

SETS AND POSETS WITH INVERSIONS

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ABSTRACT. We investigate unary operations \vee , \wedge and \diamond on a set X satisfying $x = x^{\vee\vee} = x^{\wedge\wedge}$ and $x^\diamond = x^{\vee\wedge} = x^{\wedge\vee}$ for all $x \in X$. Moreover, if in particular X is a meet-semilattice, then we also investigate the operations defined by

$$\begin{aligned} x_\blacktriangledown &= x \wedge x^\vee, & x_\blacktriangle &= x \wedge x^\wedge, & x_\blacklozenge &= x \wedge x^\diamond; \\ x_\bullet &= x^\vee \wedge x^\wedge, & x_\clubsuit &= x^\vee \wedge x^\diamond, & x_\spadesuit &= x^\wedge \wedge x^\diamond; \end{aligned}$$

and $x_\star = x \wedge x^\vee \wedge x^\wedge \wedge x^\diamond$ for all $x \in X$.

Our prime example for this is the set-lattice $\mathcal{P}(U \times V)$ of all relations on one group U to another V equipped with the operations defined such that

$$F^\vee(u) = F(-u), \quad F^\wedge(u) = -F(u) \quad \text{and} \quad F^\diamond(u) = -F(-u)$$

for all $F \subset U \times V$ and $u \in U$.

1. A few basic facts on relations and functions

A subset F of a product set $X \times Y$ is called a relation on X to Y . If in particular $F \subset X^2$, then we may simply say that F is a relation on X . In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the identity relation on X .

If F is a relation on X to Y , then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F , respectively.

Instead of $y \in F(x)$ sometimes we shall also write $x F y$. Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X] = F[D_F]$ will be called the domain and range of F , respectively.

If in particular $D_F = X$, then we say that F is a relation of X to Y , or that F is a total relation on X to Y . While, if $R_F = Y$, then we say that F is a relation on X onto Y .

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If F is a relation on X to Y , then $F = \bigcup_{x \in X} \{x\} \times F(x) = \bigcup_{x \in D_F} \{x\} \times F(x)$. Therefore, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$, or by specifying D_F and $F(x)$ for all $x \in D_F$.

For instance, if F is a relation on X to Y , then the inverse relation F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Thus, we also have $F^{-1} = \{(y, x) : (x, y) \in F\}$.

Moreover, if in addition G is a relation on Y to Z , then the composition relation $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

Now, a relation F on X may be called reflexive, transitive and antisymmetric if $\Delta_X \subset F$, $F \circ F \subset F$ and $F \cap F^{-1} \subset \Delta_X$, respectively. Moreover, a relation having all these properties may be called a partial order relation.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

The inverse f^{-1} of a function f is a function if and only if f is injective in the sense that $f(x) \neq f(y)$ for all $x, y \in D_f$ with $x \neq y$. Moreover, for a function g , we have $g = f^{-1}$ if and only if $g \circ f = \Delta_{D_f}$ and $f \circ g = \Delta_{D_g}$.

A function \star of a set X to itself is called a unary operation on X . Moreover, a function $*$ of X^2 to X is called a binary operation in X . For any $x, y \in X$, we usually write x^* or x_\star and $x * y$ in place of $\star(x)$ and $\star((x, y))$, respectively.

A unary operation \star on a set X is called an involution if $x = x^{**}$ for all $x \in X$. Hence, it is clear that \star is injective and onto X . Moreover, we can also note that a unary operation \star is an involution if and only if $\star = \star^{-1}$.

If X is a group, then for any $A, B \subset X$, we may also naturally define $A + B = \{x + y : x \in A, y \in B\}$ and $-A = \{-x : x \in A\}$. However, thus the family $\mathcal{P}(X)$ of all subsets of X is only a monoid with involution.

Finally, we note that a relation F on one group X to another Y will be called here odd, even and symmetric-valued if $F(-x) = -F(x)$, $F(-x) = F(x)$ and $-F(x) = F(x)$ hold for all $x \in X$, respectively.

2. A few basic facts on partially ordered sets

According to Birkhoff [1], a set X , equipped with a partially order relation \leq , is called a poset (partially ordered set). In this case, for any $x, y \in X$, we write $x < y$ if $x \leq y$ and $x \neq y$.

A poset X is called a chain (totally ordered set) if for any $x, y \in X$ we have either $x \leq y$ or $y \leq x$. Thus, X is a chain if and only if at least (exactly) one of the alternatives $x < y$, $x = y$ and $y < x$ holds.

If X is a poset with the relation \leq , then X is also a poset with the inverse relation \geq of \leq . This poset is denoted by X^* and called the dual of X . Thus, if in particular X is a chain, then X^* is also a chain.

If X and Y are posets, then for any $(x, y), (z, w) \in X \times Y$ we may naturally write $(x, y) \leq (z, w)$ if $x \leq z$ and $y \leq w$. Thus, $X \times Y$ is also a poset. However, if X and Y are chains, then $X \times Y$ need not be a chain.

Therefore, under the above assumptions, it is frequently more convenient to write $(x, y) \leq (z, w)$ if either $x < z$ or $x = z$ and $y \leq w$. Thus, $X \times Y$ is also a poset. Moreover, if X and Y are chains, then $X \times Y$ is also a chain.

A function f of one poset X to another Y is called increasing if $f(x) \leq f(y)$ for all $x, y \in X$ with $x \leq y$. Moreover, f is called strictly increasing if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

Decreasing functions can be defined quite similarly. Note that a function f of one poset X to another Y is decreasing if and only if it is an increasing function of X^* to Y , or equivalently of X to Y^* .

If X is a poset, then for any $\alpha \in X$ and $A \subset X$ we write $\alpha \in \text{lb}(A)$ if $\alpha \leq x$ for all $x \in A$. Moreover, we write $\min(A) = A \cap \text{lb}(A)$. Note that, by the antisymmetry of the inequality, $\min(A)$ is at most a singleton.

