PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 81(95) (2007), 85–93 DOI 102298/PIM0795085J

EXTENDABLE SHELLING, SIMPLICIAL AND TORIC h-VECTOR OF SOME POLYTOPES

Duško Jojić

Abstract. We show that the stellar subdivisions of a simplex are extendably shellable. These polytopes appear as the facets of the dual of a hypersimplex. Using this fact, we calculate the simplicial and toric h-vector of the dual of a hypersimplex. Finally, we calculate the contribution of each shelling component to the toric h-vector.

1. Introduction

A *polytopal* (*polyhedral*) complex is a finite set $\mathcal{C}$ of polytopes (including $\emptyset \in \mathcal{C}$) satisfying:

(i) if $P \in \mathcal{C}$ and Q is a face of P, then $Q \in \mathcal{C}$;
(ii) For all $P, Q \in \mathcal{C}$, $P \cap Q$ is a (possible empty) face of both P and Q.

For the definitions and properties of polytopes, see [**10**]. A *simplicial complex* is a special case of polytopal complex, the case when every polytope is a simplex. In this paper we consider only *pure* complexes $\mathcal{C}$, that is, complexes which satisfy the condition that all the maximal faces with respect to inclusion, called the *facets* of $\mathcal{C}$, have the same dimension, called the *dimension* of $\mathcal{C}$.

An example of a pure polytopal complex is the *boundary complex* $\mathcal{C}(\partial P)$ of a polytope P; the set of the faces of P except for the polytope itself.

A *shelling* of a polytopal complex $\mathcal{C}$ is a linear ordering $F_1, F_2, \dots, F_k$ of its facets which is arbitrary for $\dim \mathcal{C} = 0$ (all F_i are points), and for $\dim \mathcal{C} > 0$ has to satisfy the following conditions (see [**4**] or Section 8 in [**10**]):

(i) The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet has a shelling.
(ii) For every $i > 1$, the intersection of F_i with the previous facets is a beginning part of a shelling of the boundary complex $\mathcal{C}(\partial F_i)$, that is:

$$F_i \cap \left(\bigcup_{j<i} F_j \right) = G_1 \cup G_2 \cup \dots \cup G_l$$

for some shelling $G_1, G_2, \dots, G_l, \dots, G_m$ of $\mathcal{C}(\partial F_i)$.

2000 *Mathematics Subject Classification*: Primary 52B22; Secondary 52B05.
Key words and phrases: extendable shellability, hypersimplex, toric h-vector.

For polytopal complexes, the condition (i) is pleonastic, because the boundary of any polytope is shellable, see [**4**]. Further, for simplicial complexes, the condition (ii) can be simplified:

For every $i < j \leqslant k$ there exist some $l < j$ and a vertex v of F_j such that

$$F_i \cap F_j \subset F_l \cap F_j = F_j \setminus \{v\}. \tag{1}$$

A polytopal (simplicial) complex $\mathcal{C}$ is *extendably shellable* if every partial shelling can be extended to a complete shelling of $\mathcal{C}$. This concept is essential for the algorithmic use of shellings, but we know very few about extendable shellability. Ziegler showed in [**9**] that the boundaries of "almost all" 4-polytopes are not extendably shellable.

For a d-dimensional polytopal (simplicial) complex $\mathcal{C}$, we denote the number of i-dimensional faces of $\mathcal{C}$ by f_i, and $f(\mathcal{C}) = (f_0, f_1, \dots, f_d)$ is called the *f-vector.* A generating polynomial for the f-vector is the *f-polynomial*:

$$f(\mathcal{C}, x) = x^{d+1} + f_0 x^d + f_1 x^{d-1} + \cdots + f_{d-1} x + f_d.$$

A new invariant, the *h-vector* $h(\mathcal{C}) = (h_0, h_1, \dots, h_d, h_{d+1})$ is defined to be the coefficients of $f(C, x-1)$:

$$f(\mathcal{C}, x-1) = h_0 x^{d+1} + h_1 x^d + \cdots + h_d x + h_{d+1} = h(\mathcal{C}, x). \tag{2}$$

The h-vector of a shellable simplicial complex $\mathcal{C}$ has the following combinatorial interpretation. For a fixed shelling $F_1, F_2, \dots, F_k$ of $\mathcal{C}$, we define *the restriction R_j* of the facet F_j:

$$R_j = \{v \in V(F_j) : F_j \setminus \{v\} \subset F_i \text{ for some } 1 \leqslant i < j\},$$

that is, R_j is a minimal new face at the j-th step in the given shelling.

