PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 80(94) (2006), 253–258 DOI:10.2298/PIM0694253S

ON REGULARLY VARYING MOMENTS FOR POWER SERIES DISTRIBUTIONS

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Abstract. For the power series distribution, generated by an entire function of finite order, we obtain the asymptotic behavior of its regularly varying moments. Namely, we prove that $E_wX^\alpha\ell(X)\sim(E_wX)^\alpha\ell(E_wX)$, $\alpha>0$ $(w\to\infty)$, where $\ell(\cdot)$ is an arbitrary slowly varying function.

0. Introduction

0.1. Denote by A_ρ the class of transcendental entire functions with *positive* Taylor coefficients and of finite order ρ, $0\leqslant\rho<\infty$.

Definition 1. Let $f(w)=\sum a_nw^n$, $f\in A_\rho$. A power series distribution with parameter $w>0$, generated by f, is defined by (cf. [2])

$$P(X=n):=a_nw^n/f(w),\quad n=0,1,2,\cdots$$

Our aim is to obtain the asymptotic behavior of the k-th moment E_wX^k when $w\to\infty$, where

$$E_wX^k:=\sum n^kP(X=n)=\sum n^ka_nw^n/f(w),\quad k=1,2,\cdots$$

Note that the expectation E_wX is equal to

$$E_wX:=\sum na_nw^n/f(w)=wf'(w)/f(w). \tag{1}$$

For any k, consider the sequence of functions $f_k(w)$ defined recursively by

$$f_k(w)=wf'_{k-1}(w),\quad k=1,2,\cdots;\quad f_0(w)=f(w)=\sum a_nw^n.$$

Then $f_k(w)=\sum n^ka_nw^n\in A_\rho$ and

$$E_wX^k=f_k(w)/f(w),\quad k=1,2,\cdots \tag{2}$$

We shall derive the asymptotic behavior of E_wX^k for large w by applying our following recent result:

2000 *Mathematics Subject Classification*: Primary 60E05; Secondary 30D15.
Key words and phrases: regular variation, moments, power series distributions.
Supported by the Ministry of Science of Serbia, Grant 144021.

THEOREM 1. [5] *For an arbitrary* $f \in A_\rho$, *we have*

$$\frac{f(w)f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty).$$

independently of the order ρ.

0.2. Further generalization leads to the concept of *regularly varying moments* $E_w X^\alpha \ell(X)$ (cf. [1, p. 335]),

$$E_w X^\alpha \ell(X) := \sum n^\alpha \ell(n) a_n w^n / f(w), \tag{3}$$

where α is a positive *real* number and $\ell(\cdot)$ is a *slowly varying function.*

DEFINITION 2. A positive continuous function $\ell(\cdot)$, defined on $[x_0, \infty)$, is slowly varying if the asymptotic equivalence $\ell(tx) \sim \ell(x)$, $(x \to \infty)$, holds for each $t > 0$.

For $x \in [0, x_0)$ we can take $f(x) := f(x_0)$. Some examples of slowly varying functions are

$$\log^a x; \quad \log^b(\log x); \quad \exp(\log^c x); \quad \exp(\log x / \log\log x); \quad a, b \in R, \quad 0 < c < 1.$$

Functions $g(\cdot)$ of the form $g(x) = x^\mu \ell(x)$ are *regularly varying* with index $\mu \in R$ (cf. [1, p.18]). Each regularly varying function $x^\mu \ell(x)$ generates a regularly varying sequence of the form $\{n^\mu \ell(n)\}_{n=1}^\infty$.

The main tool for asymptotic estimation of regularly varying moments is the following theorem on matrix transforms with slowly varying sequences (cf. [4]).

THEOREM 2. *For a given complex-valued matrix* $(A_{nk})_{n,k=1}^\infty$ *define* $t_n(\rho) := \sum k^\rho |A_{nk}|$. *Suppose that for some positive constants* a, A, $t_n(\rho)$ *exists for* $-a \leqslant \rho \leqslant 1$ *and, for sufficiently large* n,

(i) $\left|\sum A_{nk}\right| \geqslant A$; *(ii)* $t_n(0) \to 1$; *(iii)* $t_n(1) \to \infty$; *(iv)* $t_n(-a) = O((t_n(1))^{-a})$.

