx$. Because of (2.1), $Bx \neq 0$, therefore $0 \in \sigma_p(B) \setminus \{0\}$. □

COROLLARY 2.2. *If $\lambda \neq 0$, then*

$$N(A - \lambda I) = N(AB - \lambda I) = A(N(B - \lambda I)),$$
$$N(B - \lambda I) = N(BA - \lambda I) = B(N(A - \lambda I))$$

and

$$\alpha(A - \lambda I) = \alpha(AB - \lambda I) = \alpha(BA - \lambda I) = \alpha(B - \lambda I).$$

PROOF. The proof of Proposition 2.1 shows that $N(A - \lambda I) \subseteq N(AB - \lambda I)$. Let $x \in N(AB - \lambda I)$, thus $ABx = \lambda x$, hence $\lambda Ax = A^2Bx = ABABx = (AB)^2x = \lambda^2 x$. This gives $x \in N(A - \lambda I)$. Hence we have $N(A - \lambda I) = N(AB - \lambda I)$. Similar arguments show that $N(B - \lambda I) = N(BA - \lambda I)$. From [**1**, Proposition 2] we see that $N(AB - \lambda I) = A(N(BA - \lambda I))$, thus $N(AB - \lambda I) = A(N(B - \lambda I))$.

It is easy to see that $N(A) \cap N(B - \lambda I) = \{0\}$, hence the restriction of A to $N(B - \lambda I)$ is injective, thus

$$\alpha(A - \lambda I) = \alpha(AB - \lambda I) = \alpha(A(N(B - \lambda I))) = \alpha(B - \lambda I). \qquad \square$$

PROPOSITION 2.3. *We have*

$$\sigma_{ap}(A) \setminus \{0\} = \sigma_{ap}(AB) \setminus \{0\} = \sigma_{ap}(BA) \setminus \{0\} = \sigma_{ap}(B) \setminus \{0\}.$$

PROOF. It suffices to show that $\sigma_{ap}(A) \setminus \{0\} \subseteq \sigma_{ap}(AB) \setminus \{0\} \subseteq \sigma_{ap}(B) \setminus \{0\}$. To this end let $\lambda \in \sigma_{ap}(A) \setminus \{0\}$. Then there is a sequence (x_n) in X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $(\lambda I - A)x_n \to 0$ $(n \to \infty)$. Let $z_n = (\lambda I - A)x_n$; hence $Ax_n = \lambda x_n - z_n$ and $z_n \to 0$. Then

$$A^2 x_n = \lambda A x_n - A z_n = \lambda(\lambda x_n - z_n) - A z_n = \lambda^2 x_n - \lambda z_n - A z_n$$

and

$$BAx_n = \lambda B x_n - B z_n,$$

thus

$$A^2 x_n = ABAx_n = \lambda ABx_n - ABz_n,$$

therefore

$$\lambda^2 x_n - \lambda ABx_n = (\lambda I + A - AB)z_n, \tag{2.2}$$

this gives $(AB - \lambda I)x_n \to 0$, hence $\lambda \in \sigma_{ap}(AB) \setminus \{0\}$. From(2.2) we get

$$\lambda^2 Bx_n - \lambda B^2 x_n = w_n,$$

where $w_n = (\lambda B + AB - B^2)z_n \to 0$ $(n \to \infty)$. Hence

$$(\lambda I - B)Bx_n = \lambda^{-1} w_n. \tag{2.3}$$

Because of (2.2) there is $m \in \mathbb{N}$ such that

$$Bx_n \neq 0 \text{ for } n \geqslant m \text{ and } (\|Bx_n\|^{-1})_{n \geqslant m} \text{ is bounded.}$$

For $n \geqslant m$ let $y_n = \|Bx_n\|^{-1} Bx_n$. Then $\|y_n\| = 1$ and, by (2.3)

$$(\lambda I - B)y_n = (\lambda \|Bx_n\|)^{-1} w_n \quad (n \geqslant m).$$

Therefore $(\lambda I - B)y_n \to 0$ $(n \to \infty)$, and so $\lambda \in \sigma_{ap}(B) \setminus \{0\}$. $\square$

REMARK. The proof of Proposition 2.3 also follows from Proposition 2.1 if we apply Berberian–Quigley functor (see e.g. [**5**, Theorem 1-5.11]).