The type of F_j in the given shelling is the cardinality of R_j, i.e., $\mathrm{type}(F_j) = |R_j|$. Now, we have that

$$h_k(\mathcal{C}) = |\{j : \mathrm{type}(F_j) = k\}|.$$

This interpretation of the h-vector was of great significance in the proof of the upper-bound theorem and in the characterization of f-vectors of simplicial polytopes, see [**10**].

The entries of the h-vector of a simplicial polytope are Betti numbers of the associated toric variety. This can be generalized to nonsimplicial polytopes, to define the *toric h-vector*, but we do not have combinatorial interpretation for the entries of this vector.

In this paper, we use the combinatorial formula for the toric h-vector of Eulerian poset given by Stanley [**8**]. To a graded poset P we associate two polynomials, $f(P,t)$ and $g(P,t)$ recursively:

(1) For the Boolean lattice B_1, the only graded poset of rank 1, we have $f(B_1, t) = g(B_1, t) = 1$.
(2) If $\mathrm{rank}(P) = n + 1 > 0$, then $f(P, t)$ has the degree n, say
$f(P, t) = h_0^T + h_1^T t + \cdots + h_{n-1}^T t^{n-1} + h_n^T t^n$, and we define
$g(P, t) = h_0^T + (h_1^T - h_0^T) t + \cdots + (h_{[n/2]}^T - h_{[n/2]-1}^T) t^{[n/2]}$.

(3) If $\operatorname{rank}(P) = n+1 > 0$ we define

$$f(P,t) = \sum_{x\in P,\ x<\hat{1}_P} g\left([\hat{0}_P, x], t\right)(t-1)^{n-r(x)}. \tag{3}$$

The toric h-vector $h^T(P) = (h_0^T, h_1^T, \dots, h_n^T)$ of an Eulerian poset P is defined as the vector of coefficients of the polynomial $f(P,t)$.

For all simplicial Eulerian posets the toric h-vector coincides with the usual h-vector, defined with relation (2). Further, for the entries of the toric h-vector of any Eulerian poset of rank $n+1$ holds $h_i^T = h_{n-i}^T$.

A poset P is *quasisimplicial* if for any coatom c of P, the interval $[\hat{0}, c]$ is simplicial, that is, for any corank 2 element x of P, the interval $[\hat{0}, x]$ is a Boolean algebra. From the formulae (3) and the fact that for all Boolean lattices B_m the equality $g(B_m, t) = 1$ holds, we can obtain that for a quasisimplicial poset P whose h-vector is $h(P) = (h_0, h_1, \dots, h_n)$ the toric h-vector of P is

$$h_i^T = \begin{cases} h_i & \text{for } i \leqslant [n/2], \\ h_{n-i} & \text{for } i > [n/2]. \end{cases} \tag{4}$$

2. Polytopes T_k^n are extendable shellable

The *stellar subdivision* of a polytope P in a face F (see [**6**] for more details) is a new polytope $conv(P \cup x^F)$, where x^F is a point of the form $y^F - \varepsilon(y^P - y^F)$, where y^P is in the interior of P, y^F is in the relative interior of F, and ε is small enough. Let T_k^n denotes a polytope obtained as a stellar subdivision of n-simplex Δ_n in a k-face S. Obviously, T_k^n is simplicial and in [**2**] we can find that

$$\partial T_k^n = (\Delta_n \setminus S) \cup \left(\partial S * \partial lk_{\Delta_n} S * \{x^S\}\right). \tag{5}$$

Remark 1. Let us denote the vertices of S with $r_1, r_2, \dots, r_{k+1}$; the vertices of $\Delta_n \setminus S$ denote with $c_1, c_2, \dots, c_{n-k}$, and let c_{n-k+1} denotes the new vertex x^S. Now, we have that the set of the vertices of T_k^n is

$$V(T_k^n) = \{r_1, r_2, \dots, r_{k+1}\} \cup \{c_1, c_2, \dots, c_{n-k}, c_{n-k+1}\}.$$