Then the asymptotic relation

$$\sum A_{nk}\ell(k) = \ell(t_n(1)) \left(\sum A_{nk}\right)(1 + o(1)) \qquad (n \to \infty),$$

holds for all slowly varying sequences $\{\ell(k)\}_{k=1}^\infty$.

0.3. Here we quote some well-known assertions we shall need in the sequel.

LEMMA 1. *If* $a(x) \sim b(x) \to \infty$, *then* $\ell(a(x)) \sim \ell(b(x)) \quad (x \to \infty)$.

LEMMA 2. [3, Vol. I, p. 36]. *Let* $g(x) = \sum a_n x^n, h(x) = \sum b_n x^n$, $g, h \in A_\rho$. *If* $a_n \sim b_n$ $(n \to \infty)$, *then* $g(x) \sim h(x)$ $(x \to \infty)$.

LEMMA 3. *Jensen's inequality:* $EX^t \geqslant (EX)^t$, $t > 1$, *and vice versa for* $0 < t < 1$.

LEMMA 4. *Lyapunov moments inequality asserts that, for* $r > s > t > 0$,

$$(EX^s)^{r-t} \leqslant (EX^r)^{s-t}(EX^t)^{r-s}.$$

1. Results

1.1. The above Theorem 2 has many applications in real or complex analysis (cf. [4]). We shall apply it here to derive the following *theorem on regularly varying moments for discrete laws.*

THEOREM 3. *Let a discrete law G be given by* $P(X_n = k) = p_{nk} \geqslant 0$, $\sum_k p_{nk} = 1$. *If* $EX_n \to \infty$ *and* $EX_n^\beta \sim C_\beta (EX_n)^\beta$ $(n \to \infty)$ *for* $\beta \in (0, B]$, $B > 1$, $C_\beta > 0$ *then, for an arbitrary slowly varying function* $\ell(\cdot)$, *the asymptotic relation*

$$EX_n^\beta \ell(X_n) \sim C_\beta (EX_n)^\beta \ell(EX_n) \quad (n \to \infty),$$

holds

a. *for each* $\beta \in (0, B-1]$;

b. *for each* $\beta \in (B-1, B]$, *if* EX_n^{B+1} *exists and* $EX_n^{B+1} = O((EX_n)^{B+1})$ $(n \to \infty)$.

PROOF. Putting $A_{nk} := p_{nk} k^\beta / C_\beta (EX_n)^\beta$, we find out that conditions (i) and (ii) of Theorem 2 are satisfied. For $\beta \in (0, B-1]$, we obtain

$$t_n(1) = EX_n^{\beta+1}/C_\beta (EX_n)^\beta \sim (C_{\beta+1}/C_\beta) EX_n \to \infty \quad (n \to \infty).$$

Also,

$$t_n(-\beta/2) = EX_n^{\beta/2}/C_\beta (EX_n)^\beta \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O(t_n(1))^{-\beta/2} \ (n \to \infty).$$

Therefore, the conditions of Theorem 2 are satisfied with $A = 1$, $a = \beta/2$ and the result for $\beta \in (0, B-1]$ follows. □

For the case $\beta \in (B-1, B]$, we need the following

LEMMA 5. *Under the condition* b *of Theorem* 3, *we have*

$$EX_n^{\beta+1} = O((EX_n)^{\beta+1}) \quad (n \to \infty), \tag{4}$$

for each $\beta \in (B-1, B]$.

PROOF. Indeed, applying Lyapunov moments inequality (Lemma 4) with $r = B+1$, $s = \beta+1$, $t = B$, we get

$$\begin{aligned} EX_n^{\beta+1} &\leqslant (EX_n^B)^{B-\beta} (EX_n^{B+1})^{\beta+1-B} \\ &= O((EX_n)^{B(B-\beta)} (EX_n)^{(B+1)(\beta+1-B)}) = O((EX_n)^{\beta+1}). \end{aligned}$$

Now, by Jensen's inequality $EX_n^{\beta+1} \geqslant (EX_n)^{\beta+1}$ i.e., $t_n(1) \geqslant EX_n/C_\beta \to \infty$ $(n \to \infty)$.