PROPOSITION 2.4. $\sigma_r(A) \setminus \{0\} = \sigma_r(AB) \setminus \{0\}$.

PROOF. Let $\lambda \in \sigma_r(A) \setminus \{0\}$. Hence $\lambda \notin \sigma_p(A)$ and $\overline{(\lambda I - A)(X)} \neq X$. Thus $N(\lambda I^* - A^*) \neq \{0\}$. By Proposition 2.1, $N(\lambda I^* - (AB)^*) = N(\lambda I^* - B^* A^*) \neq \{0\}$, hence $\overline{(\lambda I - AB)(X)} \neq X$. Since $\lambda \notin \sigma_p(AB)$(Proposition 2.1), we have $\lambda \in \sigma_r(AB) \setminus \{0\}$.

Now let $\lambda \in \sigma_r(AB) \setminus \{0\}$, hence $\lambda \notin \sigma_p(AB)$ and $\overline{(\lambda I - AB)(X)} \neq X$. It follows that$N(\lambda I^* - (AB)^*) = N(\lambda I^* - B^* A^*) \neq \{0\}$. From Proposition 2.1 we get $N(\lambda I^* - A^*) \neq \{0\}$, thus $\overline{(\lambda I - A)(X)} \neq X$. Since $\lambda \notin \sigma_p(A)$ (Proposition 2.1), $\lambda \in \sigma_r(A)$. $\square$

Corollary 2.5. *$\sigma_r(A) \setminus \{0\} = \sigma_r(B) \setminus \{0\}$.*

Proof. By [**1**, Theorem 3], $\sigma_r(AB) \setminus \{0\} = \sigma_r(BA) \setminus \{0\}$. Now use Proposition 2.4. □

Let $T \in \mathcal{L}(X)$. The number

$$\gamma(T) = \inf\Big\{ \frac{\|TX\|}{d\,(x, N(T))} : x \in X,\, x \notin N(T) \Big\}$$

is called the *minimal modulus* of T; $d\,(x, T)$ denotes the distance of x from $N(T)$. It is well known that $T(X)$ is closed if and only if $\gamma(T) > 0$ (see [**3**, Satz 55.2]).

Proposition 2.6. *$\sigma(A) = \sigma(AB)$.*

Proof. Let $\lambda \in \sigma(A) \setminus \{0\}$ and assume to the contrary that $\lambda \in \rho(AB)$. Then $\alpha(\lambda I - AB) = 0$ and $\lambda \notin \sigma_{ap}(AB)$. By Proposition 2.1 and Proposition 2.3, $\alpha(\lambda I - A) = 0$ and $\lambda \notin \sigma_{ap}(A)$. Therefore

$$\gamma(\lambda I - A) = \inf\{\|(\lambda I - A)x\| : x \in X,\, \|x\| = 1\} > 0$$

hence $(\lambda I - A)(X)$ is closed. Thus we have shown that $\lambda I - A$ is semi-Fredholm. Since $\lambda \in \rho((AB)^*) = \rho(B^*A^*)$, it follows from [**2**, Proposition 5.3] that $\lambda \in \rho(A^*B^*)$. Since $A^*B^*A^* = (A^*)^2$ and $B^*A^*B^* = (B^*)^2$, the same arguments as above show that $\alpha(\lambda I^* - A^*) = 0$ and that $\lambda I^* - A^*$ is semi-Fredholm. By [**3**, Satz 82.1] it follows now that $\beta(\lambda I - A) = \alpha(\lambda I^* - A^*) = 0$, thus $0 \in \rho(A)$, a contradiction. Hence $\sigma(A) \setminus \{0\} \subseteq \sigma(AB) \setminus \{0\}$.