From the relation (5), we conclude that all facets of T_k^n are just $(n-1)$-simplices $F_{i,j} = \operatorname{conv}(V(T_k^n) \setminus \{r_i, c_j\})$, for all $i = 1, 2, \dots, k+1$; $j = 1, 2, \dots, n-k, n-k+1$. Therefore, if we label rows and columns of a $(k+1) \times (n-k+1)$ rectangle $R_{(k+1)\times(n-k+1)}$ with vertices $r_1, r_2, \dots, r_{k+1}$ and $c_1, c_2, \dots, c_{n-k}, c_{n-k+1}$ of T_k^n, then facets of T_k^n correspond with $(k+1)(n-k+1)$ squares of $R_{(k+1)\times(n-k+1)}$.

Now, with an appropriate labelling of the vertices, we can prove that the following complexes are combinatorially equivalent: $\partial T_k^n \cong \partial(\Delta_k \times \Delta_{n-k})^* \cong \partial\Lambda_k^{n+1}$ (here Λ_k^{n+1} is a simplicial n-disk, obtained as the union of $k+1$ facets of an $(n+1)$-simplex).

Note that the facets $F_{i,j}$ and $F_{p,q}$ of T_k^n share a common $(n-2)$-simplex if and only if $i = p$ or $j = q$ (corresponding squares are in the same row or column of $R_{(k+1)\times(n-k+1)}$). Now, it is easy to verify that the lexicographical order of the facets of T_k^n

$$F_{i,j} < F_{p,q} \;\Leftrightarrow\; i < p \text{ or } i = p,\ j < q \tag{6}$$

is a shelling order. In this order we have that $type(F_{i,j}) = i + j - 2$, so we can conclude that for $0 \leqslant k \leqslant n - k$ the h-vector of the polytopes T_k^n is $h(T_k^n) = (1, 2, \dots, k, k+1, \dots, k+1, k, \dots, 1)$.

THEOREM 2. *Polytopes T_k^n are extendably shellable.*

PROOF. Let $\mathcal{F}$ be a subset of the facets of T_k^n, which we can identify with the squares in the rectangle $R_{(k+1)\times(n-k+1)}$. For all $i = 1, 2, \dots, k+1$ we let $\mathcal{F}_i = \{j \in [n-k+1] : F_{i,j} \in \mathcal{F}\}$; i.e., $\mathcal{F}_i$ is the set of the squares from $\mathcal{F}$ contained in the i-th row of $R_{(k+1)\times(n-k+1)}$. We will prove that the following statements are all equivalent:

(a) there exists a shelling order for the union of the facets contained in $\mathcal{F}$.
(b) the sets $\mathcal{F}_i$ form a chain: for all i, j we have that $\mathcal{F}_i \subseteq \mathcal{F}_j$ or $\mathcal{F}_j \subseteq \mathcal{F}_i$.
(c) there exists a shelling of T_k^n with the facets from $\mathcal{F}$ at the beginning.

(a) $\Rightarrow$ (b). Assume that for some i, i' there exists j such that $j \in \mathcal{F}_i$ and $j \notin \mathcal{F}_{i'}$. Now, from the shellability of $\mathcal{F}$, for all $j' \in \mathcal{F}_{i'}$, the intersection $F_{i,j} \cap F_{i',j'}$ is contained in an $(n-2)$-simplex, which is the intersection of $F_{i,j}$ or $F_{i',j'}$ with a facet from $\mathcal{F}$, see (1). The only facets of T_k^n with the above properties are $F_{i,j'}$ and $F_{i',j}$. As we assumed that $F_{i',j} \notin \mathcal{F}$, then shellability of $\mathcal{F}$ implies that $F_{i,j'} \in \mathcal{F}$, and therefore $\mathcal{F}_i \supseteq \mathcal{F}'_i$.