Also, by (4),

$$t_n(-\beta/2) \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O((EX_n^{\beta+1}/(EX_n)^\beta)^{-\beta/2}) = O((t_n(1))^{-\beta/2}).$$

Therefore, the conditions of Theorem 2 are satisfied and we get

$$EX_n^\beta \ell(X_n) \sim C_\beta (EX_n)^\beta \ell(EX_n^{\beta+1}/C_\beta (EX_n)^\beta) \quad (n \to \infty),$$

for each $\beta \in (B-1, B]$.

But, since

$$EX_n/C_\beta \leqslant \frac{EX_n^{\beta+1}}{C_\beta(EX_n)^\beta} = O(EX_n) \quad (n\to\infty),$$

it follows by the uniform convergence theorem for slowly varying functions (cf. [1, p.6]), that

$$\ell\left(\frac{EX_n^{\beta+1}}{C_\beta(EX_n)^\beta}\right) \sim \ell(EX_n) \quad (n\to\infty). \qquad \square$$

1.2. We turn back now to the asymptotic evaluation of regularly varying moments for power series distributions. Using Theorem 3 above, it will be shown that this evaluation is equivalent to the following *theorem on moments of power series distributions.*

THEOREM 4. *For each* $\alpha > 0$*, we have* $E_wX^\alpha \sim (E_wX)^\alpha \quad (w\to\infty)$.

For the generating entire function $f(w) = \sum a_k w^k \in A_\rho$, recall (1) and (2):

$$E_wX = \sum k a_k w^k / f(w) = wf'(w)/f(w);$$
$$E_wX^m = \sum k^m a_k w^k / f(w) = f_m(w)/f(w).$$

The proof of Theorem 4 requires some preliminary lemmas.

LEMMA 6. *The expectation* E_wX *is a monotone increasing and unbounded function in* w.

PROOF. Since

$$w\frac{d}{dw}(E_wX) = E_wX^2 - (E_wX)^2 > 0,$$

we conclude that E_wX is a monotone increasing function in w. If it is bounded, then there exists a $d > 0$ such that $E_wX < d$ for each $w > 0$. By (1) we get $f'(w)/f(w) < d/w$, and integrating we find $f(w) = O(w^d)$. Hence in this case f is a polynomial, which contradicts our assumption that f is a transcendental entire function. $\square$

LEMMA 7. *For* $m \in N$, $f_m(w) \sim w^m f^{(m)}(w) \quad (w\to\infty)$.

PROOF. Note that $f \in A_\rho$ implies $f^{(m)}, f_m \in A_\rho$, $\ m = 1, 2, \cdots$. Since, for fixed $m \in N$,

$$f_m(w) = \sum k^m a_k w^k; \quad w^m f^{(m)}(w) = \sum_{k\geqslant m} k(k-1)\cdots(k-m+1)a_k w^k;$$
$$k(k-1)\cdots(k-m+1) \sim k^m \quad (k\to\infty),$$

the result follows by Lemma 2. $\square$

LEMMA 8. *For each* $m \in N$ *we have* $E_wX^{m+1}/E_wX^m \sim E_wX \quad (w\to\infty)$.

PROOF. Applying Theorem 1, we obtain

$$\frac{E_w X^2}{(E_w X)^2} \to 1 \quad (w \to \infty), \tag{5}$$

because

$$\frac{E_w X^2}{(E_w X)^2} - \frac{1}{E_w X} = \frac{f(w) f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty),$$

and, by Lemma 6, $1/E_w X \to 0$.

