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FACTORIZATION PROPERTIES OF SUBRINGS IN TRIGONOMETRIC POLYNOMIAL RINGS

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Abstract. We explore the subrings in trigonometric polynomial rings and their factorization properties. Consider the ring S' of complex trigonometric polynomials over the field $\mathbb{Q}(i)$ (see [**11**]). We construct the subrings S'_1, S'_0 of S' such that $S'_1 \subseteq S'_0 \subseteq S'$. Then S'_1 is a Euclidean domain, whereas S'_0 is a Noetherian HFD. We also characterize the irreducible elements of S'_1, S'_0 and discuss among these structures the condition: Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$.

1. Introduction

Factorization properties of integral domains have been a common interest of algebraists, particularly for polynomial rings. In this study we investigate the factorization properties of the subrings of S' (see [**11**]). The basic concepts, notions and terminology are standard, as in [**7**].

For the factorization of exponential polynomials, J. F. Ritt developed: "If $1 + a_1e^{\alpha_1 x} + \cdots + a_ne^{\alpha_n x}$ is divisible by $1 + b_1e^{\beta_1 x} + \cdots + b_re^{\beta_r x}$ with no $b = 0$, then every β is a linear combination of $\alpha_1, \dots, \alpha_n$ with rational coefficients" [**9**, Theorem].

Getting inspired by this, G. Picavet and M. Picavet [**7**] investigated some factorization properties in trigonometric polynomial rings. Following [**7**], when we replace all α_k above by im, with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \right\},$$

$$T = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \right\}$$

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are trigonometric polynomial rings.

Following Cohn [**4**], an integral domain D is atomic if each nonzero nonunit of D is a product of irreducible elements (*atoms*) of D, and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist any infinite strictly ascending chain of principal integral ideals of D. Every PID, UFD and Noetherian domain satisfy ACCP and a domain satisfying ACCP is atomic. Grams [**6**] and Zaks [**13**] provided examples of atomic domains, which do not satisfy ACCP. Following [**12**], an integral domain D is said to be a *half-factorial domain* (HFD) if D is *atomic* and whenever $x_1 \dots x_m = y_1 \dots y_n$, where $x_1, x_2 \dots x_m$, $y_1, y_2 \dots y_n$ are irreducibles in D, then $m = n$. A UFD is obviously an HFD, but the converse fails, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [**12**], but not a UFD. Moreover a polynomial extension of an HFD is not an HFD, for example, $\mathbb{Z}\big[\sqrt{-3}\,\big][X]$ is not an HFD, as $\mathbb{Z}\big[\sqrt{-3}\,\big]$ is an HFD but not integrally closed [**5**]. Following [**2**], an integral domain D is a *finite factorization domain* (FFD) if each nonzero nonunit of D has only a finite number of non-associate divisors and hence only a finite number of factorizations up to order and associates. In general,

$$\textbf{UFD} \Longrightarrow \textbf{HFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic},$$
$$\textbf{UFD} \Longrightarrow \textbf{FFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic}.$$

But none of the above implications is reversible.

In [**7**, Theorems 2.1 and 3.1], G. Picavet and M. Picavet demonstrated that T' is a Euclidean domain and T is a Dedekind half-factorial domain. Moreover, in [**11**] we extended the study of factorization properties of trigonometric polynomials with coefficients from the field $\mathbb{Q}$ and its algebraic extension $\mathbb{Q}(i)$, instead of $\mathbb{R}$ and $\mathbb{C}$, that is we study

$$S' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}(i) \right\},$$
$$S = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q} \right\}.$$

where S' is a Euclidean domain and S is a Dedekind finite factorization domain (see [**11**, Theorem 1 & Theorem 2]).