The expressions $\text{ub}(A)$ and $\max(A)$ can be defined quite similarly. Moreover, we may write $\inf(A) = \max(\text{lb}(A))$ and $\sup(A) = \min(\text{ub}(A))$. Thus, for instance, $\inf(A) = \max(\text{lb}(A)) = \text{lb}(A) \cap \text{ub}(\text{lb}(A))$.

Therefore, by identifying singletons with their elements, we have $\alpha = \inf(A)$ if and only if $\alpha \in \text{lb}(A)$ and $\alpha \in \text{ub}(\text{lb}(A))$. That is, $\alpha \leq x$ for all $x \in A$ and $\beta \leq \alpha$ for all $\beta \in \text{lb}(A)$, i.e., for all $\beta \in X$ with $\beta \leq x$ for all $x \in A$.

A poset X is called a meet-semilattice (join-semilattice) if $x \wedge y = \inf\{x, y\}$ ($x \vee y = \sup\{x, y\}$) exists for all $x, y \in X$. Thus, X is a join-semilattice if and only if X^* is a meet-semilattice.

In particular, a poset X is called a lattice if it is both a meet-semilattice and a join-semilattice. Note that every chain X is a lattice. Namely, if $x, y \in X$ such that $x \leq y$, then we evidently have $x = x \wedge y$ and $y = x \vee y$.

Concerning increasing functions, one can easily prove the following theorems.

THEOREM 2.1. *For any function f of one poset X to another Y , the following assertions hold:*

- (1) *If f is strictly increasing, then f is increasing;*
- (2) *If f is injective and increasing, then f is strictly increasing.*

PROOF. To prove (2), assume that the conditions of (2) hold and $x, y \in X$ such that $x < y$. Then, by the definition of $<$, we have $x \leq y$ and $x \neq y$. Hence, since f is increasing and injective, we can infer that $f(x) \leq f(y)$ and $f(x) \neq f(y)$. Therefore, $f(x) < f(y)$ also holds. This shows that f is strictly increasing. \square

THEOREM 2.2. *For any function f of a chain X to a poset Y , the following assertions hold:*

- (1) *If f is strictly increasing, then f is injective;*
- (2) *If f is injective and increasing, then f^{-1} is strictly increasing.*

PROOF. To prove (2), assume that the conditions of (2) hold and $z, w \in f[X]$ such that $z < w$. Then, by the definition of $f[X]$, there exist $x, y \in X$ such that $z = f(x)$ and $w = f(y)$. Hence, since $z < w$, and thus $z \neq w$, we can see that $x \neq y$. Thus, since X is totally ordered, we have either $x < y$ or $y < x$.

However, if $y < x$ is true, then by Theorem 2.1 we have $f(y) < f(x)$, and hence $w < z$. This contradicts the assumption that $z < w$. Namely, if both $z < w$ and $w < z$ are true, then $z \leq w$ and $w \leq z$, and thus $z = w$ is also true.

Therefore, we can only have $x < y$. However, since $z = f(x)$ and $w = f(y)$, we also have $x = f^{-1}(z)$ and $y = f^{-1}(w)$ by the injectivity of f . Therefore, $f^{-1}(z) < f^{-1}(w)$ also holds. This shows that f^{-1} is strictly increasing. \square

THEOREM 2.3. *If f is an injective increasing function of one poset X onto another Y such that f^{-1} is also increasing, then for any $A \subset X$ we have*

- (1) $f(\inf(A)) = \inf(f[A])$ if at least one of these infima exists;
- (2) $f(\sup(A)) = \sup(f[A])$ if at least one of these suprema exists.

PROOF. To prove the first part of (1), suppose that $A \subset X$ such that $\alpha = \inf(A)$ exists. Then, for any $x \in A$, we have $\alpha \leq x$, and hence $f(\alpha) \leq f(x)$. Therefore, $f(\alpha) \in \text{lb}(f[A])$.

On the other hand, if $\beta \in \text{lb}(f[A])$, then for any $x \in A$, we have $\beta \leq f(x)$, and hence $f^{-1}(\beta) \leq x$. Therefore, $f^{-1}(\beta) \in \text{lb}(A)$, and thus $f^{-1}(\beta) \leq \alpha$. Hence, we can infer that $\beta \leq f(\alpha)$, and thus $f(\alpha) \in \text{ub}(\text{lb}(f[A]))$ also holds.

The above arguments show that $f(\alpha) = \max(\text{lb}(f[A])) = \inf(f[A])$, and thus the required equality is also true.

The second part of (1) can be proved quite similarly. Moreover, it can be also easily derived from the first part of (1) by taking f^{-1} and $f[A]$ in place of f and A , respectively. \square

COROLLARY 2.1. *If f is an injective increasing function of one poset X onto another Y such that f^{-1} is also increasing, then for any $x, y \in X$ we have*

- (1) $f(x \wedge y) = f(x) \wedge f(y)$ if X and Y are meet-semilattices;
- (2) $f(x \vee y) = f(x) \vee f(y)$ if X and Y are join-semilattices.

REMARK 2.1. Note that if for instance f is a function of one poset X to another Y such that $f(x \wedge y) = f(x) \wedge f(y)$ whenever $x, y \in X$ such that $x \wedge y$ exists, then f is necessarily increasing.

3. Sets with inversions

DEFINITION 3.1. Let X be a set, and assume that \vee , \wedge and \diamond are unary operations on X such that $x = x^{\vee\vee} = x^{\wedge\wedge}$ and $x^\diamond = x^{\vee\wedge} = x^{\wedge\vee}$ for all $x \in X$. Then, we say that X is a set with inversions \vee , \wedge and \diamond .

The introduction of the above definition has been suggested by the following obvious examples.