Since Theorem 1 is valid for each $f \in A_\rho$ and $f^{(m)} \in A_\rho$, $m = 1, 2, \cdots$, replacing f by $f^{(m)}$, we get

$$\frac{f^{(m+1)}(w) f^{(m-1)}(w)}{(f^{(m)}(w))^2} \to 1 \text{ i.e. } \frac{f^{(m+1)}(w)}{f^{(m)}(w)} \sim \frac{f^{(m)}(w)}{f^{(m-1)}(w)} \quad (w \to \infty). \tag{6}$$

Hence by Lemma 7 and (6),

$$\frac{E_w X^{m+1}}{E_w X^m} = \frac{f_{m+1}(w)}{f_m(w)} \sim \frac{w^{m+1} f^{(m+1)}(w)}{w^m f^{(m)}(w)} \sim \frac{w^m f^{(m)}(w)}{w^{m-1} f^{(m-1)}(w)}$$
$$\sim \frac{f_m(w)}{f_{m-1}(w)} = \frac{E_w X^m}{E_w X^{m-1}}, \ n \in N.$$

Therefore,

$$\frac{E_w X^{m+1}(w)}{E_w X^m} \sim \frac{E_w X^m}{E_w X^{m-1}} \sim \cdots \sim \frac{E_w X^2}{E_w X} \sim E_w X \quad (w \to \infty). \qquad \square$$

A simple consequence of the previous lemma is the following:

LEMMA 9. *For each $m \in N$, we have $E_w X^m \sim (E_w X)^m \quad (w \to \infty)$.*

PROOF. Indeed,

$$E_w X^m = (E_w X) \prod_{k=1}^{m-1} (E_w X^{k+1} / E_w X^k) \sim (E_w X)^m \quad (w \to \infty).$$

For the rest of the proof of Theorem 4 we apply Lemma 4.

Let $m > \alpha > m - 1$, $m \in N$. Then Lyapunov's inequality and Lemma 9 give

$$E_w X^\alpha \leqslant (E_w X^m)^{\alpha - m + 1} (E_w X^{m-1})^{n - \alpha} \sim (E_w X)^{m(\alpha - m + 1)} (E_w X)^{(m-1)(m-\alpha)}$$
$$= (E_w X)^\alpha.$$

Hence

$$\limsup_{w \to \infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \leqslant 1.$$

Now, let $r = m+1$, $s = m$, $t = \alpha$. We get $(E_w X^m)^{m+1-\alpha} \leqslant (E_w X^\alpha)(E_w X^{m+1})^{n-\alpha}$, i.e.,

$$E_w X^\alpha \geqslant (E_w X^m)^{m+1-\alpha} (E_w X^{m+1})^{\alpha - m} \sim (E_w X)^{m(m+1-\alpha)} (E_w X)^{(m+1)(\alpha - m)}$$
$$= (E_w X)^\alpha.$$

Therefore,

$$\liminf_{w\to\infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \geqslant 1,$$

and this concludes the proof of Theorem 4. □

1.3. Combining the last two theorems, we finally obtain a *theorem on regularly varying moments for power series distributions*.

THEOREM 5. *For a power series distribution generated by an entire function* $f(w) = \sum a_k w^k \in A_\rho$, *we have*

$$E_w X^\alpha \ell(X) \sim (E_w X)^\alpha \ell(E_w x),\ \ \alpha > 0 \quad (w \to \infty),$$

i.e.,

$$\sum k^\alpha \ell(k) a_k w^k / f(w) \sim (w f'(w)/f(w))^\alpha \ell(w f'(w)/f(w)) \quad (w \to \infty),$$

where $\ell(\cdot)$ *is an arbitrary slowly varying function .*

As an example we take the well-known Poisson distribution. Applying Theorem 5, we obtain

THEOREM 6. *For the Poisson law defined by* $P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda},\ \lambda > 0,\ k = 0, 1, 2, \cdots,$ *we have*

$$E X^\alpha \ell(X) := \sum k^\alpha \ell(k) \frac{\lambda^k}{k!} e^{-\lambda} \sim \lambda^\alpha \ell(\lambda),\ \ \alpha > 0 \quad (\lambda \to \infty).$$

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(Received 12 05 2006)
(Revised 19 10 2006)