PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 91(105) (2012), 125–135 DOI: 10.2298/PIM1205125N

# ORDINARY DIFFERENTIAL EQUATIONS WITH DELTA FUNCTION TERMS

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Abstract. This article is devoted to nonlinear ordinary differential equations with additive or multiplicative terms consisting of Dirac delta functions or derivatives thereof. Regularizing the delta function terms produces a family of smooth solutions. Conditions on the nonlinear terms, relating to the order of the derivatives of the delta function part, are established so that the regularized solutions converge to a limiting distribution.

## Introduction

This paper is devoted to ordinary differential equations (and systems) of the form

$$y'(t) = f(t, y(t)) + g(y(t))\delta^{(s)}, \quad y(t_0) = y_0, \tag{0.1}$$

where $\delta^{(s)}$ denotes the $s$-th derivative of the Dirac delta function. The case of constant $g(y) \equiv \alpha$ will be referred to as the *additive case*, the general case with $s = 0$ will be called the *multiplicative case*. We shall replace the delta function by a family of regularizations $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(t/\varepsilon)$ and ask under what conditions on $f$, $g$ and $s$ the family of regularized solutions admits a limit as $\varepsilon \to 0$.

In the case of partial differential equations, such weak limits have been termed *delta waves* and studied in various situations, see e.g. [**15, 16, 21**]. The interest in problem (0.1) comes also from the fact that such equations have been proven to admit solutions in the Colombeau algebra of generalized functions [**5, 8, 10**].

Equations of the type (0.1) with $s = 0$ arise in nonsmooth mechanics [**1, 3, 7, 14**] and are referred to as measure differential equations or impulsive differential equations, often considered under the form

$$y'(t) = f(t, y(t)) + \sum_{i=1}^{N} g_i(y(t))\delta(t - t_i).$$

2010 *Mathematics Subject Classification:* Primary 46F10; Secondary 34A37.

The first author was partially supported by OAED and Ministry of Science, Serbia (III-44006 and OI-174024).

Such an equation can be interpreted piecewise: the function $g_i(y(t))$ is fixed during the action of the delta distribution, i.e., $g_i(y(t))$ is substituted by $\lim_{t\to t_i, t<t_i} g_i(y(t))$ and this value depends on the classical solution up to the point $t_i$ only. In the additive case one can alternatively consider the delta function term as the derivative of a function of bounded variation. We refer to the vast literature on such equations [**2, 4, 6, 12, 13, 17**]. The point of view of distribution theory and regularizing sequences is sometimes taken as well. Here the work [**18, 19, 20**] is relevant, in which higher order linear differential equations with measures as coefficients were considered and unique solutions in the space of Borel measures with primitives being normalized to being right-continuous functions were obtained. There it was also shown that the regularized solutions converge to a solution of a measure differential equation of the same type.

Higher derivatives (with $s = 1$ or $s = 2$) in nonlinear differential equations arise, e.g., in geodesic equations and geodesic deviation equations for impulsive gravitational waves [**11, 22**] and in the calculus of variations with strongly singular potential [**9**].

In this article, we adopt the approach of regularization and taking limits. The admissible order $s$ of the delta function term is arbitrary in principle, but may be restricted by the type of sublinearity of the nonlinear function $f$ (no restriction if $f$ is bounded).

The paper is organized as follows. In the first section we establish the existence of a limiting function in the additive case when $f$ is a sublinear function of order $r$ with respect to $y$, $r < 1/s$, or $f$ is globally Lipschitz in the case $s = 1$. In both cases the limit is the sum of a function continuous in $[t_0, 0) \cup (0, \infty)$ and a multiple of $\delta^{(s-1)}$. Depending on the case, the function part may be continuous across $t = 0$ or suffer a jump (whose value we compute). Similar assertions can also be obtained for systems of differential equations.

The second section addresses the multiplicative case with $s = 0$. If $f$ and $g$ are globally Lipschitz or $f$ is arbitrary and $g$ has a sufficiently small Lipschitz constant, the existence of a limiting functions with a jump at $t = 0$ is derived. In both sections we illuminate the required conditions by means of a number of examples and counterexamples. The limiting functions do not depend on the regularization if non-negative mollifiers are used. This may or may not be the case if the non-negativity condition is violated, as is shown by various examples.

