PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 81(95) (2007), 29–43 DOI 102298/PIM0795029M

AN ATTEMPT AT FRANKL'S CONJECTURE

Petar Marković

Abstract. In 1979 Frankl conjectured that in a finite union-closed family $\mathcal{F}$ of finite sets, $\mathcal{F} \neq \{\emptyset\}$ there has to be an element that belongs to at least half of the sets in $\mathcal{F}$. We prove this when $|\bigcup \mathcal{F}| \leqslant 10$.

1. Introduction

Frankl's conjecture (sometimes also called the union-closed sets conjecture) is one of the most famous problems in combinatorics. There has been extensive research of this problem and the amount of papers published is quite large. One of the most popular lines of attack is to prove the conjecture for the first finitely many cases. There are two ways in which this was done: proving the conjecture for $|\mathcal{F}| \leqslant n$ and proving the conjecture for $|\bigcup \mathcal{F}| \leqslant m$. The second approach has managed to achieve $m \leqslant 9$ (in [**13**]). In this paper we push it one step further.

Another approach is to find small union-closed $\mathcal{G}$ families which imply that any union-closed family $\mathcal{F}$ containing $\mathcal{G}$ satisfies the Frankl's conjecture, with the element appearing in one-half of the sets being one of the elements which belong to $\bigcup \mathcal{G}$. In the present paper we include several such results, which are not new, as lemmas when we need them. Credit for these lemmas will be assigned where it is due, but we will reprove them here in order to keep the paper self-contained and to familiarize the reader with the basic technique which will be used for the most of the paper. Along these lines, Vaughan (with coauthors) has recently proved several nice results in [**24**], [**25**], [**4**] and [**26**] and these results were further improved by Morris in [**13**].

It is important to mention that the main technique used here (assigning weights to elements and sets) is mostly a rephrasing of the technique initially discovered by Poonen in [**16**]. However, the new idea is to (implicitly) use several weights at the same time. This allows us to push the bounds much further than was previously possible. This method will most probably not prove the whole conjecture. However, it does deal with the small cases efficiently, and is amenable to an algorithmic

2000 *Mathematics Subject Classification*: Primary 05D05; Secondary 05A05, 04A20.
Key words and phrases: Frankl's conjecture, union-closed sets conjecture.
Supported by Ministry of Science and Environment of Serbia, grant 144011G..

approach. We feel that in the hands of a good programmer it could probably push the bounds of the Frankl's conjecture even further.

The results in this paper were first proved as a part of the author's qualifying paper at the Vanderbilt University, in 2000. Because of the requirements for such a paper, the bibliography was quite extensive. Rather than reducing the bibliography, since there is quite a lot of activity in the field recently, we updated it. We feel that a large bibliography could be an asset to a researcher starting to work on Frankl's Conjecture.

2. Initial Results

Throughout this paper $\mathcal{F}$ will denote a finite family of finite sets closed under unions and X will denote the union of $\mathcal{F}$. We will call $\mathcal{F}$ *Frankl's* if $X = \bigcup \mathcal{F}$ contains an element which is in at least one half of the sets from $\mathcal{F}$.

Definition 2.1. We call any function $w : X \to \{x \in R \mid x \geqslant 0\}$, such that $w(a) > 0$ for some $a \in X$, a *weight function*. The weight $w(S)$, for $S \subseteq X$ is equal to $\sum_{x \in S} w(x)$. The number $0.5w(X)$ will be called the *target weight* and denoted by $t(w)$.

Lemma 2.1. *$\mathcal{F}$ is Frankl's if and only if there is a weight function w assigned to elements of $X = \bigcup \mathcal{F}$ such that $\sum_{S \in \mathcal{F}} w(S) \geqslant t(w)|\mathcal{F}|$.*

Proof. ($\Rightarrow$) Let a be an element of at least half of the sets in $\mathcal{F}$. Take the weight function w such that $w(a) = 1$ and $w(x) = 0$ for $x \neq a$. Then $t(w) = 0.5$, and the inequality is obviously satisfied.

($\Leftarrow$) Assume that $\mathcal{F}$ is not Frankl's. Let $n_a(\mathcal{F})$ be the number of occurrences of the element a in sets from $\mathcal{F}$. We take an arbitrary weight function w. Then

$$\sum_{S \in \mathcal{F}} w(S) = \sum_{S \in \mathcal{F}} \sum_{a \in S} w(a) = \sum_{a \in X} w(a) n_a(\mathcal{F}) < \sum_{a \in X} w(a) \frac{|\mathcal{F}|}{2} = t(w)|\mathcal{F}|. \qquad \square$$

The following lemma is one of the earliest results on union-closed families.

Lemma 2.2. *If $\mathcal{F}$ contains a one-element set, or a two-element set, then it is Frankl's.*

Proof. In case of a one-element set, let this set be $\{a\}$. Consider the sets in $\mathcal{F}$ that do not contain a. For each such set K, the set $K \cup \{a\}$ must also be a member of $\mathcal{F}$. Therefore we have an injection from sets in $\mathcal{F}$ that do not contain a into the sets in $\mathcal{F}$ that do contain a, so a must be a member of at least half of the sets in $\mathcal{F}$.

Now, let $\mathcal{F}$ contain a two-element set $\{a, b\}$. Let us assign weight function w to members of X such that $w(a) = w(b) = 1$ and $w(x) = 0$ for all other $x \in X$. The target weight $t(w) = 1$. If we consider the full powerset lattice of subsets of X, we can partition this lattice into intervals $[K, K \cup \{a, b\}]$, where K are subsets of X which do not contain a or b. If we just look at the sets which are in one such interval, we see that:

- If $K \in \mathcal{F}$ $(w(K) = 0)$, then $K \cup \{a, b\} \in \mathcal{F}$ $(w(K \cup \{a, b\}) = 2)$ and

- $w(K \cup \{a\}) = w(K \cup \{b\}) = 1$.

Therefore, for sets from $\mathcal{F}$ which are in this interval, we have that the average weight is at least 1. As the interval was picked arbitrarily, the same can be said for all of $\mathcal{F}$. This, by Lemma 2.1 is sufficient to show that $\mathcal{F}$ is Frankl's. □

DEFINITION 2.2. For $S, K \subseteq X$, $S \cap K = \emptyset$ we call any interval in the Boolean lattice $\mathcal{P}(X)$ of the form $[K, K \cup S]$ an *S-hypercube*. We can partition a hypercube into levels, where set is on level k if and only if k is the cardinality of its intersection with S.

Let $\mathcal{F}$ a union-closed family of sets and w a weight function. The *deficit* of a set $L \subseteq X$ with $w(L) < t(w)$ is $d(L) = t(w) - w(L)$. The *surplus* of a set $L \subseteq X$ with $w(L) > t(w)$ is $s(L) = w(L) - t(w)$ Let $\mathcal{C}$ be an S-hypercube. The deficit of $\mathcal{C}$ and is the surplus of $\mathcal{C}$ are defined to be

$$d(\mathcal{C}) = \sum_{\substack{L \in \mathcal{C} \cap \mathcal{F} \\ w(L) < t(w)}} (t(w) - w(L)) \quad \text{and} \quad s(\mathcal{C}) = \sum_{\substack{L \in \mathcal{C} \cap \mathcal{F} \\ w(L) > t(w)}} (w(L) - t(w)).$$

respectively.