EXAMPLE 3.1. Let \mathbb{R} be the set of all real numbers, and for any $x \in \mathbb{R}$ define

$$x^\vee = -x, \quad x^\wedge = \begin{cases} 0 & \text{if } x = 0, \\ x^{-1} & \text{if } x \neq 0, \end{cases} \quad \text{and} \quad x^\diamond = \begin{cases} 0 & \text{if } x = 0, \\ -x^{-1} & \text{if } x \neq 0. \end{cases}$$

Then, \mathbb{R} is a set with inversions \vee , \wedge and \diamond .

EXAMPLE 3.2. Let $\mathbb{C} = \mathbb{R}^2$, and for any $z = (u, v) \in \mathbb{C}$ define $z^\vee = (-u, v)$, $z^\wedge = (u, -v)$ and $z^\diamond = (-u, -v)$. Then, \mathbb{C} is a set with inversions \vee , \wedge and \diamond .

REMARK 3.1. Note that z^\wedge and z^\diamond are just the complex conjugate and the ordinary negative of z , respectively.

EXAMPLE 3.3. Let U and V be groups, and for any function f of U to V and $u \in U$ define $f^\vee(u) = f(-u)$, $f^\wedge(u) = -f(u)$ and $f^\diamond(u) = -f(-u)$. Then, the family V^U of all functions f of U to V is a set with inversions \vee , \wedge and \diamond .

REMARK 3.2. Note that, for any $u \in U$ and $v \in V$, we have

$$\begin{aligned} (u, v) \in f^\diamond &\Leftrightarrow v = f^\diamond(u) \Leftrightarrow v = -f(-u) \\ &\Leftrightarrow -v = f(-u) \Leftrightarrow (-u, -v) \in f \Leftrightarrow -(u, v) \in f. \end{aligned}$$

Therefore, f^\diamond is just the global negative of f .

REMARK 3.3. The global negative f^\diamond has to be carefully distinguished from the pointwise one f^\wedge despite that both can be naturally denoted by $-f$. Namely, for instance, if $\Delta = \Delta_U$ is the identity function of U , then $\Delta^\diamond = \Delta$. But, $\Delta^\wedge = \Delta$ if and only if $-u = u$, or equivalently $2u = 0$ for all $u \in U$.

EXAMPLE 3.4. Let U and V be groups, and for any relation F on X to Y and $u \in U$ define $F^\vee(u) = F(-u)$, $F^\wedge(u) = -F(u)$ and $F^\diamond(u) = -F(-u)$. Then, the family $\mathcal{P}(U \times V)$ of all relations F on U to V is a set with inversions \vee , \wedge and \diamond .

REMARK 3.4. It can be easily seen that $F^\vee = \{(-u, v) : (u, v) \in F\}$, $F^\wedge = \{(u, -v) : (u, v) \in F\}$, and $F^\diamond = \{(-u, -v) : (u, v) \in F\}$. Therefore, Example 3.4 is a generalization of not only Example 3.3, but also Example 3.2 too.

EXAMPLE 3.5. Let U be a group, and for any relation F on U define

$$F^\# = \{(-v, -u) : (u, v) \in F\}.$$

Then, the family $\mathcal{P}(U^2)$ of all relations F on U is a set with inversions -1 , \diamond and $\#$.

REMARK 3.5. It can be easily seen that

$$(F^{-1})^\vee = (F^\wedge)^{-1} \quad \text{and} \quad (F^\vee)^{-1} = (F^{-1})^\wedge.$$

Moreover, we can also note that above relations are, in general, quite different. Therefore, we cannot write \vee or \wedge in place of \diamond in the above example.

However, from the above examples we can immediately get several further examples with the help of the following

THEOREM 3.1. *If X is a set with inversions \vee , \wedge and \diamond , then*

- (1) *X is set with inversions \wedge , \vee and \diamond ;*
- (2) *X is set with inversions \vee , \diamond and \wedge .*

PROOF. To check (2), note that if $x \in X$, then we have

$$x^{\vee\diamond} = x^{\vee\vee\wedge} = x^\wedge \quad \text{and} \quad x^{\diamond\vee} = x^{\wedge\vee\vee} = x^\wedge.$$

Hence, we can see that $x^{\diamond\diamond} = x^{\diamond\vee\wedge} = x^{\wedge\wedge} = x$. Therefore, (2) is also true. \square

REMARK 3.6. By the above theorem, for instance, we can also state that X is a set with inversions \diamond , \vee and \wedge .

Moreover, as an immediate consequence of Definition 3.1 and Theorem 3.1, we can also state

THEOREM 3.2. *If X is a set with inversions \vee , \wedge and \diamond , then the operation $\square = \vee$, \wedge or \diamond is injective and onto X . Moreover, we have $\square = \square^{-1}$.*

4. Fixed points of the operations \vee , \wedge and \diamond

DEFINITION 4.1. If X is a set and \square is a unary operation on X , then we write $X_\square = \{x \in X : x = x^\square\}$.

EXAMPLE 4.1. Thus, according to Example 3.1, we have

$$\mathbb{R}_\vee = \{0\}, \quad \mathbb{R}_\wedge = \{-1, 0, 1\} \quad \text{and} \quad \mathbb{R}_\diamond = \{0\}.$$

EXAMPLE 4.2. Moreover, according to Example 3.2, we have

$$\mathbb{C}_\vee = \{0\} \times \mathbb{R}, \quad \mathbb{C}_\wedge = \mathbb{R} \times \{0\} \quad \text{and} \quad \mathbb{C}_\diamond = \{(0, 0)\}.$$

EXAMPLE 4.3. Furthermore, according to Example 3.4, for any relation F on U to V we have

$$\begin{aligned} F \in \mathcal{P}(U, V)_\diamond &\Leftrightarrow F \text{ is odd;} \\ F \in \mathcal{P}(U, V)_\vee &\Leftrightarrow F \text{ is even;} \\ F \in \mathcal{P}(U, V)_\wedge &\Leftrightarrow F \text{ is symmetric-valued.} \end{aligned}$$