## 1. The additive case

In this section, we study the equation

$$y'(t) = f(t, y(t)) + \alpha\delta^{(s)}(t), \quad y(-1) = y_0$$

on some interval $[-1, T]$ with $T > 0$. Here $\alpha \in \mathbb{R}$ and $s \geqslant 1$ is an integer ($s = 0$ will arise as a special case in the next section). Throughout, $f$ is assumed to be continuous in $(t, y)$ and locally Lipschitz with respect to $y$, uniformly on compact time intervals. We suppose that the free equation $y'(t) = f(t, y(t))$ is uniquely solvable on the whole interval $[-1, T]$ for whatever data $y(-1) = y_0 \in \mathbb{R}$, and is also uniquely solvable on $[0, T]$ for arbitrary data $y(0) = y_1$.

We are going to find limits of the family of regularized solutions $y_\varepsilon(t)$ as $\varepsilon \to 0$ when the $\delta$-distribution is substituted by some mollifier $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(t/\varepsilon)$. We shall suppose that $\phi \in C^\infty(\mathbb{R})$ has integral one, and $\operatorname{supp}\phi = [-a, b]$, $a, b \geqslant 0$.

The supports of $\phi_\varepsilon$ and its derivatives are contained in $[-a\varepsilon, b\varepsilon]$. Our first concern will be that $y_\varepsilon$ does not blow up in this interval; note that $y_\varepsilon(t)$ coincides with $\bar{y}(t)$, the classical solution to $y'(t) = f(t, y(t))$, $y(-1) = y_0$, up to $t = -a\varepsilon$, and $\lim_{\varepsilon\to 0} y_\varepsilon(-a\varepsilon) = \bar{y}(0)$. To have convergence of $y_\varepsilon(t)$ on the whole interval $[-1, T]$ we will have to verify that the limit $\lim_{\varepsilon\to 0} y_\varepsilon(b\varepsilon)$ exists, thus providing the initial data for the limiting solution for $t > 0$.

A function $g : \mathbb{R} \to \mathbb{R}$ will be called sublinear of order $r$, $0 \leqslant r < 1$, if $|g(u)| \leqslant C(1 + |u|^r)$ for some $C > 0$ and all $u \in \mathbb{R}$. In this terminology, a function which is sublinear of order 0 is bounded. In the following, $\phi_\pm \geqslant 0$ denotes the positive and negative part of $\phi$, respectively, so that $\phi = \phi_+ - \phi_-$.

THEOREM 1.1. (a) *Let $0 \leqslant r < 1$ and assume that $f$ is sublinear of order $r$ with respect to $y$, uniformly on compact intervals with respect to $t$. Let $0 < s < 1/r$ (i.e., $s$ is arbitrary if $f$ is bounded). Then the solutions to*

$$y'_\varepsilon(t) = f(t, y_\varepsilon(t)) + \alpha\phi_\varepsilon^{(s)}(t), \quad y_\varepsilon(-1) = y_0 \tag{1.1}$$

*converge to $y(t) = \bar{y}(t) + \alpha\delta^{(s-1)}(t)$, where $\bar{y}(t)$, $t \in [-1, T]$, is the classical solution to*

$$y'(t) = f(t, y(t)), \quad y(-1) = y_0.$$

(b) *Let $f$ be globally Lipschitz with respect to $y$, uniformly on compact intervals with respect to $t$ and assume that the double limits*

$$\lim_{\eta\to 0,\ y\to\pm\infty} \frac{f(\eta, y)}{y} = M_\pm$$

*exist. Let $s = 1$. Then the solutions to* (1.1) *converge to $y(t) = \bar{y}_1(t) + \alpha\delta(t)$, where $\bar{y}_1(t)$ is equal to $\bar{y}(t)$ for $t \in [-1, 0]$ and $\bar{y}_1(t)$ is the classical solution to*

$$y'(t) = f(t, y(t)), \quad y(0) = \bar{y}(0) + \beta,$$

*$t \in [0, \infty)$ with*

$$\beta = \alpha\Big(M_+\int_{-a}^{b} \phi_+(u)\,du - M_-\int_{-a}^{b} \phi_-(u)\,du\Big), \quad \text{if } \alpha > 0,$$

$$\beta = \alpha\Big(M_-\int_{-a}^{b} \phi_+(u)\,du - M_+\int_{-a}^{b} \phi_-(u)\,du\Big), \quad \text{if } \alpha < 0.$$

PROOF. We shall rewrite (1.1) in the following way. Let $y_\varepsilon(t) = y_{1\varepsilon}(t) + \alpha\phi_\varepsilon^{(s-1)}(t)$, where $y_{1\varepsilon}$ is the solution to

$$y_{1\varepsilon}{}'(t) = f\big(t, y_{1\varepsilon}(t) + \alpha\phi_\varepsilon^{(s-1)}(t)\big), \quad y_{1\varepsilon}(-1) = y_0,$$

i.e.,

$$y_{1\varepsilon}(t) = y_0 + \int_{-1}^{t} f\big(u, y_{1\varepsilon}(u) + \alpha\phi_\varepsilon^{(s-1)}(u)\big)\,du.$$

Obviously, $y_{1\varepsilon}(t) = \bar{y}(t)$ for $t \in [-1, -a\varepsilon]$, and $y_{0\varepsilon} := y_{1\varepsilon}(-a\varepsilon) \to \bar{y}(0)$ as $\varepsilon \to 0$.