It is an obvious consequence of Lemma 2.1 that if for some weight function w the sum of surpluses of the sets in $\mathcal{F}$ which have weights greater than $t(w)$ is greater than or equal to the sum of deficits of the sets in $\mathcal{F}$ which have weights less than $t(w)$, then $\mathcal{F}$ is Frankl's. Also, if for every hypercube $\mathcal{C}$, $s(\mathcal{C}) \geqslant d(\mathcal{C})$, then $\mathcal{F}$ is Frankl's.

The following proposition was proved in [**16**].

PROPOSITION 2.1. *Assume that $\mathcal{F}$ contains three different 3-element sets which are all subsets of the same four-element set. Then $\mathcal{F}$ is Frankl's.*

PROOF. Assume that $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are the three sets from the statement. Then also $\{a, b, c, d\} \in \mathcal{F}$. We are considering the weight $w(x) = 1$ for $x \in \{a, b, c, d\}$, and $w(x) = 0$ else. Take an arbitrary $\{a, b, c, d\}$-hypercube, $\mathcal{C} = [K, K \cup \{a, b, c, d\}]$.

If any set in $\mathcal{C}$ is in $\mathcal{F}$, then the top set of $\mathcal{C}$ is in $\mathcal{F}$, also. If $K \in \mathcal{F}$, then $s(\mathcal{C}) \geqslant 5$, as three of the level 3 sets are in $\mathcal{F}$. So, either $s(\mathcal{C}) \geqslant d(\mathcal{C})$, or all four level 1 sets are also in $\mathcal{F}$. But then all four level 3 sets are in $\mathcal{F}$, too, so again we have that $s(\mathcal{C}) \geqslant d(\mathcal{C})$.

If $K \notin \mathcal{F}$, then either $d(\mathcal{C}) = 0$, or a level 1 set is in $\mathcal{F}$. If latter is the case, then $s(\mathcal{C}) \geqslant 4$, as the top set and at least 2 level 3 sets are in $\mathcal{F}$ (the second as each of a, b, c, d is in at least two of the three sets from the statement of the lemma). Again, we get $s(\mathcal{C}) \geqslant d(\mathcal{C})$. □

Notice that in all the S-hypercubes we considered in the previous proof we had that if they had nonempty intersection with $\mathcal{F}$, then their top is in $\mathcal{F}$. This is a consequence of our choice of S, as we had $S \in \mathcal{F}$. We will continue with this practice, so we will not repeat later that if an S-hypercube contains a set from $\mathcal{F}$, then the top of this hypercube must also be in $\mathcal{F}$.

The results in [**26**] give a sharper bound for the number of three-element subsets of a six-element set which can be in $\mathcal{F}$, but the following will be sufficient for our purpose:

COROLLARY 2.1. *If $\mathcal{F}$ contains 11 three-element sets which are all subsets of the same six-element set, then $\mathcal{F}$ is Frankl's.*

PROOF. We will prove that in such case $\mathcal{F}$ contains three three-element sets in the same four-element set, and then we are done by Proposition 2.1.

In a six-element set there are $\binom{6}{3} = 20$ three-element subsets. They can be partitioned into 10 pairs of complements. By the pigeon-hole principle, there must be two three element sets among the 11 in $\mathcal{F}$ which are complements of each other. Let these two sets be A and B. Each of the other 9 three-element sets intersects one of A and B in a two-element set and the other in a one-element set. Without loss of generality we may assume that at least 5 of the three-element sets from $\mathcal{F}$ have a two-element intersection with A and one-element intersection with B.

Now we consider these 5 three-element sets. There must be two among them, C and D which have $C \cap B = D \cap B = \{b\}$ (pigeon-hole principle again, as there are 5 sets, and B has only 3 elements). We see that A, C, $D \in \mathcal{F}$, and all three are contained in $A \cup \{b\}$. $\square$

The next lemma was stated, without proof in [**25**].

PROPOSITION 2.2. *Suppose that $\mathcal{F}$ contains 3 three-element sets which all contain the same two elements. Then $\mathcal{F}$ is Frankl's.*

PROOF. Let us denote these three sets by $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, b, e\}$. Again we assume that $\mathcal{F}$ is not Frankl's. We pick the weight function w such that $w(a) = w(b) = 3$, $w(c) = w(d) = w(e) = 2$ and $w(x) = 0$ for all other $x \in X$. Then $t(w) = 6$. We consider an $\{a, b, c, d, e\}$-hypercube $\mathcal{C}$ with bottom set K. Let us first assume $K \in \mathcal{F}$. Then we also have the top set of the hypercube in $\mathcal{F}$, whose surplus equals the deficit of the bottom set (and these two cancel out, so we won't count them in deficits/surpluses). Three level 3 sets each with surplus 2 are guaranteed to be in $\mathcal{F}$, as well as the three level 4 sets (unions of pairs of the former), each with surplus 4. So, we are guaranteed a surplus of at least 18 from level 3 and 4 sets. We have different cases depending on the number of level 1 sets which are in $\mathcal{F}$:

(1) If there are less than two level 1 sets in $\mathcal{F}$, then the combined deficit from levels 1 and 2 can be at most 16 (upto 12 from level 2 and upto 4 from level 1). Thus, $s(\mathcal{C}) \geqslant d(\mathcal{C})$.

(2) There are two level 1 sets in $\mathcal{F}$. Then the combined deficit from level 1 is at most 8, so the combined deficit from level 2 needs to be at least 11. Thus, all three level 2 sets with weight 4 are in $\mathcal{F}$(and so is $K \cup \{c, d, e\}$), and at least five of the six weight 5 sets are in $\mathcal{F}$. By taking union of a weight 5 set with the set $K \cup \{c, d, e\}$, we get a weight 9 set (surplus 3) in $\mathcal{F}$. Now, the surpluses from levels 3 and 4 are at least 21, which is unachieveable by deficits from levels 1 and 2.

(3) There are three level 1 sets in $\mathcal{F}$. If both of weight 3 sets are among them, then weight 4 sets cancel out with weight 7 sets, as each weight 4 set in $\mathcal{F}$ implies

two of the weight 7 sets in $\mathcal{F}$. The level 1 sets and weight 5 sets have combined deficit of at most 16, which is less than 18 (surplus from weight 8 and 10 sets).

If one of the level 1 sets in $\mathcal{F}$ has weight 3 and the other two weight 2, then the total deficit from level 1 sets is 11, while the known surpluses from weight 10 and weight 8 sets add up to 18. Therefore, the level 2 sets have to produce deficit of 8. Because of the weight 3 set which is in $\mathcal{F}$, each weight 4 set in $\mathcal{F}$ implies a weight 7 set in $\mathcal{F}$. Therefore, level 2 contains at least 8 sets which are in $\mathcal{F}$. So, two of the weight 4 sets must be in $\mathcal{F}$, and by union of these two and the weight 3 set we get a weight 9 set in $\mathcal{F}$. Now, the deficit produced by level 2 sets should be 11, and, as weight 4 sets are contributing just 1 to the deficit (aech adds 1 to surplus and 2 to the deficit), this is impossible.