Concerning the set X_\vee , we can easily prove the following

THEOREM 4.1. *If X is a set with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

$$(1) \ x \in X_\vee; \quad (2) \ x^\wedge = x^\diamond; \quad (3) \ x^\vee \in X_\vee; \quad (4) \ x^\wedge \in X_\vee; \quad (5) \ x^\diamond \in X_\vee.$$

PROOF. By the corresponding definitions and Theorem 3.2, it is clear that

$$\begin{aligned} x^\vee \in X_\vee &\Leftrightarrow x^\vee = x^{\vee\vee} \Leftrightarrow x^\vee = x \Leftrightarrow x \in X_\vee, \\ x^\wedge = x^\diamond &\Leftrightarrow x^\wedge = x^{\wedge\wedge} \Leftrightarrow x = x^\vee \Leftrightarrow x \in X_\vee. \end{aligned}$$

Therefore, (3) and (2) are equivalent to (1).

Moreover, by the corresponding definitions and Theorem 3.1, it is clear that

$$\begin{aligned} x^\wedge \in X_\vee &\Leftrightarrow x^\wedge = x^{\wedge\vee} \Leftrightarrow x^\wedge = x^\diamond, \\ x^\diamond \in X_\vee &\Leftrightarrow x^\diamond = x^{\diamond\vee} \Leftrightarrow x^\diamond = x^\wedge. \end{aligned}$$

Therefore, (4) and (5) are equivalent to (2), and thus also to (1). \square

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 4.1 by using Theorem 3.1.

THEOREM 4.2. *If X is a set with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

$$(1) \ x \in X_\wedge; \quad (2) \ x^\vee = x^\diamond; \quad (3) \ x^\vee \in X_\wedge; \quad (4) \ x^\wedge \in X_\wedge; \quad (5) \ x^\diamond \in X_\wedge.$$

THEOREM 4.3. *If X is a set with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

- (1) $x \in X_\diamond$; (2) $x^\vee = x^\wedge$; (3) $x^\vee \in X_\diamond$; (4) $x^\wedge \in X_\diamond$; (5) $x^\diamond \in X_\diamond$.

Now, as an immediate consequence of the latter theorem and Example 4.3, we can also state

THEOREM 4.4. *For any relation F on one group U to another V , the following assertions are equivalent:*

- (1) F is odd; (2) $F = F^\diamond$; (3) $F^\vee = F^\wedge$;
 (4) F^\vee is odd; (5) F^\wedge is odd; (6) F^\diamond is odd.

Hence, it is clear that in particular, we also have

COROLLARY 4.1. *If f is an additive function of one group X to another Y , then $f = f^\diamond$.*

REMARK 4.1. The latter statements, under different notation, have already been established in our former paper [3].

5. Posets with inversions

DEFINITION 5.1. A poset X , with inversions \vee , \wedge and \diamond , is said to be of

- (1) $\uparrow\uparrow$ -type if both \vee and \wedge are increasing;
 (2) $\uparrow\downarrow$ -type if \vee is increasing and \wedge is decreasing.

Some further similar types of posets with inversions are to be defined analogously.

EXAMPLE 5.1. If U and V are groups, then the family $\mathcal{P}(U, V)$ of all relations on U to V , equipped with the ordinary set inclusion and the operations \vee , \wedge and \diamond defined in Example 3.4, is an $\uparrow\uparrow$ -type poset with inversions.

EXAMPLE 5.2. If U is a group, then the family $\mathcal{P}(U^2)$ of all relations on U , equipped with the ordinary set inclusion and the operations -1 , \diamond and $\#$ considered in Example 3.5, is also an $\uparrow\uparrow$ -type poset with inversions.

EXAMPLE 5.3. If U is a group and V is a partially ordered group, then the family V^U of all functions of U to V , equipped with the pointwise inequality and the operations \vee , \wedge and \diamond defined in Example 3.3, is an $\uparrow\downarrow$ -type poset with inversions.

EXAMPLE 5.4. The family \mathbb{R} of all real numbers, equipped with the usual inequality and the operations \vee , \wedge and \diamond defined in Example 3.1, is a $\downarrow\downarrow$ -type poset with inversions.

EXAMPLE 5.5. If the family \mathbb{C} of all complex numbers is equipped with either the coordinate-wise inequality or the lexicographic order considered in Section 2, then the operations \vee and \wedge defined in Example 3.2 are not monotonic. However, the operation \diamond defined there is decreasing.

Namely, if for instance $z = (0, 0)$ and $w = (1, 0)$, then $z < w$, $w^\vee = (-1, 0) < (0, 0) = z^\vee$ and $z^\wedge = (0, 0) < (1, 0) = w^\wedge$. Thus, \vee is not increasing and \wedge is not decreasing.

Moreover, if for instance $\omega = (0, 1)$, then $z < \omega$, $z^\vee = (0, 0) < (0, 1) = \omega^\vee$ and $\omega^\wedge = (0, -1) < (0, 0) = z^\wedge$. Thus, \vee is not decreasing and \wedge is not increasing.

Now, as some immediate consequences of Definition 5.1 and Theorem 3.1, we can also state the following two theorems.

THEOREM 5.1. *If X is an $\uparrow\uparrow$ -type poset with inversions \vee , \wedge and \diamond , then*

- (1) X is an $\uparrow\uparrow$ -type poset with inversions \wedge , \vee and \diamond ;
- (2) X is an $\uparrow\uparrow$ -type poset with inversions \vee , \diamond and \wedge .

THEOREM 5.2. *If X is an $\uparrow\downarrow$ -type poset with inversions \vee , \wedge and \diamond , then*

- (1) X is a $\downarrow\uparrow$ -type poset with inversions \wedge , \vee and \diamond ;
- (2) X is also an $\uparrow\downarrow$ -type poset with inversions \vee , \diamond and \wedge .