First we shall show that $y_{1\varepsilon}(t)$ is bounded for $t \in [-a\varepsilon, b\varepsilon]$. By using the sublinearity we have

$$\begin{aligned}
|y_{1\varepsilon}(t) - y_{0\varepsilon}| &\leqslant \int_{-a\varepsilon}^{t} \big|f\big(u, y_{1\varepsilon}(u) + \alpha\phi_\varepsilon^{(s-1)}(u)\big)\big|\, du \\
&\leqslant \int_{-a\varepsilon}^{t} C\big(1 + \big|y_{1\varepsilon}(u) + \alpha\phi_\varepsilon^{(s-1)}(u) - y_{0\varepsilon} + y_{0\varepsilon}\big|^r\big)\, du \\
&\leqslant Cd_\varepsilon + C|y_{0\varepsilon}|^r d_\varepsilon + C\int_{-a\varepsilon}^{b\varepsilon} |\alpha|^r |\phi_\varepsilon^{(s-1)}(u)|^r du \\
&\quad + C\int_{-a\varepsilon}^{t} |y_{1\varepsilon}(u) - y_{0\varepsilon}|^r du,
\end{aligned}$$

where $d_\varepsilon = a\varepsilon + b\varepsilon$. After the change of the variables $u/\varepsilon \mapsto u$ we have

$$\int_{-a\varepsilon}^{b\varepsilon} |\phi_\varepsilon^{(s-1)}(u)|^r du = \varepsilon^{1-rs}\int_{-a}^{b} |\phi^{(s-1)}(u)|^r du.$$

That means that for $0 < s < 1/r$ the above integral converges to zero as $\varepsilon \to 0$. When $f$ is globally Lipschitz and $s = 1$, a similar estimate holds with $r = 1$, and the integral is seen to remain bounded as $\varepsilon \to 0$. Using $|u|^r \leqslant \max\{1, |u|\}$ we get from Gronwall's inequality that

$$\max\{1, |y_{1\varepsilon}(t) - y_{0\varepsilon}|\} \leqslant C_1 e^{Cd_\varepsilon}.$$

For $0 < s < 1/r$, this in turn implies that

$$|y_{1\varepsilon}(t) - y_{0\varepsilon}| \leqslant C_2 d_\varepsilon + C_3\big(\varepsilon^{1-rs} + d_\varepsilon e^{rCd_\varepsilon}\big) \to 0$$

as $\varepsilon \to 0$ and hence $y_{1\varepsilon}(b\varepsilon) \to \bar{y}(0)$. Thus the limiting solution has no jump at 0 and $y_{1\varepsilon}(t) \to \bar{y}(t)$ for $t > 0$ as $\varepsilon \to 0$, thereby proving (a).

Let $f$ be globally Lipschitz and $s = 1$. As mentioned, an estimate as above holds with $r = 1$ and so $|y_{1\varepsilon}(t) - y_{0\varepsilon}| \leqslant \bar{\beta}$ for some $\bar{\beta} > 0$ on $[-a\varepsilon, b\varepsilon]$. We shall prove that $y_{1\varepsilon}(t)$, which is the solution to

$${y_{1\varepsilon}}'(t) = f(t, y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t)), \quad y_{1\varepsilon}(-a\varepsilon) = y_{0\varepsilon},$$

and the solution $z_\varepsilon(t)$ to

$${z_\varepsilon}'(t) = f(t, \alpha\phi_\varepsilon(t)),\ z_\varepsilon(-a\varepsilon) = y_{0\varepsilon}$$

differ at most by $d_\varepsilon C(|y_{0\varepsilon}| + \bar{\beta})$ on $[-a\varepsilon, b\varepsilon]$, where $C$ is the Lipschitz constant for $f$ on $[-a, b]$, $y \in \mathbb{R}$. It holds that

$$\begin{aligned}
|y_{1\varepsilon}(t) - z_\varepsilon(t)| &\leqslant \int_{-a\varepsilon}^{t} \big|f\big(u, y_{1\varepsilon}(u) + \alpha\phi_\varepsilon(u)\big) - f\big(u, \alpha\phi_\varepsilon(u)\big)\big| du \\
&\leqslant \int_{-a\varepsilon}^{t} C|y_{1\varepsilon}(u)| du \leqslant d_\varepsilon C(|y_{0\varepsilon}| + \bar{\beta}).
\end{aligned}$$

In particular, $y_{1\varepsilon}(b\varepsilon) - z_\varepsilon(b\varepsilon) \to 0$ as $\varepsilon \to 0$.