Again following [**7**], $\operatorname{Sin}^2 x = (1 - \operatorname{Cos} x)(1 + \operatorname{Cos} x)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $\operatorname{Cos} kx$ and $\operatorname{Sin} kx$ the two functions $x \mapsto \operatorname{Cos} kx$ and $x \mapsto \operatorname{Sin} kx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N} \setminus \{1\}$, $\operatorname{Cos} nx$ represents a polynomial in $\operatorname{Cos} x$ with degree n and $\operatorname{Sin} nx$ represents the product of $\operatorname{Sin} x$ and a polynomial in $\operatorname{Cos} x$ with degree $n-1$. Conversely by linearization formulas, it follows that any product $\operatorname{Cos}^n x \operatorname{Sin}^p x$ can be written as:

$$\sum_{k=0}^{q}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx), \text{ where } q \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{Q}.$$

Hence $S = \mathbb{Q}[\operatorname{Cos} x, \operatorname{Sin} x] \subseteq \mathbb{R}[\operatorname{Cos} x, \operatorname{Sin} x] = T$ and $S' = \mathbb{Q}(i)[\operatorname{Cos} x, \operatorname{Sin} x] \subseteq \mathbb{C}[\operatorname{Cos} x, \operatorname{Sin} x] = T'$.

We continue the investigations to find the factorization properties in trigonometric polynomial rings, begun in [**7**] and extended in [**11**]. In other words we extend this study towards finding factorization properties of subrings of trigonometric polynomial rings, by establishing S'_0 and S'_1 as subrings.

In Section 2 we explore S'_1 and S'_0, and demonstrate that the ring S'_1 is Euclidean domain ($\backsimeq (\mathbb{Q}[X])_X$), whereas S'_0 is a Notherian HFD ($\backsimeq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$). In Section 3 we discus Condition 1 (see [**8**, p. 661]) among the rings S'_1, S'_0 and S'. We also extend the Condition 1, as Condition 2.

2. The Subrings of $\mathbb{Q}(i)[\operatorname{Cos} x, \operatorname{Sin} x]$

A Construction of S'_1. We consider

$$S'_1 = \left\{\sum_{k=0}^{n}(a_k \operatorname{Cos} kx + ib_k \operatorname{Sin} kx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}\right\}.$$

Let $z = \sum_{k=0}^{n}(a_k \operatorname{Cos} kx + ib_k \operatorname{Sin} kx) \in S'_1$. As $\operatorname{Cos} x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\operatorname{Sin} x = \frac{1}{2i}(e^{ix} - e^{-ix})$, so

$$\begin{aligned} z &= \sum_{k=0}^{n}\left\{\Big(\frac{a_k + b_k}{2}\Big)e^{ikx} + \Big(\frac{a_k - b_k}{2}\Big)e^{-ikx}\right\} \\ &= e^{-inx}\left[\sum_{k=0}^{n}\left\{\Big(\frac{a_k + b_k}{2}\Big)e^{i(n+k)x} + \Big(\frac{a_k - b_k}{2}\Big)e^{i(n-k)x}\right\}\right], \end{aligned}$$

where $(a_k + b_k)/2, (a_k - b_k)/2 \in \mathbb{Q}$. Therefore any element z is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}[X]$ and $\deg(P) \leqslant 2n$.

Conversely, for $\alpha_k \in \mathbb{Q}$, $0 \leqslant k \leqslant 2n$, we have

$$e^{-inx}P(e^{ix}) = e^{-inx}\Bigg(\sum_{k=0}^{2n}\alpha_k e^{ikx}\Bigg) = \sum_{k=0}^{n-1}\big(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}\big) + \alpha_n.$$

As $e^{ix} = \operatorname{Cos} x + i \operatorname{Sin} x$, so

$$\begin{aligned} e^{-inx}P(e^{ix}) &= \sum_{k=0}^{n-1}\big\{\alpha_k(\operatorname{Cos}(n-k)x - i\operatorname{Sin}(n-k)x) \\ &\qquad + \alpha_{2n-k}(\operatorname{Cos}(n-k)x + i\operatorname{Sin}(n-k)x)\big\} + \alpha_n \\ &= \sum_{k=0}^{n-1}\big\{(\alpha_k + \alpha_{2n-k})\operatorname{Cos}(n-k)x \\ &\qquad + i(\alpha_{2n-k} - \alpha_k))\operatorname{Sin}(n-k)x\big\} + \alpha_n, \end{aligned}$$

where $\alpha_k + \alpha_{2n-k}$, $\alpha_{2n-k} - \alpha_k \in \mathbb{Q}$. Therefore S_1' contains all the elements that are of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}[X]$ has degree at most $2n$.