Moreover, as an immediate consequence of Theorem 3.2 and Corollary 2.1, we can also at once state the following

THEOREM 5.3. *If X is an $\uparrow\uparrow$ -type poset with inversions \vee , \wedge and \diamond , then for any $x, y \in X$ and $\square \in \{\vee, \wedge, \diamond\}$, we have*

- (1) $(x \wedge y)^\square = x^\square \wedge y^\square$ if X is a meet-semilattice;
- (2) $(x \vee y)^\square = x^\square \vee y^\square$ if X is a join-semilattice.

Furthermore, by using Theorems 3.2 and a dual of Theorem 2.3, we can also easily establish the following

THEOREM 5.4. *If X is an $\uparrow\downarrow$ -type lattice with inversions \vee , \wedge and \diamond , then for any $x, y \in X$ and $\square \in \{\wedge, \diamond\}$, we have*

- (1) $(x \wedge y)^\vee = x^\vee \wedge y^\vee$;
- (2) $(x \vee y)^\vee = x^\vee \vee y^\vee$;
- (3) $(x \wedge y)^\square = x^\square \vee y^\square$;
- (4) $(x \vee y)^\square = x^\square \wedge y^\square$.

6. Compound operations on meet-semilattices with inversions

DEFINITION 6.1. If X is a meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we write

$$\begin{aligned} x_\blacktriangledown &= x \wedge x^\vee, & x_\blacktriangle &= x \wedge x^\wedge, & x_\blacklozenge &= x \wedge x^\diamond; \\ x_\bullet &= x^\vee \wedge x^\wedge, & x_\blacklozenge &= x^\vee \wedge x^\diamond, & x_\blackspade &= x^\wedge \wedge x^\diamond; \end{aligned}$$

REMARK 6.1. If X is a join-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we may naturally write $x^\blacktriangledown = x \vee x^\vee$. Namely, if in particular X is a lattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have $x_\blacktriangledown \leq x \leq x^\blacktriangledown$.

Concerning the above operations, we can easily prove the following

THEOREM 6.1. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \vee is increasing, then for any $x \in X$ we have*

- (1) $x_\blacktriangledown = (x^\vee)_\blacktriangledown = (x_\blacktriangledown)^\vee$;
- (2) $x_\blacklozenge = (x^\vee)_\blacktriangle = (x_\blacktriangle)^\vee$;
- (3) $x_\bullet = (x^\vee)_\blacklozenge = (x_\blacklozenge)^\vee$;
- (4) $x_\blacklozenge = (x^\vee)_\bullet = (x_\bullet)^\vee$;
- (5) $x_\blacktriangle = (x^\vee)_\blackspade = (x_\blackspade)^\vee$;
- (6) $x_\blackspade = (x^\vee)_\blacklozenge = (x_\blacklozenge)^\vee$.

PROOF. By the corresponding definitions and Theorem 3.1, we have

$$\begin{aligned}(x^\vee)_\blacktriangledown &= x^\vee \wedge x^{\vee\vee} = x^\vee \wedge x = x_\blacktriangledown; & (x^\vee)_\blacktriangle &= x^\vee \wedge x^{\vee\wedge} = x^\vee \wedge x^\diamond = x_\blacktriangle; \\ (x^\vee)_\blacklozenge &= x^\vee \wedge x^{\vee\diamond} = x^\vee \wedge x^\wedge = x_\bullet; & (x^\vee)_\blackspade &= x^{\vee\wedge} \wedge x^{\vee\diamond} = x^\diamond \wedge x^\wedge = x_\blackspade.\end{aligned}$$

Moreover, by Theorem 3.2 and Corollary 2.1, we also have

$$\begin{aligned}(x_\blacktriangledown)^\vee &= (x \wedge x^\vee)^\vee = x^\vee \wedge x^{\vee\vee} = x^\vee \wedge x = x_\blacktriangledown; \\ (x_\blacktriangle)^\vee &= (x \wedge x^\wedge)^\vee = x^\vee \wedge x^{\wedge\vee} = x^\vee \wedge x^\diamond = x_\blacktriangle; \\ (x_\bullet)^\vee &= (x \wedge x^\diamond)^\vee = x^\vee \wedge x^{\diamond\vee} = x^\vee \wedge x^\wedge = x_\bullet; \\ (x_\blackspade)^\vee &= (x^\wedge \wedge x^\diamond)^\vee = x^{\wedge\vee} \wedge x^{\diamond\vee} = x^\diamond \wedge x^\wedge = x_\blackspade.\end{aligned}$$

Therefore, assertions (1)–(3) and (6) are true.

Moreover, from (2) and (3), we can immediately infer that

$$\begin{aligned}(x_\blackspade)^\vee &= (x_\blacktriangle)^{\vee\vee} = x_\blacktriangle; & (x_\bullet)^\vee &= (x_\blacklozenge)^{\vee\vee} = x_\blacklozenge; \\ (x^\vee)_\blackspade &= (x^{\vee\vee})_\blacktriangle = x_\blacktriangle; & (x^\vee)_\bullet &= (x^{\vee\vee})_\blacklozenge = x_\blacklozenge.\end{aligned}$$

Therefore, assertions (5) and (4) are also true. \square

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 6.1 by using Theorem 3.1.