In the case when the limits $M_\pm = \lim_{\eta\to 0,\, y\to\pm\infty} \frac{f(\eta,y)}{y}$ exist, we can compute the value of the limiting jump. Indeed,

$$\begin{aligned} z_\varepsilon(b\varepsilon) &= y_{0\varepsilon} + \int_{-a\varepsilon}^{b\varepsilon} f\big(u, \alpha\phi(u/\varepsilon)\varepsilon^{-1}\big)\, du \\ &= y_{0\varepsilon} + \varepsilon \int_{-a}^{b} f\big(\varepsilon u, \alpha\phi(u)\varepsilon^{-1}\big)\, du \\ &= y_{0\varepsilon} + \varepsilon \int_{[-a,b]\cap\{|\phi(u)|\leqslant\varepsilon\}} f\big(\varepsilon u, \alpha\phi(u)\varepsilon^{-1}\big)\, du \\ &\qquad + \int_{[-a,b]\cap\{|\phi(u)|>\varepsilon\}} \frac{f\big(\varepsilon u, \alpha\phi(u)\varepsilon^{-1}\big)}{\alpha\,\phi(u)\,\varepsilon^{-1}}\,\alpha\phi(u)\, du. \end{aligned}$$

The first integral is less than or equal to

$$\varepsilon(b-a) \sup_{t\in[-\varepsilon a,\varepsilon b],|y|\leqslant|\alpha|} |f(t,y)|$$

and converges to 0 as $\varepsilon \to 0$; the second integral converges to $\alpha(M_+ \int_{-a}^{b} \phi_+(u)\, du - M_- \int_{-a}^{b} \phi_-(u)\, du)$ by Lebesgue's dominated convergence theorem, if $\alpha > 0$, and similarly for $\alpha < 0$. □

Remark 1.1. If the mollifier $\phi$ is nonnegative, then the above jump does not depend on it, i.e., it equals $\alpha M_+$.

Example 1.1. Let $f(y) = y\sin(\log(1+y^2))$. Then $f$ is globally Lipschitz, but does not have limits $M_+$ and $M_-$ required in (b). We let $\alpha = 1$ and choose the mollifier $\phi \geqslant 0$ such that $\phi \equiv 1$ on an interval $I \subset [-a,b]$ and

$$\int_{[-a,b]\smallsetminus I} \phi(u)\, du = 1/4, \quad \text{length}(I) = 3/4.$$

Then

$$\begin{aligned} z_\varepsilon(b\varepsilon) &= \int_{-a\varepsilon}^{b\varepsilon} \frac{1}{\varepsilon}\phi\Big(\frac{u}{\varepsilon}\Big) \sin\Big(\log\Big(1 + \frac{1}{\varepsilon^2}\phi^2\Big(\frac{u}{\varepsilon}\Big)\Big)\Big) du \\ &= \int_{-a}^{b} \phi(u) \sin\Big(\log\Big(1+\frac{1}{\varepsilon^2}\phi^2(u)\Big)\Big) du \\ &= \frac{3}{4}\sin\Big(\log\Big(1+\frac{1}{\varepsilon^2}\Big)\Big) + \int_{[-a,b]\smallsetminus I} \phi(u)\sin\Big(\log\Big(1+\frac{1}{\varepsilon^2}\phi^2(u)\Big)\Big) du. \end{aligned}$$

We can choose a subsequence $\varepsilon_k \to 0$ so that $\sin\big(\log\big(1 + 1/\varepsilon_k^2\big)\big) = (-1)^k$. Since the second term is less then or equal to $1/4$ in absolute value, we see that the sequence $z_\varepsilon(b\varepsilon)$ is oscillating with a jump of height at least 1. So there does not exist a limiting solution on the right-hand side of $t = 0$.

The assumption that $f$ is globally Lipschitz is not necessary for the existence of a limit of the regularized solutions, as can be seen from the following example (even in the case $s = 1$).