Now let $\lambda \in \sigma(AB) \setminus \{0\}$ and assume that $\lambda \in \rho(A)$. Then $\alpha(\lambda I - A) = 0$ and $\lambda \notin \sigma_{ap}(A)$. Proposition 2.1 and Proposition 2.3 show that $\alpha(\lambda I - AB) = 0$ and $\lambda \notin \sigma_{ap}(AB)$. As in the first part of the proof we conclude that $\gamma(\lambda I - AB) > 0$. Thus $\lambda I - AB$ is semi-Fredholm. Since $\lambda \in \rho(A^*)$, the same arguments as above give $\alpha(\lambda I^* - (AB)^*) = 0$ and $\lambda I^* - (AB)^*$ is semi-Fredholm. From $\beta(\lambda I - AB) = \alpha(\lambda I^* - (AB)^*) = 0$ we get the contradiction $\lambda \in \rho(AB)$.

So far we have $\sigma(A) \setminus \{0\} = \sigma(AB) \setminus \{0\}$. It remains to show that $0 \in \sigma(AB)$. Assume to the contrary that there is $C \in \mathcal{L}(X)$ with $ABC = I = CAB$. Then $N(B) = \{0\}$ and $B^2C = B$, therefore $BC = I$, hence $A = I$, a contradiction. □

Corollary 2.7. *$\sigma(A) = \sigma(B)$.*

Proof. Since $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ [**2**, Proposition 5.3], Proposition 2.6 shows that $\sigma(A) \setminus \{0\} = \sigma(B) \setminus \{0\}$. Because of $0 \in \sigma(A) \cap \sigma(B)$, we have $\sigma(A) = \sigma(B)$. □

Proposition 2.8. *$\sigma_c(A) \setminus \{0\} = \sigma_c(AB) \setminus \{0\}$.*

Proof. By Proposition 2.1, Proposition 2.4 and Proposition 2.6

$$\begin{aligned}\sigma_c(A) \setminus \{0\} &= \sigma(A) \setminus [\sigma_p(A) \cup \sigma_r(A) \cup \{0\}] \\ &= \sigma(AB) \setminus [\sigma_p(AB) \cup \sigma_r(AB) \cup \{0\}] = \sigma_c(AB) \setminus \{0\}.\end{aligned}$$ □

Corollary 2.9. *$\sigma_c(A) \setminus \{0\} = \sigma_c(B) \setminus \{0\}$.*

Proof. Use Proposition 2.8 and [**1**, Theorem 3]. □

In what follows $\mathcal{A}$ denotes a complex unital Banach algebra. For $a \in \mathcal{A}$ we write $\sigma(a)$ for the spectrum of a and λ_a for the bounded linear operator on $\mathcal{A}$ given by $\lambda_a(x) = ax$ $(x \in \mathcal{A})$.

PROPOSITION 2.10. *Let $a, b \in \mathcal{A}$.*
(1) $\sigma(a) = \sigma(\lambda_a)$;
(2) $\lambda_{ab} = \lambda_a \lambda_b$;
(3) *if* $aba = a^2$ *and* $bab = b^2$, *then* $\lambda_a\lambda_b\lambda_a = \lambda_a^2$, $\lambda_b\lambda_a\lambda_b = \lambda_b^2$, *and*

$$\sigma(a) = \sigma(b) = \sigma(ab) = \sigma(ba).$$

PROOF. (1) [**2**, Proposition 3.19].
(2) Clear.
(3) follows from (1), (2), Proposition 2.6 and Corollary 2.7. □