CONCLUSION 1. A consequence of the above construction is : $S_1' = \{e^{-inx}P(e^{ix}), n \in \mathbb{N}, \text{ where } P(X) \in \mathbb{Q}[X] \text{ and } \deg(P) \leqslant 2n\}$. So we have an isomorphism $f : (\mathbb{Q}[X])_X \to S_1'$ through the substitution morphism $X \to e^{ix}$. Therefore $S_1' \backsimeq (\mathbb{Q}[X])_X$.

THEOREM 2.1. *S_1' is a Euclidean domain having nonzero elements of $\mathbb{Q}$ as units and irreducible elements, up to units, trigonometric polynomials of the form* $\operatorname{Cos} x + i \operatorname{Sin} x - a$, *where $a \in \mathbb{Q} \setminus \{0\}$.*

PROOF. $(\mathbb{Q}[X])_X$ is a localization of $\mathbb{Q}[X]$ by a multiplicative system generated by a prime because X is a prime in $\mathbb{Q}[X]$ [**1**, Example 1.8 (b)]. Also $\mathbb{Q}[X]$ is a Euclidean domain. Therefore $(\mathbb{Q}[X])_X$ is a Eucledean domain [**10**, Proposition 7]. Now use the isomorphism $S_1' \backsimeq (\mathbb{Q}[X])_X$ in Conclusion 1. □

A Construction of S_0'. Let $z = \sum_{k=0}^{n}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx)$, $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{Q}(i)$, such that $a_n = \alpha + \gamma + i\beta$ and $b_n = -\beta + i(\alpha - \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{Q}$; obviously $z \in S'$. We define S_0' to be the set of all the polynomials of the form $\sum_{k=0}^{n}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx)$, $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{Q}(i)$ and $a_n = \alpha + \gamma + i\beta$, $b_n = -\beta + i(\alpha - \gamma)$. Let z be a polynomial from S_0'. We may write

$$\begin{aligned} z &= a_0 + \sum_{k=1}^{n-1}(a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) + \big\{(\alpha+\gamma+i\beta)\operatorname{Cos} nx + (-\beta + i(\alpha-\gamma))\operatorname{Sin} nx\big\} \\ &= a_0 + \sum_{k=1}^{n-1}\Big\{\Big(\frac{a_k' + b_k'' + i(a_k'' - b_k')}{2}\Big)e^{ikx} + \Big(\frac{a_k' - b_k'' + i(a_k'' + b_k')}{2}\Big)e^{-ikx}\Big\} \\ &\qquad + (\alpha + i\beta)e^{inx} + \gamma e^{-inx}, \end{aligned}$$

where $a_k = a_k' + ia_k''$, $b_k = b_k' + ib_k''$ and $a_k', a_k'', b_k', b_k'' \in \mathbb{Q}$, $a_0 \in \mathbb{Q}(i)$. Setting $\alpha_k' = \frac{1}{2}\big(a_k' + b_k'' + i(a_k'' - b_k')\big)$ and $\beta_k' = \frac{1}{2}\big(a_k' - b_k'' + i(a_k'' + b_k')\big)$, we have

$$z = e^{-inx}\Big[a_0 e^{inx} + \sum_{k=1}^{n-1}\big\{\alpha_k' e^{i(n+k)x} + \beta_k' e^{i(n-k)x}\big\} + (\alpha + i\beta)e^{i2nx} + \gamma\Big],$$

where $\alpha_k', \beta_k', a_0 \in \mathbb{Q}(i)$ and $\alpha, \beta, \gamma \in \mathbb{Q}$. So z is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{Q}(i)[X]$ and $\deg(P) \leqslant 2n$.