THEOREM 6.2. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \wedge is increasing, then*

$$\begin{aligned}(1) \ x_\blackspade &= (x^\wedge)_\blacktriangledown = (x_\blacktriangledown)^\wedge; & (2) \ x_\blacktriangle &= (x^\wedge)_\blacktriangle = (x_\blacktriangle)^\wedge; \\ (3) \ x_\bullet &= (x^\wedge)_\blacklozenge = (x_\blacklozenge)^\wedge; & (4) \ x_\blacklozenge &= (x^\wedge)_\bullet = (x_\bullet)^\wedge; \\ (5) \ x_\blackspade &= (x^\wedge)_\blackspade = (x_\blackspade)^\wedge; & (6) \ x_\blacktriangledown &= (x^\wedge)_\blackspade = (x_\blackspade)^\wedge.\end{aligned}$$

THEOREM 6.3. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \diamond is increasing, then*

$$\begin{aligned}(1) \ x_\blackspade &= (x^\diamond)_\blacktriangledown = (x_\blacktriangledown)^\diamond; & (2) \ x_\blackspade &= (x^\diamond)_\blacktriangle = (x_\blacktriangle)^\diamond; \\ (3) \ x_\blacklozenge &= (x^\diamond)_\blacklozenge = (x_\blacklozenge)^\diamond; & (4) \ x_\bullet &= (x^\diamond)_\bullet = (x_\bullet)^\diamond; \\ (5) \ x_\blacktriangle &= (x^\diamond)_\blackspade = (x_\blackspade)^\diamond; & (6) \ x_\blacktriangledown &= (x^\diamond)_\blackspade = (x_\blackspade)^\diamond.\end{aligned}$$

7. A further important operation on meet-semilattices with inversions

DEFINITION 7.1. If X is a meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we write $x_\star = x \wedge x^\vee \wedge x^\wedge \wedge x^\diamond$.

REMARK 7.1. If X is a join-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we may also naturally write $x^\star = x \vee x^\vee \vee x^\wedge \vee x^\diamond$.

A simple computation gives the following

THEOREM 7.1. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$x_\star = x_\blacktriangledown \wedge x_\blackspade = x_\blacktriangle \wedge x_\blackspade = x_\bullet \wedge x_\blacklozenge.$$

PROOF. By the corresponding definitions and the commutativity and associativity of the operation \wedge , we have

$$x_{\star} = x \wedge x^{\vee} \wedge x^{\wedge} \wedge x^{\diamond} = (x^{\vee} \wedge x^{\wedge}) \wedge (x \wedge x^{\diamond}) = x_{\bullet} \wedge x_{\blacklozenge}.$$

The proof of the other equalities are even more obvious. \square

Now, in addition to Theorems 6.1–6.3, we can easily prove the following

THEOREM 7.2. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$(1) x_{\blacktriangledown} = x_{\blacktriangledown\blacktriangledown}; \quad (2) x_{\blacklozenge} = x_{\blacklozenge\blacklozenge} = x_{\blacklozenge\blacktriangledown}; \quad (3) x_{\star} = x_{\blacktriangledown\blacksquare} = x_{\blacksquare\blacktriangledown}$$

with $\blacksquare \in \{\blacktriangle, \blacklozenge, \bullet, \clubsuit\}$.

PROOF. By the corresponding definitions and Theorems 6.1–6.3 and 7.1, we have

$$\begin{aligned} x_{\blacktriangledown\blacktriangledown} &= x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\vee} = x_{\blacktriangledown} \wedge x_{\blacktriangledown} = x_{\blacktriangledown}; & x_{\blacktriangledown\blacktriangle} &= x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\wedge} = x_{\blacktriangledown} \wedge x_{\blacklozenge} = x_{\star}; \\ x_{\blacktriangledown\blacklozenge} &= x_{\blacktriangledown} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\blacktriangledown} \wedge x_{\blacklozenge} = x_{\star}; & x_{\blacktriangledown\bullet} &= (x_{\blacktriangledown})^{\vee} \wedge (x_{\blacktriangledown})^{\wedge} = x_{\blacktriangledown} \wedge x_{\blacklozenge} = x_{\star}; \\ x_{\blacktriangledown\clubsuit} &= (x_{\blacktriangledown})^{\vee} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\blacktriangledown} \wedge x_{\blacklozenge} = x_{\star}; & x_{\blacktriangledown\blacklozenge} &= (x_{\blacktriangledown})^{\wedge} \wedge (x_{\blacktriangledown})^{\diamond} = x_{\blacklozenge} \wedge x_{\blacklozenge} = x_{\blacklozenge}; \end{aligned}$$

and quite similarly

$$\begin{aligned} x_{\blacktriangle\blacktriangledown} &= x_{\blacktriangle} \wedge (x_{\blacktriangle})^{\vee} = x_{\blacktriangle} \wedge x_{\clubsuit} = x_{\star}; \\ x_{\blacklozenge\blacktriangledown} &= x_{\blacklozenge} \wedge (x_{\blacklozenge})^{\vee} = x_{\blacklozenge} \wedge x_{\bullet} = x_{\star}; & x_{\bullet\blacktriangledown} &= x_{\bullet} \wedge (x_{\bullet})^{\vee} = x_{\bullet} \wedge x_{\blacklozenge} = x_{\star}; \\ x_{\clubsuit\blacktriangledown} &= x_{\clubsuit} \wedge (x_{\clubsuit})^{\vee} = x_{\clubsuit} \wedge x_{\blacktriangle} = x_{\star}; & x_{\blacklozenge\blacktriangledown} &= x_{\blacklozenge} \wedge (x_{\blacklozenge})^{\vee} = x_{\blacklozenge} \wedge x_{\blacklozenge} = x_{\blacklozenge}. \end{aligned} \quad \square$$

Analogously to Theorem 7.1, we can also easily prove the following theorems.

THEOREM 7.3. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$(1) x_{\blacktriangle} = x_{\blacktriangle\blacktriangle}; \quad (2) x_{\clubsuit} = x_{\blacktriangle\clubsuit} = x_{\clubsuit\blacktriangle}; \quad (3) x_{\star} = x_{\blacktriangle\blacksquare} = x_{\blacksquare\blacktriangle}$$

with $\blacksquare \in \{\blacklozenge, \bullet, \clubsuit\}$.