If the level 1 sets in $\mathcal{F}$ are $K \cup \{c\}$, $K \cup \{d\}$ and $K \cup \{e\}$, then all three weight 4 sets are also in $\mathcal{F}$. The surpluses and deficits of the known sets cancel out and the only remaining sets in $\mathcal{C}$ which may produce deficit are of weight 5. However, if any those three that contain a is in $\mathcal{F}$, that implies that the weight 9 set $K \cup \{a, c, d, e\}$ is in $\mathcal{F}$, while the three containing b similarly union to $K \cup \{b, c, d, e\}$ with $\{c, d, e\}$. So $s(\mathcal{C}) \geqslant d(\mathcal{C})$ in this case, as well.

(4) There are four level 1 sets in $\mathcal{F}$. If both $K \cup \{a\}$ and $K \cup \{b\}$ are in $\mathcal{F}$, then without loss of generality we may assume that $K \cup \{e\}$ is not in $\mathcal{F}$. Each of the weight 4 sets in $\mathcal{F}$ implies two of the weight 7 sets in $\mathcal{F}$(as before), so we may disregard these two groups. The deficits of the level 1 sets add up to 14, while the surpluses of weight 8 and weight 10 sets add up to 18. So, at least five of the weight 5 sets should be in $\mathcal{F}$. But that would imply one of them contains e, and that unions with weight 2 sets to a weight 9 set. Then the deficits of weight 5 sets should now be at least 8, which is impossible.

If the missing level 1 set is $K \cup \{a\}$ or $K \cup \{b\}$ (say, the latter), then all three weight 4 sets, all three weight 7 sets which contain a and the set $K \cup \{a, c, d, e\}$ are forced to be in $\mathcal{F}$. The deficits of level 1 and weight 4 sets add up to 21, while the surpluses of level 3 and 4 sets which we know are in $\mathcal{F}$ add up to 24. So, at least 4 weight 5 sets should be in $\mathcal{F}$. But one of them then contains b, which implies $K \cup \{b, c, d, e\} \in \mathcal{F}$. Now the surpluses add up to 27, which can not be surpassed by the deficits.

(5) There are five level 1 sets in $\mathcal{F}$. Then $\mathcal{C} \subseteq \mathcal{F}$, and $s(\mathcal{C}) = d(\mathcal{C})$

Now we investigate the case when $K \notin \mathcal{F}$. The number of weight 2 sets in $\mathcal{F}$ is less then or equal to the number of weight 10 sets in $\mathcal{F}$, so these two groups cancel each other out. Our cases are:

(5.1) Both $K \cup \{a\}, K \cup \{b\} \in \mathcal{F}$. Then all three weight 8 sets are in $\mathcal{F}$, and each weight 4 set in $\mathcal{F}$ implies two of the weight 7 sets in $\mathcal{F}$, so we may disregard these two groups. The surpluses of weight 8 sets, plus the level 5 set add up to 12, while the deficits of the weight 3 and weight 5 sets can be at most 12.

(5.2) Both $K \cup \{a\}, K \cup \{b\} \notin \mathcal{F}$. Each weight 5 set in $\mathcal{F}$ implies that one of the weight 8 sets in $\mathcal{F}$, and if there are at least three weight 5 sets in $\mathcal{F}$, then all three weight 8 sets are in $\mathcal{F}$, so the surplus of weight 8 sets is greater or equal to the deficit of weight 5 sets. The deficit of the weight 4 sets can be at most 6, which is covered by the surplus of the top set. So, in this case also, $s(\mathcal{C}) \geqslant d(\mathcal{C})$.

(5.3) One of the sets $K \cup \{a\}$, $K \cup \{b\}$ is in $\mathcal{F}$ (say, $K \cup \{a\}$), and the other is not. Then all three weight 8 sets are in $\mathcal{F}$, and the combined surplus of them and the top set is 12. The deficit of $K \cup \{a\}$ is 3, and each of the weight 4 sets in $\mathcal{F}$, when unioned with $K \cup \{a\}$ produces a weight 7 set in $\mathcal{F}$. So, the total deficit of the weight 4 sets may be considered to be at most 3, and the deficit of the weight 5 sets is at most 6. Thus, these deficits are covered by the surpluses of the weight 8 sets and the top set.

This shows that in all possible cases $s(\mathcal{C}) \geqslant d(\mathcal{C})$, so $\mathcal{F}$ is Frankl's. □

To proceed, we define a relation '$\preceq$' on elements of X. For now we assume that $\mathcal{F}$ is a counterexample to the conjecture with $|X|$ minimal.

Definition 2.3 (McKenzie). For $a, b \in X$ we say that $a \preceq b$ iff for all $K \in \mathcal{F}$, if $b \in K$ then $a \in K$.

Fact 2.1. *The relation '$\preceq$' is a partial order.*

Proof. It is obvious that this is a pre-order (a reflexive, transitive relation). If it is not a partial order, then there are elements a, b in X such that $a \preceq b$ and $b \preceq a$, i.e., they either both belong to a set in $\mathcal{F}$, or neither of them does. But then, by identifying these two elements as one new element, we get that the new family $\mathcal{F}'$ is also a counterexample, and that $|\bigcup \mathcal{F}'| < |X|$. This contradicts the minimality of $|X|$. □

Lemma 2.3 (McKenzie). *For every element $a \in X$ which is maximal in the partial order $\preceq$ defined above, $X - \{a\}$ belongs to $\mathcal{F}$.*

Proof. We know that $(\forall b \in X - \{a\})(a \npreceq b)$. So for every $b \in X - \{a\}$ there is a set $K_b \in \mathcal{F}$ so that $b \in K_b$ and $a \notin K_b$. Then $X - \{a\} = \bigcup\limits_{b \in X - \{a\}} K_b \in \mathcal{F}$ □

McKenzie has also noticed that there must be at least two maximal elements in this partial order in a minimal counterexample, but we do not need that result for the purposes of this paper. We do need the following simple corollary:

Corollary 2.2. *If there exists a counterexample $\mathcal{F}$ to the Frankl's conjecture with $|\bigcup \mathcal{F}| = m$ and let $m' \geqslant m$. Then there exists a counterexample $\mathcal{G}$ to the Frankl's conjecture such that $|\bigcup \mathcal{G}| = m'$ and $\mathcal{G}$ contains a set of the size $m' - 1$.*

Proof. If m' is the minimal size of the largest set in a counterexample, then the Lemma 2.3 is providing us with the desired set. So, assume that there is a counterexample $\mathcal{H}$ with $|\bigcup \mathcal{H}| = m'' < m'$ and that m'' is minimal. We take any element of the set in $\mathcal{H}$ with size $m'' - 1$, guaranteed by Lemma 2.3. We replace this element by several elements (as in the proof of Lemma 2.4) to get a counterexample to Frankl's conjecture with $|\bigcup \mathcal{G}| = m'$ and such that $\mathcal{G}$ contains a $m' - 1$-element set. □

Lemma 2.4. *If every family $\mathcal{F}$ with $|X| = k$ is Frankl's, then every family $\mathcal{F}'$ with top set X', $|X'| < |X|$ is Frankl's.*