Conversely, for $\alpha_0 \in \mathbb{Q}$, and $\alpha_k \in \mathbb{Q}(i)$, $1 \leqslant k \leqslant 2n$, we have

$$\begin{aligned} e^{-inx}P(e^{ix}) &= e^{-inx}\big(\alpha_0 + \alpha_1 e^{ix} + \cdots + \alpha_{2n}e^{i2nx}\big) \\ &= \alpha_0 e^{-inx} + \sum_{k=1}^{2n-1}\alpha_k e^{-i(n-k)x} + \alpha_{2n}e^{inx} \\ &= \alpha_0 e^{-inx} + \alpha_{2n}e^{inx} + \sum_{k=1}^{n-1}\big(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}\big) + \alpha_n \\ &= \alpha_0(\operatorname{Cos} nx - i \operatorname{Sin} nx) + \alpha_{2n}(\operatorname{Cos} nx + i \operatorname{Sin} nx) \end{aligned}$$

$$+\sum_{k=1}^{n-1}\big\{\alpha_k\big(\operatorname{Cos}(n-k)x - i\operatorname{Sin}(n-k)x\big) \\ + \alpha_{2n-k}(\operatorname{Cos}(n-k)x + i\operatorname{Sin}(n-k)x)\big\} + \alpha_n.$$

Take $\alpha_k = \alpha'_k + i\alpha''_k$, $\alpha_{2n-k} = \alpha'_{2n-k} + i\alpha''_{2n-k}$ and $\alpha_{2n} = \alpha'_{2n} + i\alpha''_{2n}$. Thus

$$\begin{aligned} e^{-inx}P(e^{ix}) &= (\alpha_0 + \alpha'_{2n} + i\alpha''_{2n})\operatorname{Cos} nx + (-\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0))\operatorname{Sin} nx \\ &\quad + \sum_{k=1}^{n-1}\big\{(\alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}))\operatorname{Cos}(n-k)x \\ &\qquad + (\alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k))\operatorname{Sin}(n-k)x\big\} + \alpha_n \\ &= a_n\operatorname{Cos} nx + b_n\operatorname{Sin} nx + \sum_{k=1}^{n-1}\big\{a_k\operatorname{Cos}(n-k)x + b_k\operatorname{Sin}(n-k)x\big\} + \alpha_n, \end{aligned}$$

where

$$\begin{aligned} a_n &= \alpha_0 + \alpha'_{2n} + i\alpha''_{2n}, & a_k &= \alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}), \\ b_n &= -\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0), & b_k &= \alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k). \end{aligned}$$

So, every element of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{Q}(i)[X]$ and $\deg(P) \leqslant 2n$ is in S'_0.

Conclusion 2. A consequence of above construction is: $S'_0 = \big\{e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}+X\mathbb{Q}(i)[X]$ and $\deg(P) \leqslant 2n\big\}$. So we have an isomorphism $f : (\mathbb{Q}+X\mathbb{Q}(i)[X])_X \to S'_0$ through the substitution morphism $X \to e^{ix}$. Therefore $S'_0 \backsimeq (\mathbb{Q} + X\mathbb{Q}(i)[X])_X$.

Theorem 2.2. *The integral domain S'_0 is a Noetherian HFD having nonzero elements of $\mathbb{Q}(i)$ as units and trigonometric polynomials $\operatorname{Cos} x + i\operatorname{Sin} x - a$, where $a \in \mathbb{Q}(i) \setminus \{0\}$ are irreducible elements, up to units.*

Proof. Since X is a prime in $\mathbb{Q} + X\mathbb{Q}(i)[X]$ [**1**, Example 1.8(b)], we have that $(\mathbb{Q}+X\mathbb{Q}(i)[X])_X$ is a localization of $\mathbb{Q}+X\mathbb{Q}(i)[X]$ by a multiplicative system generated by a prime. Also $\mathbb{Q} + X\mathbb{Q}(i)[X]$ is a Notherian HFD [**3**, Theorem 4], [**2**, Proposition 3.1]. Therefore $(\mathbb{Q} + X\mathbb{Q}(i)[X])_X$ is an HFD [**1**, Corollary 2.5] and Notherian [**14**, Corollary 1, p. 224]. Hence the isomorphism $S'_0 \backsimeq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$ in Conclusion 2 gives the result. □

The following is an analogue of [**11**, Corollary 1] and gives a factorization in S'_0 instead of S'.