EXAMPLE 1.2. The dissipative case $f(t,y) = -y^3$. We analyze the case $y_0 > 0$ only and assume here that $\phi \geqslant 0$. Obviously any solution to $y' = -y^3$ decreases for positive $y$ and increases when $y$ is negative since $f$ is decreasing with respect to $y$. By using the comparison theorem one can see that the solution $y_{1\varepsilon}$ to

$$y_{1\varepsilon}'(t) = f(t, y_{1\varepsilon}(t) + \phi_\varepsilon(t)), \quad y_{0\varepsilon} := \bar{y}(-a\varepsilon) > 0$$

is less or equal to the solution to

$$v'(t) = f(t, v + g_\varepsilon(t)), \quad v(-a\varepsilon) = y_{0\varepsilon},$$

where $g_\varepsilon(t) \leqslant \phi_\varepsilon(t)$ and

$$g_\varepsilon(t) = \begin{cases} 0, & t < -\bar{a}_\varepsilon \\ \bar{\xi}_\varepsilon, & t \in [-\bar{a}_\varepsilon, \bar{b}_\varepsilon] \\ 0, & t > \bar{b}_\varepsilon, \end{cases}$$

for some $0 < \bar{a}_\varepsilon \leqslant a\varepsilon$, $0 < \bar{b}_\varepsilon \leqslant b\varepsilon$ such that $\bar{b}_\varepsilon + \bar{a}_\varepsilon = \bar{\xi}_\varepsilon^{-1}/2$ with $\bar{\xi}_\varepsilon \to \infty$ as $\varepsilon \to 0$. This means that $y_{1\varepsilon}(t) \leqslant v(t)$, where

$$v'(t) = \begin{cases} -v^3, \ v(-a\varepsilon) = y_{0\varepsilon}, & t \in [-a\varepsilon, -\bar{a}_\varepsilon] \\ -(v + \bar{\xi}_\varepsilon)^3, \ v(-\bar{a}_\varepsilon) = \bar{y}_{0\varepsilon}, & t \in [-\bar{a}_\varepsilon, \bar{b}_\varepsilon] \\ -v^3, \ v(\bar{b}_\varepsilon) = \tilde{y}_{0\varepsilon}, & t \in [\bar{b}_\varepsilon, b\varepsilon]. \end{cases}$$

Observe that $y_{0\varepsilon} \approx \bar{y}_{0\varepsilon}$; here $\bar{y}_{0\varepsilon}$, $\tilde{y}_{0\varepsilon}$ are the terminal values of the solutions on the preceding intervals. The solution on the second interval is given by

$$v(t) = (\bar{y}_{0\varepsilon} + \bar{\xi}_\varepsilon)(1 + 2(t + \bar{a}_\varepsilon)(\bar{y}_{0\varepsilon} + \bar{\xi}_\varepsilon)^2)^{-1/2} - \bar{\xi}_\varepsilon.$$

This means that $v(\bar{b}_\varepsilon)$ behaves like $\sqrt{\bar{\xi}_\varepsilon} - \bar{\xi}_\varepsilon$, which goes to $-\infty$ as $\varepsilon \to 0$. (This is then true of $v(t)$ for $\bar{b}_\varepsilon \leqslant t \leqslant b\varepsilon$ as well.) The same is true for $y_{1\varepsilon}(b\varepsilon)$, i.e., $y_{1\varepsilon}(b\varepsilon) = \tilde{\xi}_\varepsilon \to -\infty$ as $\varepsilon \to 0$. The function $\phi_\varepsilon(t) = 0$ for $t \geqslant b\varepsilon$, and $y_{1\varepsilon}$ is the solution to

$$y_{1\varepsilon}'(t) = -{y_{1\varepsilon}}^3, \quad y_{1\varepsilon}(b\varepsilon) = \tilde{\xi}_\varepsilon,$$

that is,

$$y_{1\varepsilon}(t) = -(\tilde{\xi}_\varepsilon^{-2} + 2(t - b\varepsilon))^{-1/2}$$

and $y_{1\varepsilon}(t) \to -(2t)^{-1/2}$, $t > 0$. That gives that the solution to

$$y_\varepsilon'(t) = -y_\varepsilon^3(t) + \phi_\varepsilon'(t)$$

converges to $\bar{y}_1(t) + \delta(t)$ where

$$\bar{y}_1(t) = \begin{cases} y_0(1 + 2(t+1)y_0^2)^{-1/2}, & t < 0 \\ -(2t)^{-1/2}, & t > 0. \end{cases}$$

It is not true that if $y' = f(t,y)$, $y(-1) = y_0$ has a unique global solution in $[-1, \infty)$, then the solutions to $y_\varepsilon' = f(t, y_\varepsilon) + \phi_\varepsilon''$ with the same initial data have a limiting function. This can be seen from the following example.