Let $\mathcal{K}(X)$ denote the ideal of all compact operators in $\mathcal{L}(X)$ and let $\widehat{\mathcal{L}}$ denote the quotient algebra $\mathcal{L}(X)/(X)$. By $\widehat{T}$ we denote the coset $T + \mathcal{K}(X) \in \widehat{\mathcal{L}}$ $(T \in \mathcal{L}(X))$. Observe that $\widehat{\mathcal{L}}$ is a Banach algebra with unit $\widehat{I}$. Satz 81.2 in [**3**] shows that for $T \in \mathcal{L}(X)$ we have

$$T \quad \text{is Fredholm} \iff 0 \notin \sigma(\widehat{T}).$$

Hence

$$\sigma_F(T) = \sigma(\widehat{T}). \tag{2.4}$$

Since $\widehat{A}\widehat{B}\widehat{A} = \widehat{A}^2$ and $\widehat{B}\widehat{A}\widehat{B} = \widehat{B}^2$, an immediate consequence of Proposition 2.10 and (2.4) is

COROLLARY 2.11. $\sigma_F(A) = \sigma_F(B) = \sigma_F(AB) = \sigma_F(BA)$.

The proof of Theorem 1.2 is now complete. □

If $T \in \mathcal{L}(X)$ is Fredholm then the *index* $\text{ind}(T)$ of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.

COROLLARY 2.12. *Let $\lambda \notin \sigma_F(A)$.*
(1) *If* $\lambda \neq 0$, *then* $\text{ind}(\lambda I - A) = \text{ind}(\lambda I - AB) = \text{ind}(\lambda I - BA) = \text{ind}(\lambda I - B)$.
(2) *If* $\lambda = 0$, *then* $\text{ind}(A) = \text{ind}(B) = \text{ind}(AB) = \text{ind}(BA) = 0$.

PROOF. (1) Because of [**1**, Theorem 6] it suffices to show that $\text{ind}(\lambda I - A) = \text{ind}(\lambda I - BA)$. By Corollary 2.2 and [**3**, Satz 82.1] we have

$$\begin{aligned}
\text{ind}(\lambda I - A) &= \alpha(\lambda I - A) - \alpha(\lambda I^* - A^*)\\
&= \alpha(\lambda I - AB) - \alpha(\lambda I^* - A^*B^*)\\
&= \alpha(\lambda I - AB) - \alpha(\lambda I^* - (BA)^*)\\
&= \alpha(\lambda I - AB) - \beta(\lambda I - BA)\\
&= \alpha(\lambda I - BA) - \beta(\lambda I - BA) = \text{ind}\, I - BA).
\end{aligned}$$

(2) We have $\widehat{A} = \widehat{B} = \widehat{AB} = \widehat{BA} = \widehat{I}$, thus, by [**3**, Satz 82.5] we get

$$\text{ind}(A) = \text{ind}(B) = \text{ind}(AB) = \text{ind}(BA) = \text{ind}(I) = 0.$$

□

An operator $T \in \mathcal{L}(X)$ is called a *Riesz operator* if $\sigma_F(T) = \{0\}$. From Corollary 2.11 we have:

COROLLARY 2.13. *The following assertions are equivalent:*

(1) *A is a Riesz operator;*
(2) *B is a Riesz operator;*
(3) *AB is a Riesz operator;*
(4) *BA is a Riesz operator.*

References

[1] B. A. Barnes, *Common operator properties of the linear operators RS and SR*, Proc. Amer. Math. Soc. 126 (1998), 1055–1061.
[2] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer-Verlag, 1973.
[3] H. Heuser, *Funktionalanalysis*, 2nd ed., Teubner, 1986.
[4] V. Rakočević, *A note on a theorem of I. Vidav*, Publ. Inst. Math. (Beograd) 68(82) (2000), 105–107.
[5] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, 1960.
[6] Ch. Schmoeger, *On the operator equations $ABA = A^2$ and $BAB = B^2$*, Publ. Inst. Math. (Beograd) 78(92) (2005), 127–133.
[7] I. Vidav, *On idempotent operators in a Hilbert space*, Publ. Inst. Math. (Beograd) 4(18) (1964), 157–163.

Institut für Analysis
Universität Karlsruhe (TH)
Englerstraße 2
76128 Karlsruhe
Germany
christoph.schmoeger@math.uni-karlsruhe.de

(Received 15 03 2006)
(Revised 06 09 2006)