Corollary 2.1. *Let $z = \sum_{k=0}^{n}(a_k\operatorname{Cos} kx + b_k\operatorname{Sin} kx)$, $n \in \mathbb{N} \setminus \{1\}$, $a_k, b_k \in \mathbb{Q}(i)$ with $(a_n, b_n) \neq (0,0)$, such that $a_n = \alpha+\gamma+i\beta$ and $b_n = -\beta+i(\alpha-\gamma)$, where $\alpha, \beta, \gamma \in \mathbb{Q}$. Let d be a common divisor of the integers k such that $(a_k, b_k) \neq (0,0)$. Then z has a unique factorization*

$$\lambda(\operatorname{Cos} nx - i\operatorname{Sin} nx)\prod_{j=1}^{2n/d}(\operatorname{Cos} dx + i\operatorname{Sin} dx - \alpha_j), \ \ \textit{where } \lambda, \alpha_j \in \mathbb{Q}(i) \setminus \{0\}.$$

PROOF. Since $S_0' \subset S'$, the proof follows by [**11**, Corollary 1]. $\square$

REMARK 2.1. The factorization in S_1' is an analogue of Corollary 2.1.

Now onwards the symbol $\cap$ in all diagrams will represent the inclusion $\subseteq$.

REMARK 2.2. $\mathbb{Q} + X\mathbb{Q}(i)[X]$ is a Noetherian HFD wedged between two Euclidean domains $\mathbb{Q}[X]$ and $\mathbb{Q}(i)[X]$, that is $\mathbb{Q}[X] \subseteq Q + X\mathbb{Q}(i)[X] \subseteq \mathbb{Q}(i)[X]$ and the localization of all these by a multiplicative system generated by X preserves their factorization properties as

$$\begin{array}{ccccc} \mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\ \cap & & \cap & & \cap \\ (\mathbb{Q}[X])_X & \subseteq & (\mathbb{Q} + X\mathbb{Q}(i)[X])_X & \subseteq & (\mathbb{Q}(i)[X])_X. \end{array}$$

Using Conclusion 1, Conclusion 2 and [**11**, Theorem 1], we have

$$\begin{array}{ccccc} \mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\ \cap & & \cap & & \cap \\ S_1' & \subseteq & S_0' & \subseteq & S', \end{array}$$

where S_0' is a Noetherian HFD wedged between two Euclidean domains S_1' and S'.

REMARK 2.3. (a) Consider the domain extension $\mathbb{Q}[X] \subseteq (Q[X])_X$. As $X\mathbb{Q}[X]$ is a maximal ideal of $\mathbb{Q}[X]$ and $X\mathbb{Q}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}[X])^e = (\mathbb{Q}[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}[X])^e \backsimeq S_1'$ by Conclusion 1.

(b) If we consider the domain extension $\mathbb{Q}+X\mathbb{Q}(i)[X] \subseteq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$. We observe that $X\mathbb{Q}(i)[X]$ is a maximal ideal of $\mathbb{Q}+X\mathbb{Q}(i)[X]$ and $X\mathbb{Q}(i)[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}(i)[X])^e = (\mathbb{Q} + X\mathbb{Q}(i)[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}(i)[X])^e \backsimeq S_0'$ by Conclusion 2.