PROOF. Assume not. Then there is a family $\mathcal{F}'$ with l-element top set, $l < k$ which is not Frankl's. Consider the family $\mathcal{F}''$, which is constructed from $\mathcal{F}'$ by taking an element a of X', and replacing each occurrence of this element in a set of $\mathcal{F}'$ with $k - l + 1$ new elements, $a_0, a_1, \ldots, a_{k-l}$. The family $\mathcal{F}''$ is union closed, because $\mathcal{F}'$ was, and every element is less than half of the sets (the old ones because they were such in $\mathcal{F}'$, and the a_is because a was in less than half of the sets of $\mathcal{F}'$). $\square$

3. Results for $|X| = 10$

LEMMA 3.1. *If $|X| = 10$ and $\mathcal{F}$ contains two three-element sets with a two-element intersection, then $\mathcal{F}$ is Frankl's.*

PROOF. Similarly as in the proof of Lemma 2.4, we suppose $\mathcal{F}$ is not Frankl's, so we may assume that $\mathcal{F}$ contains no one- or two-element sets. Let $\{a, b, c\}$ and $\{a, b, d\}$ be the two sets in $\mathcal{F}$. We consider the weight function w, with $w(a) = w(b) = 8$, $w(c) = w(d) = 6$ and $w(x) = 1$ for $x \in X - \{a, b, c, d\}$. We have $t(w) = 17$. Let $\mathcal{C}$ be an $\{a, b, c, d\}$-hypercube with bottom set K. We consider the cases:

(1) $|K| = 1$. In such hypercubes only levels 2, 3 and 4 may contain sets from $\mathcal{F}$. The surplus of the top set $K \cup \{a, b, c, d\}$ is 12, and there are only five sets with weights less than $t(w)$, all of them on level 2. $K \cup \{c, d\}$ has deficit 4, while the other four have deficit 2. So, $d(\mathcal{C}) \leqslant 12$, and $d(\mathcal{C}) \leqslant s(\mathcal{C})$.

(2) $|K| = 2$. Here the sum of deficits of the sets on level 2 is 7. If we consider the number of level 1 sets, we have subcases:

1. There are no level 1 sets in $\mathcal{F}$. Then the surplus of the top set is 13, while the sum of deficits can be at most 7 (five level 2 sets).
2. There is one level 1 set in $\mathcal{F}$. That set implies that at least one of the sets $K \cup \{a, b, c\}$ and $K \cup \{a, b, d\}$ is in $\mathcal{F}$(each has surplus 7). Thus, $s(\mathcal{C}) \geqslant 20$, and as the deficit of a level 1 set is at most 9, $d(\mathcal{C}) \leqslant 16$.
3. There are two level 1 sets in $\mathcal{F}$. That implies that both of the sets $K \cup \{a, b, c\}$ and $K \cup \{a, b, d\}$ are in $\mathcal{F}$, so $s(\mathcal{C}) \geqslant 27$. The combined deficit of the level 1 sets is at most 18, and that means that $d(\mathcal{C}) \leqslant 25$.
4. There at least three level 1 sets in $\mathcal{F}$. Then they form three 3-element sets with common two-element intersection. Then $\mathcal{F}$ is Frankl's by the Proposition 2.2.

(3) $|K| = 3$. If $K \notin \mathcal{F}$, the surplus of the top set is 14. The sets producing deficit are on the level 1 (two with weight 9 and two with weight 11) and the set $K \cup \{c, d\}$, with weight 15. Thus, at least two of the level 1 sets must be in $\mathcal{F}$. That implies that $K \cup \{a, b, c\}$ and $K \cup \{a, b, d\}$ are both in $\mathcal{F}$. The surplus is now at least 30, and that is equal to the combined deficits of all the sets in $\mathcal{C}$.

If $K \in \mathcal{F}$ we would like to prove that, although such hypercubes may have deficit greater than the surplus, $d(\mathcal{C}) \leqslant s(\mathcal{C}) + 2$. We see that the top set and the bottom one cancel each other out, and that $K \cup \{a, b, c\}$ and $K \cup \{a, b, d\}$ are both in $\mathcal{F}$. Their combined surplus is 16, so we must have at least three level 1 sets in

$\mathcal{F}$, or the two which are in $\mathcal{F}$ must be $K \cup \{c\}$ and $K \cup \{d\}$. In the second case, the only remaining set producing deficit is $K \cup \{c, d\}$, and now $d(\mathcal{C}) = s(\mathcal{C}) + 2$. If there are all four level 1 sets in $\mathcal{F}$, then $\mathcal{C} \subseteq \mathcal{F}$, and $s(\mathcal{C}) = d(\mathcal{C})$. If the three level 1 sets which are in $\mathcal{F}$ contain both $K \cup \{c\}$ and $K \cup \{d\}$, then the union of the three of them gives us another level 3 set in $\mathcal{F}$, with weight 23, and surplus 6. Thus, level 3 sets have surplus at least equal to the deficit of the level 1 sets, and the set $K \cup \{c, d\}$ produces extra deficit of 2, again. The last case is when the three level 1 sets in $\mathcal{F}$ contain both of the sets $K \cup \{a\}$ and $K \cup \{b\}$. Then the combined deficit of the level 1 sets is 20, and we have that $K \cup \{a, b\} \in \mathcal{F}$, with surplus 2. Together with the surplus of 16 coming from the two level 3 sets, this gives us a combined surplus of 18 and the third case when deficit is greater by 2 than the surplus. If, however, $K \cup \{c, d\} \in \mathcal{F}$, this adds 2 to deficit, but the sets $K \cup \{a, c, d\}$ and $K \cup \{b, c, d\}$ are forced to be in $\mathcal{F}$which increases the surplus by 12.

(4) $|K| = 4$ or $|K| = 5$. In these cases we just imitate the proof for $|K| = 3$, as the numbers work even better, and just check the three cases when $K \in \mathcal{F}$ and $d(\mathcal{C})$ was $s(\mathcal{C}) + 2$ in the case $|K| = 3$. All three corresponding cases have $d(\mathcal{C}) < s(\mathcal{C})$ now, so in general, $d(\mathcal{C}) \leqslant s(\mathcal{C})$.

(5) The top and bottom hypercubes we deal with together. The only set with a deficit in the bottom hypercube is $\emptyset$, whose deficit is equal to the surplus of X. We also have three sets with surplus in the bottom hypercube, namely $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, b, c, d\}$. Their combined surplus is 21. No other sets from bottom hypercube may be in $\mathcal{F}$, else $\mathcal{F}$ is Frankl's by Proposition 2.1.

We would like to prove that the remaining sets in the top hypercube have surplus greater than or equal to deficit. The sets in the top hypercube with deficit are the bottom one (deficit 11) and the level 1 sets (two with deficit 5 and two with deficit 3).

If the bottom set is in $\mathcal{F}$, then so are the two level 3 sets with surplus 11 each. Thus, we would have to have at least 3 of the level 1 sets in $\mathcal{F}$. They imply that three level 2 sets are in $\mathcal{F}$, and the surpluses of them are at least 7 ($X - \{a, b\}$ with surplus 1 and the other two with surpluses at least 3 each). So all the level 1 sets are in $\mathcal{F}$, which implies that all the top hypercube is in $\mathcal{F}$, and surpluses of the level 2 and 3 sets add up to 58, which is more than the deficits of the level 0 and 1 sets (27).

If the bottom set is not in $\mathcal{F}$, any level 1 set in $\mathcal{F}$ implies that at least one level 3 set with surplus 11 is in $\mathcal{F}$, too. If there are at least two level 1 sets then both of those sets with surplus 11 are in $\mathcal{F}$, and their combined surplus is greater then the combined deficit of all the level 1 sets.