THEOREM 7.4. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$(1) x_{\blacklozenge} = x_{\blacklozenge\blacklozenge}; \quad (2) x_{\bullet} = x_{\blacklozenge\bullet} = x_{\bullet\blacklozenge}; \quad (3) x_{\star} = x_{\blacklozenge\blacksquare} = x_{\blacksquare\blacklozenge}$$

with $\blacksquare \in \{\clubsuit, \spadesuit\}$.

THEOREM 7.5. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$(1) x_{\blacklozenge} = x_{\bullet\bullet}; \quad (2) x_{\star} = x_{\bullet\blacksquare} = x_{\blacksquare\bullet}$$

with $\blacksquare \in \{\clubsuit, \spadesuit\}$.

THEOREM 7.6. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

$$(1) x_{\blacktriangle} = x_{\clubsuit\clubsuit}; \quad (2) x_{\star} = x_{\clubsuit\spadesuit} = x_{\spadesuit\clubsuit}; \quad (3) x_{\blacktriangledown} = x_{\clubsuit\spadesuit}.$$

Finally, we note that the following theorem is also true.

THEOREM 7.7. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

- (1) $x_{\star} = (x^{\square})_{\star} = (x_{\star})^{\square}$ with $\square \in \{\vee, \wedge, \diamond\}$;
- (2) $x_{\star} = x_{\blacktriangleleft\star} = x_{\blacktriangleright\star}$ with $\blacksquare \in \{\blacktriangledown, \blacktriangle, \blacklozenge, \blacklozenge, \blackclubsuit, \blackspadesuit, \star\}$.

8. Maximality properties of the compound operations

THEOREM 8.1. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \vee is increasing, then for any $x \in X$ we have $x_{\blacktriangledown} = \max\{y \in X_{\vee} : y \leq x\}$.*

PROOF. Define $A = \{y \in X_{\vee} : y \leq x\}$. Then, we can easily see that $x_{\blacktriangledown} \in A$. Namely, by the corresponding definitions, we have $x_{\blacktriangledown} = x \wedge x^{\vee} \leq x$. Moreover, by Theorem 6.1, we also have $x_{\blacktriangledown} = (x_{\blacktriangledown})^{\vee}$, and thus $x_{\blacktriangledown} \in X_{\vee}$.

On the other hand, if $y \in A$, then we have $y \in X_{\vee}$ and $y \leq x$. Hence, by using the corresponding definitions, we can infer that $y = y^{\vee} \leq x^{\vee}$. Therefore, $y \leq x \wedge x^{\vee} = x_{\blacktriangledown}$, and thus $x_{\blacktriangledown} \in \text{ub}(A)$. This shows that $x_{\blacktriangledown} = \max(A)$, and thus the required equality is also true. \square

From the above theorem, by using Theorems 6.1–6.3 and 7.2 and 7.7, we can immediately derive the following corollaries.

COROLLARY 8.1. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \vee is increasing, then for any $x \in X$ we have*

- (1) $x_{\blacktriangledown} = \max\{y \in X_{\vee} : y \leq x^{\vee}\}$;
- (2) $x_{\blackspadesuit} = \max\{y \in X_{\vee} : y \leq x^{\square}\}$ with $\square \in \{\wedge, \diamond\}$.

COROLLARY 8.2. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

- (1) $x_{\blacktriangledown} = \max\{y \in X_{\vee} : y \leq x_{\blacktriangledown}\}$;
- (2) $x_{\star} = \max\{y \in X_{\vee} : y \leq x_{\blacksquare}\}$ with $\blacksquare \in \{\blacktriangle, \blacklozenge, \blacklozenge, \blackclubsuit, \blackspadesuit\}$.

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 8.1 by using Theorem 3.1.

THEOREM 8.2. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \wedge is increasing, then for any $x \in X$ we have $x_{\blacktriangle} = \max\{y \in X_{\wedge} : y \leq x\}$.*

Hence, by Theorems 6.2, 6.1 and 7.3 and 7.7, it is clear that we also have the following corollaries.

COROLLARY 8.3. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \wedge is increasing, then for any $x \in X$ we have*

- (1) $x_{\blacktriangle} = \max\{y \in X_{\wedge} : y \leq x^{\wedge}\}$;
- (2) $x_{\blackclubsuit} = \max\{y \in X_{\wedge} : y \leq x^{\square}\}$ with $\square \in \{\vee, \diamond\}$.

COROLLARY 8.4. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

- (1) $x_{\blacktriangle} = \max\{y \in X_{\wedge} : y \leq x_{\blacktriangle}\};$
- (2) $x_{\star} = \max\{y \in X_{\wedge} : y \leq x_{\blacksquare}\}$ with $\blacksquare \in \{\blacklozenge, \bullet, \clubsuit, \star\}.$

THEOREM 8.3. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \diamond is increasing, then for any $x \in X$ we have $x_{\blacklozenge} = \max\{y \in X_{\diamond} : y \leq x\}.$*

Hence, by Theorems 6.3, 6.1, 6.2 and 7.4, it is clear that we also have the following corollaries.