(c) On the same lines we can apply the same result to the domain extension $\mathbb{Q}(i)[X] \subseteq (Q(i)[X])_X$. In this case $X\mathbb{Q}(i)[X]$ is a maximal ideal of $\mathbb{Q}(i)[X]$ and $X\mathbb{Q}(i)[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{Q}(i)[X])^e = (\mathbb{Q}(i)[X])_X$ [**14**, Corollary 2]. Hence $(X\mathbb{Q}(i)[X])^e \backsimeq S'$ by [**11**, Theorem 1].

DEFINITION 2.1. Let J' be the subset of S_1' defined by

$$J' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + i b_k \operatorname{Sin} kx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q} \text{ and } a_n = b_n \right\}.$$

DEFINITION 2.2. Let I' be the subset of S_0' defined by

$$I' = \left\{ \sum_{k=0}^{n} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} kx) : n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q}(i) \text{ and } a_n = \alpha + i\beta, b_n = -\beta + i\alpha \right\}.$$

LEMMA 2.1. *For the maximal ideal* $X\mathbb{Q}[X]$ (*respectively* $X\mathbb{Q}(i)[X]$) *of* $\mathbb{Q}[X]$ (*respectively* $\mathbb{Q} + X\mathbb{Q}(i)[X]$) *we have* $(X\mathbb{Q}[X])_X \backsimeq J'$ (*respectively* $(X\mathbb{Q}(i)[X])_X \backsimeq I'$).

PROOF. Follows by Conclusion 1 (respectively Conclusion 2). $\square$

3. Conditions satisfied by ring extensions

In this section we discuss two special conditions. First one, known as Condition 1, is borrowed from [**8**] and the second one is derived from Condition 1. Moreover, we study a few interesting results about these conditions and trigonometric polynomial ring extensions satisfying them.

CONDITION 1. Let $A \subseteq B$ be a unitary (commutative) ring extension. For every $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$ [**8**, page 661].

EXAMPLE 3.1. Following [**8**, Example 1.1]; (a) If the ring extension $A \subseteq B$ satisfies Condition 1, then the ring extension $A+XB[X] \subseteq B[X]$ (or $A+XB[[X]] \subseteq B[[X]]$) also satisfies Condition 1.

(b) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, then so does the ring extension $A \subseteq C$.

(c) If B is a fraction ring of A, then the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfies Condition 1 is the generalization of localization.

(d) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1.

CONDITION 2. Let A, A_1, B and B_1 be unitary (commutative) rings such that

$$\begin{array}{ccc} A & \subseteq & B \\ \cap & & \cap \\ A_1 & \subseteq & B_1 \end{array}.$$

Then for each $x \in B_1$ there exist $x' \in U(B)$ and $x'' \in A_1$ such that $x = x'x''$.

LEMMA 3.1. *Let $A \subseteq B$ be a unitary (commutative) ring extension which satisfies Condition 1. If N is a multiplicative system in A, then the ring extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition 2.*

PROOF. Since the ring extension $A \subseteq B$ satisfies Condition 1. Therefore for each $a \in B$ there exist $b \in U(B)$ and $c \in A$ such that $a = bc$. Obviously $N^{-1}A \subseteq N^{-1}B$. Let $x = \frac{a}{s} \in N^{-1}B$, where $a \in B$, $s \in N$. This implies $x = \frac{bc}{s} = b\frac{c}{s}$, where $b \in U(B)$ and $\frac{c}{s} \in N^{-1}A$. □

EXAMPLE 3.2. (a) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 2, then so does the ring extension $A \subseteq C$.

(b) By Lemma 3.1 the ring extensions $S_1' \subseteq S_0'$ and $S_0' \subseteq S'$ satisfy Condition 2 so does the ring extension $S_1' \subseteq S'$.

(c) If the ring extension $A \subseteq B$ satisfies Condition 1, then obviously it satisfies Condition 2.