We have proved that the top and bottom hypercube together have surplus of at least 20 greater than the deficit. Thus, at least 11 of the 'bad' hypercubes with deficit by 2 greater then the surplus are in $\mathcal{F}$. But all such have their bottom set, a three-element set disjoint from the set $\{a, b, c, d\}$ in $\mathcal{F}$. Thus $\mathcal{F}$ contains 11 three element sets, all subsets of the six-element set $X - \{a, b, c, d\}$. By the Corollary 2.1, $\mathcal{F}$ is Frankl's. $\square$

LEMMA 3.2. *Let $|X| = 10$ and $\mathcal{F}$ contain two intersecting three-element sets. Then $\mathcal{F}$ is Frankl's.*

PROOF. Again, we may assume that $\mathcal{F}$ does not contain one- or two-element sets. We know also that two three-element sets in $\mathcal{F}$ can not intersect in a two-element set by Lemma 3.1. So, let $\{a, b, c\}$, $\{a, d, e\} \in \mathcal{F}$. The weight function we choose is $w(a) = 7$, $w(b) = w(c) = w(d) = w(e) = 5$ and $w(x) = 1$ for all other $x \in X$. The target weight $t(w) = 16$. We consider an $\{a, b, c, d, e\}$-hypercube $\mathcal{C}$ with bottom set K. Let us define a few groups of sets we will use often in this proof:

- $L_{10} = \{K \cup \{b\}, K \cup \{c\}, K \cup \{d\}, K \cup \{e\}\}$
- $L_{20} = \{K \cup \{a, b\}, K \cup \{a, c\}, K \cup \{a, d\}, K \cup \{a, e\}\}$
- $L_{21} = \{K \cup \{b, d\}, K \cup \{b, e\}, K \cup \{c, d\}, K \cup \{c, e\}\}$
- $L_{22} = \{K \cup \{b, c\}, K \cup \{d, e\}$
- $L_{30} = \{K \cup \{a, b, c\}, K \cup \{a, d, e\}$
- $L_{31} = \{K \cup \{a, b, d\}, K \cup \{a, b, e\}, K \cup \{a, c, d\}, K \cup \{a, c, e\}\}$
- $L_{32} = \{K \cup \{b, c, d\}, K \cup \{b, c, e\}, K \cup \{b, d, e\}, K \cup \{c, d, e\}\}$
- $L_{40} = \{K \cup \{a, b, c, d\}, K \cup \{a, b, c, e\}, K \cup \{a, b, d, e\}, K \cup \{a, c, d, e\}\}$

Notice that

$$|L_{40} \cap \mathcal{F}| \geqslant \max\{|L_{10} \cap \mathcal{F}|, |L_{21} \cap \mathcal{F}|\} \tag{3.1}$$

We consider possible sizes of K and have cases:

(1) $|K| = 1$. Then the surplus of the top set of $\mathcal{C}$ is 12, and we do not have level 0 and level 1 sets in $\mathcal{F}$. Moreover, if there are more that two level 2 sets in $\mathcal{F}$, then two of them must form a couple of three-element sets with common two elements, so $\mathcal{F}$ is Frankl's by Lemma 3.1. On the other hand, if there are at most two level 2 sets in $\mathcal{F}$, then their combined deficit is at most 10, which is less than surplus of the top set. All the sets on levels 3 and 4 have weight at least $t(w) = 16$, so in these hypercubes $s(\mathcal{C}) \geqslant d(\mathcal{C})$.

(2) $|K| = 2$. Then $K \notin \mathcal{F}$, and by Lemma 3.1 there can be at most 1 level 1 set in $\mathcal{F}$. The surplus of the top set is 13.

If there are no level 1 sets in $\mathcal{F}$, then the only sets in $\mathcal{C}$ with weights less than $t(w)$ are on the level 2 and their combined deficit must be at least 14. Any set from L_{22} (deficit 4) which is in $\mathcal{F}$ implies that a set from L_{30} (surplus 3) is also in $\mathcal{F}$, which means that the sets from L_{22} contribute to the total deficit by at most 2. The sets from L_{21} have deficit 4 each, while the sets from L_{40} have surplus 8 each, so we may disregard these two groups. Finally, the combined deficit of the sets from L_{20} is at most 8, so in this case $s(\mathcal{C}) \geqslant d(\mathcal{C})$.

Let us now assume that one level 1 set is in $\mathcal{F}$. If this set is $K \cup \{a\}$ (deficit 7), then both sets in L_{30} and the top set of the hypercube are in $\mathcal{F}$, which means $s(\mathcal{C}) \geqslant 19$. So, the deficit of the level 2 sets must be at least 13. Any set from L_{21} (deficit 4) which is in $\mathcal{F}$ implies that a set from L_{31} (surplus 3) is also in $\mathcal{F}$. The total deficit from these two groups is at most 4. It means that L_{20} and L_{22} have to contribute by 9 to the total deficit. Hence, at least 1 set from L_{22} is in $\mathcal{F}$. If there is precisely 1 of them, say $K \cup \{b, c\} \in \mathcal{F}$, then there must be at least 3 sets from

L_{20} in $\mathcal{F}$, so one of them contains neither b, nor c. This set together with $K \cup \{b, c\}$ implies that a set from L_{40} is in $\mathcal{F}$, which adds 8 to the total surplus, and that can not be equaled by remaining set from L_{20}. Thus, both sets from L_{22} are in $\mathcal{F}$. But, then the set $K \cup \{b, c, d, e\} \in \mathcal{F}$ (surplus 6), which means that the total deficit of the sets from L_{20} must be at least 7, so all four are in $\mathcal{F}$. This means that again we have a set from L_{40} in $\mathcal{F}$, and that $s(\mathcal{C})$ is again greater than $d(\mathcal{C})$.

Now, let the the level 1 set in $\mathcal{F}$ be from L_{10}, say $K \cup \{b\}$ (deficit 9). Then the sets $K \cup \{a, b, c\}$ and $K \cup \{a, b, d, e\}$ are both in $\mathcal{F}$, so the total surplus of them and the top set is at least 24. Hence, the sets on level 2 must contribute at least 16 to the total deficit. The deficits of the sets in L_{21} (4 each) which are in $\mathcal{F}$ can be cancelled by surpluses of the sets $K \cup \{a, b, c, d\}$ and $K \cup \{a, b, c, e\}$ (8 each), as any one of L_{21} sets which is in $\mathcal{F}$implies that one of $K \cup \{a, b, c, d\}$ and $K \cup \{a, b, c, e\}$ is in $\mathcal{F}$, while any three L_{21} sets in $\mathcal{F}$imply that both $K \cup \{a, b, c, d\}$ and $K \cup \{a, b, c, e\}$ are in $\mathcal{F}$. Thus, the deficit 16 must come entirely from L_{20} and L_{22}, which means that all six of those sets must be in $\mathcal{F}$. But then the set $K \cup \{a, d, e\}$ (surplus 3) is also in $\mathcal{F}$, which again pushes the surplus of $\mathcal{C}$ above the deficit.