(b) $\Rightarrow$ (c). We may assume that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots \supseteq \mathcal{F}_{k+1}$ (because we can relabel the vertices of S). We define the following linear order for the facets of T_k^n:

$$F_{i,j} < F_{k,l} \Leftrightarrow \text{ either } j \in \mathcal{F}_i,\ l \notin \mathcal{F}_k \text{ or } \ i < k \vee i = k,\ j < l. \tag{7}$$

If $F_{a,b}$ precedes to $F_{c,d}$ in this order we consider the two possibilities:

1° $a \leqslant c$, when $b \notin \mathcal{F}_a, d \notin \mathcal{F}_c$ or $b \in \mathcal{F}_a$, $d \in \mathcal{F}_c$ or $b \in \mathcal{F}_a$, $d \notin \mathcal{F}_c$, and then

$$F_{a,b} \cap F_{c,d} \subseteq F_{a,d} \cap F_{c,d} = F_{c,d} \setminus \{r_a\};$$

2° $c < a$, when $b \in \mathcal{F}_a$, $d \notin \mathcal{F}_c$, and then $F_{a,b} \cap F_{c,d} \subseteq F_{c,b} \cap F_{c,d} = F_{c,d} \setminus \{c_b\}$.

Note that $F_{a,d}$ (and $F_{c,b}$ in the second case) precedes to $F_{c,d}$ in the given order. Therefore, a shelling order for the facets of T_k^n is defined by (7).

Finally, the implication $(c) \Rightarrow (a)$ is obvious. □

It is shown in [**5**] that the cross polytope $C_n^\vartriangle$ (the convex hull of the standard basis vectors in $\mathbb{R}^n$ and their negatives), is not extendably shellable in dimension 12 or higher. The cross polytope can be obtained as the dual of the product of segments, or by successive stellar subdivisions of a simplex, so we cannot generalize Theorem 2 in this way.

3. Simplicial and toric h-vector of $\Delta^*_{n-1}(k)$

The hypersimplex $\Delta_{n-1}(k)$ is a polytope in $\mathbb{R}^n$ obtained as the intersection of the n-cube $C_n = [0,1]^n$ with the hyperplane $H_k = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = k\}$. In particular, $\Delta_{n-1}(1)$ and $\Delta_{n-1}(n-1)$ are $(n-1)$-simplices. As the polytopes $\Delta_{n-1}(k)$ and $\Delta_{n-1}(n-k)$ are isomorphic, we can assume that $0 \leqslant k \leqslant n-k$. The face lattice $L(\Delta_{n-1}(k))$ of a hypersimplex is described with the following

LEMMA 3. *Let*

$$\begin{aligned} L_{n-1}(k) = \{(A,B) : A \subset B \subseteq [n];\ |A| < k;\ |B| > k\} \\ \cup\{(A,A) : A \subseteq [n];\ |A| = k\} \cup \{\hat{0}\}. \end{aligned}$$

If we define an order on $L_{n-1}(k)$ *with* $(A,B) \leqslant (C,D) \Leftrightarrow A \supseteq C,\ B \subseteq D$, *then* $L(\Delta_{n-1}(k)) \cong L_{n-1}(k)$.

PROOF. Note that H_k does not intersect the edges of C_n in their relative interior. Therefore, the only vertices of $\Delta_{n-1}(k)$ are vertices of C_n contained in H_k. The correspondence

$$A = \{a_1, a_2, \ldots, a_k\} \subseteq [n], \quad (A,A) \leftrightarrow T_A = (t_1, t_2, \ldots, t_n), \quad t_i = \begin{cases} 1, & i \in A; \\ 0, & i \notin A \end{cases}$$

gives us a bijection between the atoms of $L_{n-1}(k)$ and the vertices of $\Delta_{n-1}(k)$.

For any pair (A,B) of the subsets of $[n]$, such that

$$A \subset B \subseteq [n]; \quad |A| \leqslant k-1; \quad |B| \geqslant k+1, \tag{8}$$

the linear functional $x \mapsto \sum_{i \in A} x_i - \sum_{j \notin B} x_j$ reaches the maximum on the $\Delta_{n-1}(k)$ at the $(|B|-|A|-1)$-dimensional face

$$S_{(A,B)} = \Big\{x \in C_n : \sum_{i=1}^n x_i = k;\ \forall i \in A, x_i = 1;\ \forall j \notin B, x_j = 0\Big\}.$$

So, we establish a bijection between all of pairs (A,B) for which the relation (8) holds and all $(|B|-|A|-1)$-faces of $\Delta_{n-1}(k)$. Also, it is easy to see that the correspondence $(A,B) \mapsto S_{(A,B)}$ defines a poset isomorphism between $L_{n-1}(k)$ and the face lattice $L(\Delta_{n-1}(k))$. $\square$

Note that the face lattice of the dual polytope $\Delta^*_{n-1}(k)$ can be obtained by applying the E_t-construction on the Boolean lattice B_n, see in [**7**].