Example 1.3. The solutions to

$$y_\varepsilon'(t) = (1 + y_\varepsilon^2)^{1/2} + \phi_\varepsilon''(t), \quad y(-1) = y_0$$

do not converge to any function defined for $t > 0$ as $\varepsilon \to 0$; in fact, they diverge to $\infty$, uniformly on every interval $[t_0, \infty)$, $t_0 > 0$. Indeed, the solution to

$$y_{1\varepsilon}{}'(t) = (1 + (y_{1\varepsilon} + \phi_\varepsilon{}')^2)^{1/2}, \quad y_{1\varepsilon}(-1) = y_0$$

is increasing and its derivative is greater than $C\varepsilon^{-2}$ on an interval of length $\mathcal{O}(\varepsilon)$, on which $\phi_\varepsilon'$ is strictly different from 0. That means that the value of the function at $t = b\varepsilon$ is greater or equal to a constant times $\varepsilon^{-1}$. After this point the function $y_{1\varepsilon}$ continues to increase, and that implies blow up to $\infty$, uniformly on each interval $[t_0, \infty)$ bounded away from zero.

Remark 1.2. Theorem 1.1 can be easily extended to the case of $(n \times n)$-systems, where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$. One just has to replace the absolute value in the estimates by the norm $\|y\| = \max\{|y_1|, \dots, |y_n|\}$ in $\mathbb{R}^n$. The function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is sublinear of order $r$ with respect to $y$ if there exists a constant $C$ such that $\|f(t, y)\| \leqslant C(1 + \|y\|^r)$. With these modifications, part (a) of Theorem 1.1 remains valid. As for part (b), suitable limiting behavior on $f$ has to be imposed as the components $y_i$ tend to infinity. Various versions of part (b) can be obtained in this way; we leave out the details.

## 2. The multiplicative case

This section is devoted to the equation

$$y'(t) = f(t, y(t)) + g(y(t))\delta(t), \quad y(-1) = y_0$$

where $f$ satisfies the same general assumptions as in Section 1 and $g \in C^1(\mathbb{R})$.

Proposition 2.1. *Let $f$ and $g$ be globally Lipschitz (uniformly on compact time intervals) and assume that $G(y) = \int dy/g(y)$ is invertible. Then the solutions to*

$$y_\varepsilon{}'(t) = f(t, y_\varepsilon(t)) + g(y_\varepsilon(t))\phi_\varepsilon(t), \quad y_\varepsilon(-1) = y_0 \tag{2.1}$$

*converge to the limiting function*

$$\bar{y}(t) = \begin{cases} \bar{y}_1(t), & t \leqslant 0 \\ \bar{y}_2(t), & t > 0 \end{cases}$$

*where $\bar{y}_1$ is the solution to*

$$\bar{y}_1'(t) = f(t, \bar{y}_1(t)), \quad \bar{y}_1(-1) = y_0$$

*and $\bar{y}_2$ is the solution to*

$$\bar{y}_2'(t) = f(t, \bar{y}_2(t)), \quad \bar{y}_2(0) = G^{-1}(G(\bar{y}_1(0)) + 1).$$

Proof. We have only to check what is going on for $t \in [-a\varepsilon, b\varepsilon]$. Let $y_{0\varepsilon} := y_\varepsilon(-a\varepsilon) = \bar{y}_1(-a\varepsilon)$. Then

$$y_\varepsilon(t) = y_{0\varepsilon} + \int_{-a\varepsilon}^{t} \big(f(s, y_\varepsilon(s)) + g(y_\varepsilon(s))\,\phi_\varepsilon(s)\big)ds,$$

$$|y_\varepsilon(t)| \leqslant |y_{0\varepsilon}| + \int_{-a\varepsilon}^{t} \big(L_f + L_g|\phi_\varepsilon(s)|\big)|y_\varepsilon(s)|ds + \int_{-a\varepsilon}^{t} \big(|f(s,0)| + |g(0)||\phi_\varepsilon(s)|\big)ds,$$

where $L_f$ and $L_g$ are the Lipschitz constants for $f$ and $g$, respectively. Letting $C_\phi = \int_{-a}^{b} |\phi(t)|dt$, Gronwall's inequality gives