(3) $|K| = 3$ and $K \in \mathcal{F}$. There can be at most two such hypercubes, otherwise there would be two three-element sets in $\mathcal{F}$ which intersect in a two-element set. We wish to prove that in such hypercubes $d(\mathcal{C}) \leqslant s(\mathcal{C}) + 4$. Assume not. Both L_{30} sets must be in $\mathcal{F}$, and combined surplus of them and the top set is 22, while the deficit of K is 13. Any L_{10} set in $\mathcal{F}$ has the deficit 8, while any L_{40} set has surplus 9. As $|L_{40} \cap \mathcal{F}| \geqslant |L_{10} \cap \mathcal{F}|$, we may disregard these two groups. If $K \cup \{a\} \in \mathcal{F}$, the deficit increases to 19. But, then $|L_{31} \cap \mathcal{F}| \geqslant |L_{21} \cap \mathcal{F}|$ and the deficit of a L_{21} set is 3, while the surplus of a L_{21} set is 4, so we may disregard these two groups, as well. Thus, the sets in L_{20} (each with deficit 1) and L_{22} (each with deficit 3) must contribute by 8 to the total deficit, so both sets in L_{22} must be in $\mathcal{F}$. But, then the set $K \cup \{b, c, d, e\} \in \mathcal{F}$, and the surplus is increased by 7. The deficit from the sets in L_{20} and L_{22} should now be at least 15, which is impossible. We get $d(\mathcal{C}) \leqslant s(\mathcal{C}) + 4$ in this case.

If $K \cup \{a\} \notin \mathcal{F}$, then the level 2 sets must contribute to the deficit by at least 14 (the surpluses of the top set and L_{30} sets adds up to 22, while the deficit of K is 13; the sets in L_{10} and L_{40} are still disregarded). The deficits of sets in L_{20} are 1 each, and the deficits of the sets in L_{21} and L_{22} are 3 each. So, at least 4 of the sets in L_{21} and L_{22} are in $\mathcal{F}$. The sets in L_{21} and L_{22} can be split into three disjoint pairs, such that the union of any two is $K \cup \{b, c, d, e\}$. Thus, $K \cup \{b, c, d, e\} \in \mathcal{F}$, and the surplus is increased by 7. Now, the sets on level 2 must contribute at least 21 to the total deficit which means that all of L_{21} and L_{22} sets are in $\mathcal{F}$, as well as three of the L_{20} sets. But, then we get that (by unions of L_{20} and L_{21} sets) all four of L_{31} sets are in $\mathcal{F}$, and the surplus of each is 4. Again, $d(\mathcal{C}) \leqslant s(\mathcal{C}) + 4$.

(4) $|K| = 3$ and $K \notin \mathcal{F}$. The surplus of the top set is 14, the sets in L_{10} and L_{40} are disregarded. If $K \cup \{a\} \in \mathcal{F}$ (deficit 6), then both L_{30} sets are in $\mathcal{F}$ (surplus 4 each). In this case again $|L_{31} \cap \mathcal{F}| \geqslant |L_{21} \cap \mathcal{F}|$ and we can disregard these two groups. Then the sets in L_{20} and L_{22} can have total deficit at most 16, which means that $d(\mathcal{C}) \leqslant s(\mathcal{C})$. If $K \cup \{a\} \notin \mathcal{F}$, then the sets in $L_{21} \cap \mathcal{F}$ contribute to

the total deficit at most 12, so L_{20} and L_{22} should contribute by at least 3. But, if any of them is in $\mathcal{F}$, then a set from L_{30} (surplus 4) is in $\mathcal{F}$. Thus, the total deficit of the L_{20} and L_{22} sets should be at least 7, so either both of the L_{22} sets, or all four L_{20} sets should be in $\mathcal{F}$. In both cases, the other set from L_{30} is in $\mathcal{F}$, and the total deficit of the sets from L_{20} and L_{22} which are in $\mathcal{F}$ should be at least 10. But, then all six of them are in $\mathcal{F}$, which means that $K \cup \{b, c, d, e\} \in \mathcal{F}$ (as the union of the two L_{22} sets). Therefore, in all cases $d(\mathcal{C}) \leqslant s(\mathcal{C})$ in such hypercubes $\mathcal{C}$.

(5) $|K| = 4$. Then each set has weight by 1 greater than the corresponding set in a hypercube with three-element bottom set. So, because of the previous case, we only need to investigate the hypercubes $\mathcal{C}$ with at most 3 sets in $\mathcal{F}$ (as $d(\mathcal{C}') \leqslant s(\mathcal{C}')+4$ in hypercubes $\mathcal{C}'$ with three-element bottom set), and with $K \in \mathcal{F}$. However, such hypercubes do not exist, as $K \in \mathcal{F}$ implies that both L_{30} sets and the top set of $\mathcal{C}$ are in $\mathcal{F}$.

(6) $|K| = 0$ and $|K| = 5$. The bottom and the top hypercube will be shown to have combined surplus at least 8 greater than the deficit, which will cover for the deficits from the two 'bad' hypercubes with three-element bottom sets. In the bottom hypercube, we may assume $\emptyset \in \mathcal{F}$ (deficit 16), and we know that the sets $\{a, b, c\}$, $\{a, d, e\}$ and $\{a, b, c, d, e\}$ are all in $\mathcal{F}$, with combined surpluses adding up to 13. Any other set in the bottom hypercube which might be in $\mathcal{F}$ must be on the level 4 and have weight greater than $t(w)$. In the top hypercube we have $X \in \mathcal{F}$, with surplus 16. So, we need to show that combined deficits of the sets in the top hypercube which are in $\mathcal{F}$ can be by at most 5 greater than the surpluses of the sets in the top hypercube which are in $\mathcal{F}$ other than X. Assume not. By the inequality (3.1) and the fact that the surplus of a set from a set from L_{40} is 11, while the deficit of a set from L_{10} is 6, and the deficit of a set from L_{21} is 1, we can disregard the sets from L_{40}, L_{10} and L_{21}. The remaining sets in the top hypercube with the weight less than $t(w)$ are the bottom ($X - \{a, b, c, d, e\}$, deficit 11), the set $X - \{b, c, d, e\}$ with deficit 4, and the two L_{22} sets, each with deficit 1. Their combined deficits can be at most 17. But if any of the first two is in $\mathcal{F}$, then so are both of the L_{30} sets, each with surplus 6. This finishes the proof. $\square$

LEMMA 3.3. *If $|X| = 10$ and $\mathcal{F}$ contains no three-element sets, then $\mathcal{F}$ is Frankl's.*

PROOF. Assume not. Then $\mathcal{F}$ does not contain one- or two-element sets, either. If there are no four-element sets in $\mathcal{F}$, then the average cardinality of a set in $\mathcal{F}$ is at least 5 (i.e., we pick for our weight function any nonzero constant function), so we are done. Otherwise, let $\{a, b, c, d\} \in \mathcal{F}$. We pick the weight function w, defined by $w(a) = w(b) = w(c) = w(d) = 2.5$ and $w(x) = 1$ for all other $x \in X$. The target weight $t(w) = 8$. Let us consider an $\{a, b, c, d\}$-hypercube $\mathcal{C}$ with the bottom set K. We have cases:

(1) $|K| = 1$. Then any set from $\mathcal{C}$ that may be in $\mathcal{F}$ has weight at least 8.5.