REMARK 4. For a vertex T_A of $\Delta_{n-1}(k)$, denote by F_A its corresponding facet in $\Delta^*_{n-1}(k)$. For $1 < k < n$, note that $\Delta_{n-1}(k)$ has $2n$ facets R_i and C_j:

$$\begin{aligned} R_i &= \{x \in \Delta_{n-1}(k) : x_i = 1\} = S_{(\{i\},[n])} = \operatorname{conv}\big(\{T_A : i \in A\}\big), \\ C_j &= \{x \in \Delta_{n-1}(k) : x_j = 0\} = S_{(\emptyset,[n]\setminus\{j\})} = \operatorname{conv}\big(\{T_A : j \notin A\}\big). \end{aligned}$$

Therefore, $\Delta^*_{n-1}(k)$ has 2n vertices: $V(\Delta^*_{n-1}(k)) = \{r_1, \ldots, r_n, c_1, \ldots c_n\}$. The vertex T_A in $\Delta_{n-1}(k)$ is contained in the facets R_i, for $i \in A$ and C_j, for $j \notin A$. Therefore, the set of the vertices of F_A is $V(F_A) = \{r_i : i \in A\} \cup \{c_j : j \notin A\}$.

An edge $E = S_{(A\setminus\{i_0\},A\cup\{j_0\})}$ of $\Delta_{n-1}(k)$, which contains T_A, is contained in the facets R_i, for $i \in A \setminus \{i_0\}$ and C_j, for $j \in A^c \setminus \{j_0\}$. So, we have that the maximal face of F_A, which corresponds to E, is an $(n-3)$-simplex $G_{r_{i_0},c_{j_0}}(A) = \operatorname{conv}(V(F_A) \setminus \{r_{i_0}, c_{j_0}\})$.

For $A = \{a_1, a_2, \ldots, a_k\} \subset [n]$, we denote the i-th row of a $k\times(n-k)$ rectangle with $r_{a_{k+1-i}}$. If $A^c = \{b_1, b_2, \ldots, b_{n-k}\}$, then the j-th columns we denote with c_{b_j}. Now, from Remark 1 it follows that all of the facets of $\Delta^*_{n-1}(k)$ are combinatorially equivalent with T^{n-2}_{k-1}.

Note that $G_{r_a,c_b}(A)$ and $G_{r_b,c_a}(A\setminus\{a\}\cup\{b\})$ denote the same faces of $\Delta^*_{n-1}(k)$ (the common facets of F_A and $F_{A\setminus\{a\}\cup\{b\}}$). So, with

$$G_{r_a,c_b}(A), \text{ for } A \subset [n],\ |A| = k;\ a < b,\ a \in A,\ b \notin A,$$

we list all of $(n-3)$-faces of $\Delta^*_{n-1}(k)$.

Now, we consider a lexicographic order of the facets of $\Delta^*_{n-1}(k)$:

$$F_A <_L F_B \Leftrightarrow A <_L B \Leftrightarrow \min(A \mathbin{\triangle} B) \in A. \tag{9}$$

If we denote with Γ_A the intersection of the facet F_A with the previous facets in this order, we can prove that

$$\Gamma_A = F_A \cap \Bigg(\bigcup_{F_{A'} <_L F_A} F_{A'} \Bigg) = \bigcup_{a\in A,\, b\in[n]\setminus A,\, b<a} G_{r_a,c_b}(A).$$

Now, we use well known bijection between k-subsets of an n-set (facets of $\Delta^*_{n-1}(k)$), and all shortest paths from lower left-hand corner, and ending at the upper right-hand corner of a $k \times (n-k)$ rectangle. The squares above this path correspond with facets of F_A contained in Γ_A.

From Theorem 2, it follows that there exists a shelling for facets from Γ_A, and this shelling can be extended to the shelling of the whole F_A, and therefore $<_L$ is a shelling order for the facets of $\Delta^*_{n-1}(k)$.