$$|y_\varepsilon(t)| \leqslant (|y_{0\varepsilon}| + d_\varepsilon C_f + C_g C_\phi)\exp(d_\varepsilon L_f + L_g C_\phi) \leqslant C < \infty,$$

i.e., $|y_\varepsilon(t)|$ remains bounded for $t \in [-a\varepsilon, b\varepsilon]$. (Here $C_f$ and $C_g$ are bounds on $|f|$ and $|g|$ in the appropriate regions.) Using this fact we will prove that the solution $z_\varepsilon(t)$ to

$$z_\varepsilon{}'(t) = g(z_\varepsilon(t))\phi_\varepsilon(t), \quad z_\varepsilon(-a\varepsilon) = y_{0\varepsilon} \tag{2.2}$$

satisfies

$$\sup_{t\in[-a\varepsilon, b\varepsilon]} |y_\varepsilon(t) - z_\varepsilon(t)| \to 0,\ \varepsilon \to 0.$$

In particular, this will be true for $t = b\varepsilon$. Indeed,

$$|y_\varepsilon(t) - z_\varepsilon(t)| \leqslant \int_{-a\varepsilon}^{t} |f(s, y_\varepsilon(s))|ds + \int_{-a\varepsilon}^{t} L_g|y_\varepsilon(s) - z_\varepsilon(s)||\phi_\varepsilon(s)|ds.$$

By Gronwall's inequality,

$$|y_\varepsilon(t) - z_\varepsilon(t)| \leqslant \int_{-a\varepsilon}^{t} |f(s, y_\varepsilon(s))|ds\ \exp(L_g C_\phi) \leqslant d_\varepsilon(C_f + CL_f)\exp(L_g C_\phi) \to 0$$

as $\varepsilon \to 0$. This means that, in the limit, the jump of $y_\varepsilon$ is equal to the jump of $z_\varepsilon$ at zero. But the latter can be easily found. By integrating (2.2) we obtain

$$\int_{z_\varepsilon(-a\varepsilon)}^{z_\varepsilon(b\varepsilon)} g(z)^{-1}dz = \int_{-a\varepsilon}^{b\varepsilon} \phi_\varepsilon(t)\,dt = 1$$

and $G(z_\varepsilon(b\varepsilon)) = G(z_\varepsilon(-a\varepsilon)) + 1$. This gives the desired result. □

The next result shows that the conditions on $f$ can be relaxed to the local Lipschitz case, provided that $g$ has a sufficiently small (global) Lipschitz constant. As in Section 1, we require again that the free equation $y'(t) = f(t, y(t))$ is uniquely solvable on the whole interval $[-1, T]$ for whatever data $y(-1) = y_0 \in \mathbb{R}$, and is also uniquely solvable on $[0, T]$ for arbitrary data $y(0) = y_1$.

Proposition 2.2. *Let $f$ be locally Lipschitz, uniformly on compact time intervals, let $g$ be globally Lipschitz with constant $L_g$. Set $C_\phi = \int_{-a}^{b} |\phi(t)|dt$ and assume that $1/g$ has an invertible primitive $G$. If $L_g C_\phi < 1$, then the assertions of Proposition 2.1 remain valid, at least on the interval $[-1, T]$.*

PROOF. We keep the notations from Proposition 2.1; in particular, we let $y_{0\varepsilon} = \bar{y}_1(-a\varepsilon)$. Choose $\eta$ such that $\eta > |g(\bar{y}_1(0))|C_\phi/(1 - L_g C_\phi)$. Let $\mathcal{B}_\varepsilon$ be the set of continuous functions on $[-a\varepsilon, b\varepsilon]$ such that

$$\sup_{t\in[-a\varepsilon,b\varepsilon]} |u(t) - y_{0\varepsilon}| \leqslant \eta,$$

and define the operator $M$ on $\mathcal{B}_\varepsilon$ by

$$Mu(t) = y_{0\varepsilon} + \int_{-a\varepsilon}^{t} \big(f(s, u(s)) + g(u(s))\phi_\varepsilon(s)\big)ds.$$