(2) $|K| = 2$. The only sets in $\mathcal{C}$ that can be in $\mathcal{F}$ and have weight less than $t(w)$ are those on the level 2, each with deficit 1. The top set has surplus 4, so at

least 5 of the level 2 sets must be in $\mathcal{F}$. But that implies that all four of the level 3 sets (each with surplus 1.5) must be in $\mathcal{F}$, so $d(\mathcal{C}) \leqslant s(\mathcal{C})$.

(3) $|K| = 3$. The sets in such a hypercube that have weight less than $t(w)$ are all on level 1 (each with deficit 2.5), and the top set has surplus 5. So, at least three level 1 sets must be in $\mathcal{F}$. However, the union of three such gives a level 3 set in $\mathcal{F}$, so $s(\mathcal{C}) \geqslant 7.5$. Then we must have all four level 1 sets in $\mathcal{F}$, which in turn implies all four level 3 sets are in $\mathcal{F}$. In all cases $d(\mathcal{C}) \leqslant s(\mathcal{C})$.

(4) $|K| = 4$ or $|K| = 5$. We prove just for $|K| = 4$, as the case $|K| = 5$ is easier (each set has weight by 1 greater than the corresponding set in a hypercube with a four-element bottom set). The sets with weight less than $t(w)$ are K (deficit 4) and the level 1 sets (each with deficit 1.5). The surplus of the top set is 6, so K and at least two of the level 1 sets are in $\mathcal{F}$. The two level 1 sets imply a level 2 set (surplus 1) in $\mathcal{F}$, so then $s(\mathcal{C}) \geqslant 7$. Thus, at least three level 1 sets must be in $\mathcal{F}$. But, then at least three level 2 sets and a level 3 set are in $\mathcal{F}$, too, which means that $s(\mathcal{C}) \geqslant 12.5$. This is more than the combined deficits of levels 0 and 1, so again $d(\mathcal{C}) \leqslant s(\mathcal{C})$.

(5) $|K| = 0$ and $|K| = 6$. These two hypercubes together can contain at most two sets with weights less than $t(w)$, namely $\emptyset$ and $X - \{a, b, c, d\}$. The combined deficits of those equals the combined surpluses of X and $\{a, b, c, d\}$, which we know are in $\mathcal{F}$. □

Lemma 3.4. *If $\mathcal{F}$ contains precisely one or precisely three 3-element sets and $|X| = 10$, then $\mathcal{F}$ is Frankl's.*

Proof. Again we assume the opposite, and may assume $\mathcal{F}$ contains no one- or two-element sets. We first consider the case when $\mathcal{F}$ contains precisely three three-element sets. Let these sets be $\{a, b, c\}$, $\{d, e, f\}$ and $\{g, h, i\}$. The weight function will be $w(a) = w(b) = w(c) = 3$, with $w(x) = 1$ for all other $x \in X$, and the target weight $t(w) = 8$. As before, we consider an $\{a, b, c\}$-hypercube $\mathcal{C}$ with least set K and have cases:

(1) $|K| = 1$. The only set in such a hypercube which may be in $\mathcal{F}$ is the top one, which has weight 10.

(2) $|K| = 2$. Again, the only sets in such a hypercube which may be in $\mathcal{F}$ are on levels 2 and 3, and they have weight at least $t(w) = 8$.

(3) $|K| = 3$ and $K \notin \{\{d, e, f\}, \{g, h, i\}\}$. The sets in such a hypercube with weights less than $t(w)$ are on level 1, with deficits 2 each. The top set has surplus 4, so all three level 1 sets must be in $\mathcal{F}$. But, then all three level 2 sets are in $\mathcal{F}$, too, which means $s(\mathcal{C}) = 7 > 6 \geqslant d(\mathcal{C})$.

(4) $4 \leqslant |K| \leqslant 6$, and $K \neq \{d, e, f, g, h, i\}$. We prove it for $|K| = 4$, as in the other cases we have corresponding sets with larger weights, so an analogous proof will work. K has deficit 4, and the surplus of the top set is 5, so there must be at least two level 1 sets (each with deficit 1) in $\mathcal{F}$. But, then their union gives at least one level 2 set in $\mathcal{F}$, so $s(\mathcal{C}) = 7 \geqslant d(\mathcal{C})$.

(5) $|K| = 0$ and $|K| = 7$. The sets in these two hypercubes which may be in $\mathcal{F}$and have weight less than $t(w)$ are $\emptyset$ and $X - \{a, b, c\}$ with aggregate deficit 9, while the sets which we know are in $\mathcal{F}$, X and $\{a, b, c\}$, have aggregate surplus 9.

(6) $K_1 = \{d, e, f\}$, $K_2 = \{g, h, i\}$ and $K_3 = \{d, e, f, g, h, i\}$. All three bottom sets and all three top sets are in $\mathcal{F}$, the deficits of former adding up to 12, while the surpluses of the latter add up to 15. Moreover, the number of level i sets in the hypercube with bottom set K_3 which are in $\mathcal{F}$ is greater than or equal to the number of level i sets in each of the other two hypercubes, for $i = 1, 2$. The only sets in these three hypercubes, other than the bottom ones, which have weights less than $t(w)$ are the level 1 sets of the hypercubes with bottom sets K_1 and K_2.

If the hypercube with bottom set K_3 contains no level 1 sets in $\mathcal{F}$, then there are no sets in these three hypercubes with weight less than $t(w)$ other then K_1, K_2 and K_3, so deficits add to 12, while surpluses are at least 15.

If the hypercube with bottom set K_3 contains exactly one level 1 set in $\mathcal{F}$(surplus 1), then the deficit of the level 1 sets in the other two hypercubes are at most 4, so the combined deficits are at most 16, and combined surpluses are at least 16.

If the hypercube with bottom set K_3 contains exactly two level 1 sets in $\mathcal{F}$ (combined surplus 2), then it also contains a level 2 set in $\mathcal{F}$ (surplus 4). The deficits of the level 1 sets in the other two hypercubes can be at most 8, so the combined deficits are at most 20, and combined surpluses are at least 22.

If the hypercube with bottom set K_3 contains exactly three level 1 sets in $\mathcal{F}$, all three level 2 sets of this hypercube are also in $\mathcal{F}$. The aggregate surplus of these three hypercubes must be at least 30, which is more than the deficits of all the sets in these hypercubes with weight less than $t(w)$ (24).

The case when there is exactly one three-element set in $\mathcal{F}$ is dealt with in exactly the same fashion (same weights etc.), except that there are no 'special' hypercubes from the last case of the proof for three three-element sets in $\mathcal{F}$. $\square$

Now we consider a counterexample to Frankl's conjecture with $|X| = 10$. By the Corollary 2.2, $\mathcal{F}$ must contain a nine-element set. Alternatively, we could have just used the Lemma 2.3 and quoted [**13**] result that there are no counterexamples with a 9-element largest set.