PROPOSITION 3.1. *Let $A \subseteq B$ and $A_1 \subseteq B_1$ be unitary (commutative) ring extensions, where $A \subseteq A_1$ and $B \subseteq B_1$. Let M be a commen ideal of A, B, A_1 and B_1 for which the extension $A_1/M \subseteq B_1/M$ satisfies Condition 2. Assume for each $\alpha \in U(B_1/M)$ there exists $a \in U(B)$ such that $p(a) = \alpha$, where $p : B_1 \to B_1/M$ is the canonical surjection; then $A_1 \subseteq B_1$ satisfies Condition 2.*

Proof. Let $b \in B_1$. We represent the class of b by $\hat{b}$ in B_1/M. Using Condition 2, we have $\hat{b} = \hat{b}'\hat{b}''$, with $\hat{b}' \in U(B/M)$, $\hat{b}'' \in A_1/M$. By hypothesis $b' \in U(B)$, since $\hat{b}'' \in A_1/M$, for $b'' \in A_1$, we have $b = b'b'' + m = b'(b'' + b'^{-1}m)$ with $m \in M$. Thus $b'' + b'^{-1}m \in A_1$. □

Lemma 3.2. *Let $A \subseteq B$ and $A_1 \subseteq B_1$ be unitary (commutative) ring extensions, where $A \subseteq A_1$ and $B \subseteq B_1$. Let M be an ideal of A_1 that is also an ideal in B_1. If for each $b \in B_1 \setminus M$ there exists $m \in M$ such that $b + m \in U(B)$, then the extension $A \subseteq B$ satisfies Condition 2.*

Proof. If $b \in M$, then $b = 1.b$. Let $b \in B_1 \setminus M$, then there exists $m \in M$ with $b + m \in U(B)$. So we can write, $b = (b+m)(b+m)^{-1}b$ and $(b+m)^{-1}b \in A_1$, because $(b+m)^{-1}b = 1 + m'$ with $m' \in M$. □

Proposition 3.2. *Let $A \subseteq B_1 \subseteq B_2$ be a unitary (commutative) ring extension such that $A \subseteq B_2$ satisfies Condition 2. If for each $x \in U(B_1)$, we have $x \in A$ or $x^{-1} \in A$ then $B_1 = N^{-1}A$, where $N = U(B_1) \cap A$.*

Proof. The inclusion $N^{-1}A \subseteq B_1$ is obvious. Let $x \in B_2$. We can write $x = x'x''$, where $x' \in U(B_1)$, $x'' \in A$. If $x' \in A$ then $x' \in A \cap U(B_1) = N$ and $x \in A$. If $x'^{-1} \in A$ then $x'^{-1} \in A \cap U(B_1) = N$ and $x = \frac{x''}{x'^{-1}} \in N^{-1}A$. □

Remark 3.1. Consider the following commutative inclusion diagram which follows from Remark 2.2.

$$\begin{array}{ccccc}
\mathbb{Q}[X] & \subseteq & \mathbb{Q} + X\mathbb{Q}(i)[X] & \subseteq & \mathbb{Q}(i)[X] \\
\cap & \searrow & \cap & \searrow & \cap \\
S_1' & \subseteq & S_0' & \subseteq & S'.
\end{array}$$

Now the following table concludes our discussion on Condition 1 and Condition 2 among trigonometric polynomial ring extensioins.

Ring Extension	Condition 1	Condition 2
$\mathbb{Q}[X] \subseteq \mathbb{Q} + X\mathbb{Q}(i)[X]$	No	No
$\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq \mathbb{Q}(i)[X]$	Yes	Yes
$\mathbb{Q}[X] \subseteq S_1'$	Yes	Yes
$\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq S_0'$	Yes	Yes
$\mathbb{Q}(i)[X] \subseteq S'$	Yes	Yes
$S_1' \subseteq S_0'$	No	Yes
$S_0' \subseteq S'$	No	Yes

By transitivity the domain extensions $\mathbb{Q}[X] \subseteq S_0'$, $\mathbb{Q} + X\mathbb{Q}(i)[X] \subseteq S'$ and $S_1' \subseteq S'$ also satisfy Condition 2.

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