Further, motivated by (6), we can define the linear order of the $(n-3)$-faces of $\Delta^*_{n-1}(k)$:

$$G_{r_a,c_b}(A) < G_{r_k,c_l}(A') \Leftrightarrow \begin{cases} A <_L A', & \text{or} \\ A = A',\ r_a > r_k, & \text{or} \\ A = A',\ r_a = r_k \text{ and } c_b < c_l. \end{cases}$$

It is easy to prove that this is a shelling order for $(n-3)$-skeleton of $\Delta^*_{n-1}(k)$. Also, we can note in the above list, that from a facet F_A we take just the facets which does not appear in Γ_A, i.e., facets whose corresponding squares are bellow corresponding path. When the simplex $G_{r_a,c_b}(A)$ corresponds with the square (i,j) in a $k\times(n-k)$ rectangle, then its type in described shelling is $i+j-2$. Therefore, the square (i,j) below the corresponding path, contributes the one to $i+j-2$ entry of h-vector of $\Delta^{*(n-3)}_{n-1}(k)$.

If we define a $k \times (n-k)$ matrix $A_{k,n-k}$ with $a_{i,j} =$ number of paths in $R_{k\times(n-k)}$, in which the square (i,j) are below, then we have that

$$h_k(\Delta_{n-1}^{*(n-3)}(k)) = \sum_{i+j=k+2} a_{i,j}.$$

By considering the possibilities for the first step in a such path, it is easy to note that the matrices $A_{k,n-k}$ satisfy the following recursive relations:

$$A_{k,n-k} = \begin{bmatrix} 0 & \\ 0 & \\ \vdots & A_{k,n-k-1} \\ 0 & \end{bmatrix} + \begin{bmatrix} & A_{k-1,n-k} & & \\ \binom{n-1}{k-1} & \binom{n-1}{k-1} & \cdots & \binom{n-1}{k-1} \end{bmatrix} \tag{10}$$

THEOREM 5. *For all $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n-k$ it holds:*

$$h(\Delta_{n-1}^{*}(k)) = \begin{pmatrix} \binom{n}{0} \\ \binom{n}{0} + \binom{n}{1} \\ \vdots \\ \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k-1} \\ \vdots \\ \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k-1} \\ \vdots \\ \binom{n}{0} \end{pmatrix} - \binom{n}{k} \begin{pmatrix} \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \begin{smallmatrix} n-k \\ \text{zeros} \end{smallmatrix} \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \tag{11}$$

PROOF. From Euler–Poincaré formula we know that the last entry of h-vector is always equal 1. The above combinatorial interpretation for the h-vector of the $(n-3)$-skeleton of $\Delta_{n-1}^*(k)$ and relation (10) gives us that:

$$h^{(n-3)}(\Delta_{n-1}^*(k)) = \begin{pmatrix} 0 \\ h^{(n-4)}(\Delta_{n-2}^*(k)) \end{pmatrix} + \begin{pmatrix} h^{(n-4)}(\Delta_{n-2}^*(k-1)) \\ 0 \end{pmatrix} + \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \begin{smallmatrix} k-1 \\ \text{zeros} \end{smallmatrix} \\ \binom{n-1}{k-1} \\ \vdots \\ \binom{n-1}{k-1} \end{pmatrix}$$

Let us to denote with $\widetilde{h}(P)$ "the reduced" h vector of a polytope P, i.e., $h(P) = (\widetilde{h}(P), 1)$. Now we use Stanley's trick to compute the h-vector of $\Delta_{n-1}^*(k)$. From the above we obtain the following recursive relations

$$\begin{aligned} \widetilde{h}\big(\Delta_{n-1}^*(k)\big) = \big(0, \widetilde{h}(\Delta_{n-2}^*(k))\big) + \big(\widetilde{h}(\Delta_{n-2}^*(k-1)), 0\big) \\ + \big(\underbrace{0, \dots, 0}_{k-1}, \tbinom{n-1}{k-1}, 0, \dots, 0, 1 - \tbinom{n-1}{k-1}\big). \end{aligned}$$

Note that $\binom{n-1}{k-1}-1$ is the last entry of $h^{(n-4)}(\Delta^*_{n-2}(k-1))$. The formulae (11) will follow from the above relation and beginning conditions; $\Delta^*_2(1)$ i $\Delta^*_2(2)$ are 2-simplexes. $\square$