LEMMA 3.5. *Let $|X| = 10$ and $\mathcal{F}$ contain precisely two three-element sets, which do not intersect, and a nine-element set. Then $\mathcal{F}$ is Frankl's.*

PROOF. Assume not. Then $\mathcal{F}$ contains no one- or two-element sets. Let the two three-element sets mentioned in the statement be $\{a, b, c\}$ and $\{d, e, f\}$. We consider the $\{a, b, c\}$-hypercube $\mathcal{C}_1$ with bottom set $\{d, e, f\}$ and the $\{d, e, f\}$-hypercube $\mathcal{C}_2$ with bottom set $\{a, b, c\}$. Without loss of generality we may assume that the number of level 1 sets in $\mathcal{C}_1$ which are in $\mathcal{F}$ is not less than the number of level 1 sets in $\mathcal{C}_2$ which are in $\mathcal{F}$. Then we consider weight function w such that $w(a) = w(b) = w(c) = 3$ and $w(x) = 1$ for all other $x \in X$. The proof goes the same as when there is precisely one three-element set in $\mathcal{F}$, with the exception of the hypercube $\mathcal{C}_2$. The level 1 sets in this hypercube which are in $\mathcal{F}$ produce deficit of 2 each. However, the level 1 sets in the $\{d, e, f\}$-hypercube $\mathcal{C}_2$ are top sets of their

$\{a, b, c\}$-hypercubes, which have $|K| = 1$ and no other set from these hypercubes is in $\mathcal{F}$. As each such hypercube has surplus 2 and there are at least as many of them as there are level 1 sets in $\mathcal{C}_1$ which are in $\mathcal{F}$, the deficit of level 1 sets of $\mathcal{C}_1$ is covered. The level 2 sets of $\mathcal{C}_1$ have weight 9 (surplus 1), so we need not consider them, either. Finally, $\{d, e, f\}$ has deficit 5, while $\{a, b, c, d, e, f\}$ has surplus 3. So, we need only to cover the total deficit of 2 from $\mathcal{C}_1$. This we do with the nine-element set we assume is in $\mathcal{F}$. Let this set be L. If $\{a, b, c\} \nsubseteq L$, then L is in the top hypercube and has weight 13 (surplus 5). Other sets of the top and bottom hypercubes which have deficit have already been covered by surpluses of $\{a, b, c\}$ and X, so this set covers the deficit of $\mathcal{C}_2$. If $\{a, b, c\} \subseteq L$, then L is the top set of an $\{a, b, c\}$-hypercube with a six-element bottom set. The bottom set of this hypercube is the only set in it which has weight less than $t(w)$ (deficit 2), so the surplus 7 of L is sufficient to cover both the deficit of 2 from this bottom set and deficit 2 fron $\mathcal{C}_2$. $\square$

THEOREM 3.1. *A finite union-closed family $\mathcal{F}$ of finite sets with largest set having at most ten elements contains an element a such that a is a member of at least half of the sets in $\mathcal{F}$.*

Acknowledgements

The author wishes to thank several people for help in getting these results. Ralph Mckenzie is responsible for the relation '$\leqslant$' from Definition 2.3 and the proof of Lemma 2.3. Miklós Maróti suggested assigning the weights to the sets after he heard some of the initial proofs (which worked for the case $|X| = 9$). Ivica Bošnjak found an error in one of the proofs and proofread the paper.

References

[1] T. Abe, *Strong semimodular lattices and Frankl's conjecture,* Algebra Universalis **44**:3–4 (2000), 379–382.

[2] T. Abe, *Excess of a lattice,* Graphs Combin. **18**:3 (2002), 395–402.

[3] T. Abe and B. Nakano, *Lower semimodular types of lattices: Frankl's conjecture holds for lower quasi-semimodular lattices*, Graphs Combin. **16**:1 (2000), 1–16.

[4] E. Brown and T. P. Vaughan, *Configurations with subset restrictions,* J. Combin. Math. Combin. Comput. **48** (2004), 197–215.

[5] K. Dohmen, *A new perspective on the union-closed sets conjecture,* Ars Combin. **58** (2001), 183–185.

[6] D. Duffus and B. Sands, *An inequality for the sizes of prime filters of finite distributive lattices,* Discrete Math. **201**:1–3 (1999), 89–99.

[7] M. El-Zahar, *A graph-theoretic version of the union-closed sets conjecture*, J. Graph Theory **26**:3 (1997), 155–163.

[8] L. F. Fitina and J-C. Renaud, *On union-closed sets and Conway's sequence*, Bull. Austral. Math. Soc. **47**:2 (1993), 321–332.

[9] W. Gao and H. Yu, *Note on the union-closed sets conjecture*, Ars Combin. **49** (1998), 280–288.

[10] R. T. Johnson and T. P. Vaughan, *On union-closed families, I*, J. Combin. Theory Ser. A **84**:2 (1998), 242–249.

[11] G. Lo Faro, *A note on the union-closed sets conjecture*, J. Austral. Math. Soc. Ser A **57**:2 (1994), 230–236.

[12] G. Lo Faro, *Union-closed sets conjecture: improved bounds*, J. Combin. Math. Combin. Comp. **16** (1994), 97–102.

[13] R. Morris, *FC-families and improved bounds for Frankl's conjecture,* European J. Combin. **27**:2 (2006), 269–282.

[14] T. Nishimura and S. Takashi, *Around Frankl's conjecture*, Sci. Rep. Yokohama Nat. Univ. Sect. I Math. Phys. Chem. **43** (1996), 15–23.

[15] R. M. Norton and D. G. Sarvate, *A note of the union-closed sets conjecture*, J. Austral. Math. Soc. Ser A **16**:3 (1993), 411–413.

[16] B. Poonen, *Union-closed families*, J. Combin. Theory Ser. A **59**:2 (1992), 253–268.

[17] D. Reimer, *An average set size theorem,* Combin., Probab. and Comput. **12**:1 (2003), 89–93.

[18] J-C. Renaud, *Is the union-closed sets conjecture the best possible?*, J. Austral. Math. Soc. Ser A **51** (1991), no. 2, 276–283.

[19] J-C. Renaud, *A second approximation to the boundary function on union-closed collections*, Ars Combin. **41** (1995), 177–188.

[20] J-C. Renaud and D. G. Sarvate, *On the union-closed sets conjecture*, Ars Combin. **27** (1989), 149–153.

[21] J-C. Renaud and D. G. Sarvate, *Improved bounds for the union-closed sets conjecture*, Ars Combin. **29** (1990), 181–185.

[22] I. Rival (ed.), *Graphs and Order*, NATO Advanced Science Institute Series C: Mathematical and Physical Sciences, 147, Reidel, Dordrecht–Boston, 1985, p. 525.

[23] R. P. Stanley, *Enumerative Combinatorics, Vol. I*, The Wadsworth and Brooks/Cole Mathematics Series, The Wadsworth and Brooks/Cole Advanced Books and Software, Monterey, 1986.

[24] T. P. Vaughan, *Families implying the Frankl conjecture*, European J. Combin. **23**:7 (2002), 851–860.

[25] T. P. Vaughan, *A note on the union-closed sets conjecture*, J. Combin. Math. Combin. Comput. **45** (2003), 95–108.

[26] T. P. Vaughan, *Three-sets in a union-closed family*, J. Combin. Math. Combin. Comput. **49** (2004), 73–84.

[27] P. Winkler, *Union-closed sets conjecture*, Austral. Math. Soc. Gaz. **14** (1987), 99.

[28] P. Wójcik, *Density of union-closed families*, Discrete Math. **105**:1–3 (1992), 259–267.

Prirodno-matematički fakultet
21000 Novi Sad
Serbia
pera@im.ns.ac.yu