We are going to show that $M$ is a contraction on $\mathcal{B}_\varepsilon$ for all sufficiently small $\varepsilon$. First, if $u \in \mathcal{B}_\varepsilon$, then

$$\begin{aligned} |Mu(t) - y_{0\varepsilon}| &\leqslant \left|\int_{-a\varepsilon}^{t} (f(s, u(s)) + g(u(s))\phi_\varepsilon(s))\, ds\right| \\ &\leqslant d_\varepsilon C_f + L_g C_\phi |y_{0\varepsilon} - \bar{y}_1(0)| + L_g C_\phi \eta + |g(\bar{y}_1(0))|C_\phi \leqslant \eta \end{aligned}$$

for sufficiently small $\varepsilon$. Here the constant $C_f$ denotes the maximum of $|f|$ on $[-a\varepsilon_0, b\varepsilon_0] \times [y_{0\varepsilon} - \eta, y_{0\varepsilon} + \eta]$ for some fixed $\varepsilon_0$ and $\varepsilon \leqslant \varepsilon_0$. We have used the decomposition

$$g(u(s)) = g(u(s)) - g(y_{0\varepsilon}) + g(y_{0\varepsilon}) - g(\bar{y}_1(0)) + g(\bar{y}_1(0))$$

and that $L_g C_\phi \eta + |g(\bar{y}_1(0))|C_\phi < \eta$ by definition. Next, if $u, v \in \mathcal{B}_\varepsilon$, then

$$\begin{aligned} |Mu(t) - Mv(t)| &\leqslant \left|\int_{-a\varepsilon}^{t} \big(f(s, u(s)) - f(s, v(s))\big)ds\right| \\ &\quad + \left|\int_{-a\varepsilon}^{t} \big(g(u(s)) - g(v(s))\big)\phi_\varepsilon(s)\, ds\right| \\ &\leqslant (d_\varepsilon L_f + L_g C_\phi) \sup_{s\in[-a\varepsilon,b\varepsilon]} |u(s) - v(s)|. \end{aligned}$$

By assumption, the constant on the right-hand side is less than 1 for sufficiently small $\varepsilon$, thus $M$ is a contraction on $\mathcal{B}_\varepsilon$.

But the solution $y_\varepsilon(t)$ of equation (2.1) is the unique fixed point. This shows that $y_\varepsilon(t)$ is bounded by $\eta$ on $[-a\varepsilon, b\varepsilon]$. The rest of the proof is the same as in Proposition 2.1. □

Let us remark that if $\phi$ is nonnegative, then $C_\phi = 1$ and the condition of Proposition 2.2 reduces to $L_g < 1$.

EXAMPLE 2.1. The case $s = 0$ in Section 1 can be seen as the special case of Proposition 2.2, where $g(y) \equiv \alpha$. It follows that under the assumptions on $f$ above, the family of regularized solutions $y_\varepsilon$ to

$$y_\varepsilon{}'(t) = f(t, y_\varepsilon(t)) + \alpha\phi_\varepsilon, \quad y_\varepsilon(-1) = y_0$$

converges to

$$\bar{y}(t) = \begin{cases} \bar{y}_1(t), & t \leqslant 0 \\ \bar{y}_2(t), & 0 < t \leqslant T \end{cases}$$

where $\bar{y}_1$ is the solution to $y' = f(t,y)$, $y(-1) = y_0$ and $\bar{y}_2$ is the solution to $y' = f(t,y)$, $y(0) = y_0 + \alpha$.

We note that here the limiting function is also a distributional solution; this cannot be asserted in the situation of Theorem 1.1 in general.

Remark 2.1. For completeness, we remark that if the singularity is in the same point where the initial condition is given, then the solution depends on the regularization as a rule. This can be seen from simplest linear equations. In fact, the solutions to

$$y'_\varepsilon(t) = y_\varepsilon(t)\phi_\varepsilon(t), \quad y_\varepsilon(0) = y_0$$

are given by and converge to

$$y_\varepsilon(t) = y_0 \exp\left(\int_0^{t/\varepsilon} \phi(s)\,ds\right) \to y_0 \exp\left(\int_0^{\infty} \phi(s)\,ds\right)$$

for $t > 0$. Depending on the support of $\phi$, the integral $\int_0^\infty \phi(s)\,ds$ can have any real value (any value between 0 and 1 for non-negative $\phi$). The same holds in the additive case

$$y'_\varepsilon(t) = y_\varepsilon(t) + \phi_\varepsilon(t), \quad y_\varepsilon(0) = y_0$$

with

$$y_\varepsilon(t) = y_0 e^t + \int_0^{t/\varepsilon} e^{t-\varepsilon s}\phi(s)\,ds \to e^t\left(y_0 + \int_0^{\infty} \phi(s)\,ds\right).$$

**Acknowledgements.** We are grateful to Uladzimir Hrusheuski for carefully reading the manuscript and making a number of valuable suggestions for improvement.

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(Received 29 08 2011)
(Reivised 13 12 2011)

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