Now, we consider the toric h-vector of $\Delta^*_{n-1}(k)$. The polytopes $\Delta^*_{n-1}(k)$ are quasisimplicial, and from the formulae (4), for $0 \leqslant k \leqslant n-k$ we have that

$$\begin{aligned} h^T(\Delta^*_{n-1}(k)) = \big(&\tbinom{n}{0}, \tbinom{n}{0}+\tbinom{n}{1}, \dots, \tbinom{n}{0}+\tbinom{n}{1}+\dots+\tbinom{n}{k-1}, \\ &\dots, \tbinom{n}{0}+\tbinom{n}{1}+\dots+\tbinom{n}{k-1}, \dots, \tbinom{n}{0}+\tbinom{n}{1}, \tbinom{n}{0}\big). \end{aligned}$$

Bayer showed in [**1**] how the shelling of the ordinary polytope could be used to compute the toric h-vector. Here, we are able to compute the toric h-vector of $\Delta^*_{n-1}(k)$ from the shelling order described in (9). For this we need the following theorem, see [**3**].

Theorem 6. *For a d-disk Γ whose h-vector is $h(\Gamma) = (h_0, h_1, \dots, h_d, 0)$ and its boundary $(d-1)$-sphere $\partial\Gamma$ the following equality holds*

$$h_i(\partial\Gamma) = h_0 + h_1 + \dots + h_i - h_{d+1-i} - \dots - h_{d+1}, \ \textit{for } 0 < i \leqslant [d/2].$$

With $h^T_A(t)$ we denote the contributions to $h(\Delta^*_{n-1}(k), t)$ of the faces that appear for the first time when we add facet F_A:

$$h^T_A(t) = \sum_{G \subseteq F_A,\, G \notin \bigcup_{B<A} F_B} g(G,t)(t-1)^{n-\dim G}.$$

Theorem 7. *For $1 \leqslant a_1 < a_2 < \dots < a_k \leqslant n$ and $A = \{a_1, a_2, \dots, a_k\}$ we have $h^T_A(t) = t^{n-k-1}\sum_{i=1}^k t^{2i-a_i}$.*

Proof. If we denote with Λ_A the intersection of the facet F_A with the facets that came after F_A in the shelling order given in the relation (9), new faces are exactly those from $\Lambda_A \setminus \partial\Lambda_A$. Therefore, from the definition of h^T_A we obtain:

$$h^T_A = \sum_{F \in \Lambda_A} g(F,t)(t-1)^{n-1-\dim F} - \sum_{F \in \partial\Lambda_A} g(F,t)(t-1)^{n-1-\dim F} + g(F_A, t).$$

The face posets of Λ_A and $\partial\Lambda_A$ are simplicial, and therefore we can compute their contribution to the h^T_A from the usual (simplicial) h-vector of Λ and $\partial\Lambda$. So, we have

$$h^T_A = (t-1)h(\Lambda_A, t) - (t-1)^2 h(\partial\Lambda_A, t) + t^{k-1} + t^{k-2} + \dots + t + 1. \tag{12}$$

In a shelling order for Λ_A, the reverse order of the one given in (6), the contribution of the squares from the $(k+1-i)$-th row (the row denoted with r_{a_i}) to the h-vector of Λ_A is

$$h_{a_i} = (\overbrace{0,0,\dots,0}^{i-1}, \overbrace{1,1,\dots,1}^{n-k-a_i+i}, \overbrace{0,\dots,0}^{k-2i+a_i}).$$

If $i-1 \leqslant k-2i+a_i$, from Theorem 6, we know that the contribution of a_i to the h-vector of $\partial\Lambda_A$ is

$$(\overbrace{0,0,\dots,0}^{i-1}, 1, 2, \dots, r, r \dots, r, r, \dots, 2, 1, \overbrace{0,0,\dots,0}^{i-1})$$

When we put this in (12), we obtain that the contribution of the a_i to h_A^T is exactly $t^{n+2i-k-a_i-1}$.

A similar calculation goes for the case $i-1 > k-2i+a_i$, and the theorem follows. $\square$

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Prirodno-matematički fakultet
78000 Banja Luka
Bosnia and Herzegovina
ducci68@blic.net