COROLLARY 8.5. *If X is a meet-semilattice with inversions \vee , \wedge and \diamond such that \diamond is increasing, then for any $x \in X$ we have*

- (1) $x_{\blacklozenge} = \max\{y \in X_{\diamond} : y \leq x^{\diamond}\};$
- (2) $x_{\bullet} = \max\{y \in X_{\diamond} : y \leq x^{\square}\}$ with $\square \in \{\vee, \wedge\}.$

COROLLARY 8.6. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ we have*

- (1) $x_{\blacklozenge} = \max\{y \in X_{\diamond} : y \leq x_{\blacklozenge}\};$
- (2) $x_{\star} = \max\{y \in X_{\diamond} : y \leq x_{\blacksquare}\}$ with $\blacksquare \in \{\clubsuit, \spadesuit, \star\}.$

9. Some further results on the fixed points of the operations \vee , \wedge and \diamond

THEOREM 9.1. *If \square is an increasing involution on a poset X , then*

$$X_{\square} = \{x \in X : x \leq x^{\square}\} = \{x \in X : x^{\square} \leq x\}.$$

PROOF. Define $A = \{x \in X : x \leq x^{\square}\}.$ Then, by the definition of X_{\square} and the reflexivity of the inequality in X , it is clear that $X_{\square} \subset A.$

Moreover, if $x \in A$, then we have $x \in X$ and $x \leq x^{\square}.$ Hence, by using the corresponding properties of \vee , we can infer that $x^{\square} \leq x^{\square\square} = x.$ Therefore, we actually have $x = x^{\square},$ and thus $x \in X_{\square}.$ This shows that $A \subset X_{\square}.$

Therefore, $X_{\square} = A,$ and thus the first part of the theorem is true. The second part of the theorem can be proved quite similarly. \square

THEOREM 9.2. *If \square is an increasing involution on a meet-semilattice X and $x_{\blacksquare} = x \wedge x^{\square}$ for all $x \in X$, then $X_{\square} = X_{\blacksquare}.$*

PROOF. If $x \in X_{\square},$ then $x_{\blacksquare} = x \wedge x^{\square} = x \wedge x = x,$ and thus $x \in X_{\blacksquare}.$ Therefore, $X_{\square} \subset X_{\blacksquare}.$

While, if $x \in X_{\blacksquare},$ then $x = x_{\blacksquare} = x \wedge x^{\square} \leq x^{\square}.$ Hence, by Theorem 9.1, we can see that $x \in X_{\square}.$ Therefore, $X_{\blacksquare} \subset X_{\square}$ is also true. \square

REMARK 9.1. Note that if X and \square are as in the above theorems, then X is already an $\uparrow\uparrow$ -type poset with inversions either \square, Δ_x and \square or \square, \square and $\square.$

Therefore, the above theorems can be immediately derived from the corresponding results for the operation $\vee.$

THEOREM 9.3. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

- (1) $x \in X_{\vee};$
- (2) $x^{\vee} = x_{\blacktriangledown};$
- (3) $x^{\wedge} = x_{\clubsuit};$
- (4) $x^{\diamond} = x_{\spadesuit}.$

PROOF. By Theorems 4.1, 9.2 and 6.1–6.3, we can see that

$$\begin{aligned} x \in X_{\vee} &\Leftrightarrow x^{\vee} \in X_{\vee} \Leftrightarrow x^{\vee} \in X_{\blacktriangledown} \Leftrightarrow x^{\vee} = (x^{\vee})_{\blacktriangledown} \Leftrightarrow x^{\vee} = x_{\blacktriangledown}; \\ x \in X_{\wedge} &\Leftrightarrow x^{\wedge} \in X_{\wedge} \Leftrightarrow x^{\wedge} \in X_{\blacktriangledown} \Leftrightarrow x^{\wedge} = (x^{\wedge})_{\blacktriangledown} \Leftrightarrow x^{\wedge} = x_{\blacktriangledown}; \\ x \in X_{\diamond} &\Leftrightarrow x^{\diamond} \in X_{\vee} \Leftrightarrow x^{\diamond} \in X_{\blacktriangledown} \Leftrightarrow x^{\diamond} = (x^{\diamond})_{\blacktriangledown} \Leftrightarrow x^{\diamond} = x_{\blacktriangledown}. \quad \square \end{aligned}$$

Analogously to the above theorem, we can also easily prove the following theorems which can also be derived from Theorem 9.3 by using Theorem 3.1.

THEOREM 9.4. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

$$(1) x \in X_{\wedge}; \quad (2) x^{\wedge} = x_{\blacktriangleright}; \quad (3) x^{\wedge} = x_{\blacktriangleright}; \quad (4) x^{\diamond} = x_{\blacktriangleright}.$$

THEOREM 9.5. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then for any $x \in X$ the following assertions are equivalent:*

$$(1) x \in X_{\diamond}; \quad (2) x^{\diamond} = x_{\blacktriangleright}; \quad (3) x^{\wedge} = x_{\blacktriangleright}; \quad (4) x^{\diamond} = x_{\blacktriangleright}.$$

Finally, we note that, in addition to Theorem 9.2, the following theorem can also be easily proved.

THEOREM 9.6. *If X is an $\uparrow\uparrow$ -type meet-semilattice with inversions \vee , \wedge and \diamond , then*

$$\begin{aligned} (1) X_{\bullet} &= X_{\vee} \cap X_{\wedge}; \quad (2) X_{\blacktriangleright} = X_{\vee} \cap X_{\diamond}; \\ (3) X_{\blacktriangleright} &= X_{\wedge} \cap X_{\diamond}; \quad (4) X_{\star} = X_{\vee} \cap X_{\wedge} \cap X_{\diamond}. \end{aligned}$$

PROOF. If $x \in X_{\vee} \cap X_{\wedge}$, then $x \in X_{\vee}$ and $x \in X_{\wedge}$. Therefore, $x_{\bullet} = x^{\vee} \wedge x^{\wedge} = x \wedge x = x$, and thus $x \in X_{\bullet}$. Consequently, $X_{\vee} \cap X_{\wedge} \subset X_{\bullet}$.

While, if $x \in X_{\bullet}$, then $x = x_{\bullet} = x^{\vee} \wedge x^{\wedge}$. Therefore, $x \leq x^{\vee}$ and $x \leq x^{\wedge}$. Hence, by using Theorem 9.1, we can infer that $x \in X_{\vee}$ and $x \in X_{\wedge}$, and thus $x \in X_{\vee} \cap X_{\wedge}$. Consequently, $X_{\bullet} \subset X_{\vee} \cap X_{\wedge}$ also holds.

Therefore, (1) is true. The proofs of (2)–(4) are quite similar. \square

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