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WARPED PRODUCT CR-SUBMANIFOLDS OF LORENTZIAN β-KENMOTSU MANIFOLDS

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Abstract. We study the geometry of warped product submanifolds of Lorentzian β-Kenmotsu manifolds. We obtain a characterization result for CR-warped products.

1. Introduction

The notion of warped product manifolds was introduced by Bishop and O'Neill [2] and then it was studied by many mathematicians and physicists. These manifolds are the generalization of the Riemannian product manifolds. Chen [4] has studied the geometry of warped product submanifolds in Kaehler manifold and showed that the warped product submanifold of the type $N_\perp \times_f N_\top$ is trivial. Later, many research articles have recently appeared exploring the existence or nonexistence of warped product submanifolds in known spaces [1, 6, 9].

Matsumoto [8] introduced the notion of Lorentzian almost paracontact metric manifolds. Later on, many geometers studied submanifolds of Lorentzian almost paracontact manifolds [5, 10].

As Kenmotsu manifolds are themselves warped product spaces, it is interesting to study warped product submanifolds in Kenmotsu manifolds. In this paper we consider the warped product submanifolds of the types $M = N_\perp \times_f N_\top$ and $M = N_\top \times_f N_\perp$ in an arbitrary Lorentzian β-Kenmotsu manifold $\bar{M}$, where $N_\top$ and $N_\perp$ are the invariant and anti-invariant submanifolds of $\bar{M}$, respectively.

2. Preliminaries

Let $\bar{M}$ be a $(2n+1)$-dimensional manifold of class C^∞ endowed with an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ, which is called the structure vector field, a 1-form η and a Lorentzian metric g with signature $(-, +, \cdots, +)$ satisfying:

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$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \tag{2.2}$$

for any vector fields X, Y on $\bar{M}$. Then such a structure (ϕ, ξ, η, g) is termed as *Lorentzian paracontact structure* and the manifold $\bar{M}$ with a Lorentzian paracontact structure is called a *Lorentzian paracontact manifold* [8].

Our purpose is to define the warped product submanifolds of a Lorentzian β-Kenmotsu manifold, that is a manifold with a paracontact structure and a compatible Lorentzian metric g, $(\bar{M}^{2n+1}, \phi, \xi, \eta, g)$ satisfying (2.1) and (2.2) with the following additional condition:

$$(\bar{\nabla}_X \phi)Y = \beta\{g(\phi X, Y)\xi + \eta(Y)\phi X\}, \tag{2.3}$$

for any $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the Levi–Civita connection with respect to the Lorentzian metric g. Thus, a Lorentzian paracontact manifold satisfying (2.3) is called a *Lorentzian β-Kenmotsu* manifold [10]. From (2.3), it is easy to obtain that

$$\bar{\nabla}_X \xi = \beta\{X + \eta(X)\xi\} \tag{2.4}$$

Now, let M be a submanifold of $\bar{M}$. Let TM be the Lie algebra of vector fields in M and $T^\perp M$ the set of all vector fields normal to M. If ∇ is the Levi–Civita connection on M, then Gauss–Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{2.6}$$

for any $X, Y \in TM$ and any $N \in T^\perp M$, where $\nabla^\perp$ is the induced connection in the normal bundle, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A are related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.7}$$

where g denotes the metric on $\bar{M}$ as well as the induced metric on M [11].

For any $X \in TM$, we write

$$\phi X = PX + FX \tag{2.8}$$

where PX is the tangential component of ϕX and FX is the normal component of ϕX, respectively. Similarly, for any vector field N normal to M, we put

$$\phi N = BN + CN \tag{2.9}$$

where BN and CN are tangential and normal components of ϕN, respectively. The covariant derivatives of the tensor fields P and F are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^\perp_X FY - F\nabla_X Y \tag{2.11}$$

for all $X, Y \in TM$.

A submanifold M, of a Lorentzian β-Kenmotsu manifold $\bar{M}$ is called *CR-submanifold* if it admits a differentiable invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant i.e., $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$ with $\phi(D_p) \subseteq D_p$ and $\phi(D_p^{\perp}) \subset T_p^{\perp} M$, for every $p \in M$. A CR-submanifold is known to be invariant, anti-invariant and proper if $D^{\perp} = 0$, $D = 0$ and $D \neq 0 \neq D^{\perp}$, respectively.

Note that ξ is time like vector field and all vector field in $D \oplus D^{\perp}$ are space like.

Let M be an m-dimensional CR-submanifold of $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold $(\bar{M}^{2n+1}, \phi, \xi, \eta, g)$. Then, $F(T_pM)$ is a subspace of $T_p^{\perp} M$. Then for every $p \in M$, there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$T_p\bar{M} = T_pM \oplus F(T_pM) \oplus \mu_p.$$

3. Warped Product Submanifolds

The study of warped product manifolds was initiated by Bishop and O'Neill [2]. They defined these manifolds as follows:

Definition 3.1. Let (B, g_1) and (F, g_2) be two semi-Riemannian manifolds with metric g_1 and g_2 respectively and f a positive differentiable function on B. The warped product of B and F is the manifold $B \times_f F = (B \times F, g)$, where

$$g = g_1 + f^2 g_2.$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $B \times F$ onto B and F, respectively.

A warped product manifold $B \times_f F$ is said to be *trivial* if the warping function f is constant. We recall that on a warped product manifold, one has

$$\nabla_U V = \nabla_V U = (U \ln f)V \tag{3.1}$$

for any vector fields U tangent to B and V tangent to F [2].

Throughout the paper, we denote by $N_{\top}$ and $N_{\perp}$, the invariant and anti-invariant submanifolds of a Lorentzian β-Kenmotsu manifold $\bar{M}$, respectively. Then their warped product CR-submanifolds are one of the following forms:

$$\text{(i)}\ \ M = N_{\perp} \times_f N_{\top}, \quad \text{(ii)}\ \ M = N_{\top} \times_f N_{\perp}.$$

For case (i), when $\xi \in TN_{\top}$, we have the following theorem.

Theorem 3.1. *There does not exist a warped product CR-submanifold $M = N_{\perp} \times_f N_{\top}$ of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that $N_{\top}$ is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti-invariant submanifold of $\bar{M}$.*

Proof. Let $M = N_{\perp} \times_f N_{\top}$ be a warped product CR-submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that $N_{\top}$ is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. Then by equation (3.1), we get

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X$$

for any vector fields Z and X tangent to $N_\perp$ and $N_\top$, respectively.

In particular,

$$\nabla_Z \xi = (Z \ln f)\xi, \tag{3.2}$$

whereas by (2.4), (2.5) and the fact that ξ is tangent to $N_\top$, we have

$$\nabla_Z \xi = \beta Z, \quad h(Z, \xi) = 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that $Z \ln f = 0$, for all $Z \in TN_\perp$ i.e., f is constant for all $Z \in TN_\perp$. This completes the proof. $\square$

Now, the other case, when ξ tangent to $N_\perp$ is dealt in the following two results.

LEMMA 3.1. *Let $M = N_\perp \times_f N_\top$ be a warped product CR-submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that ξ is tangent to $N_\perp$. Then*

$$\text{(i)}\ \xi \ln f = \beta, \quad \text{(ii)}\ g(h(X, \phi X), FZ) = -\{\beta\eta(Z) + Z \ln f\}\|X\|^2,$$

for any $X \in TN_\top$ and $Z \in TN_\perp$.

PROOF. If $\xi \in TN_\perp$ then for any $X \in TN_\top$, we have

$$\nabla_X \xi = (\xi \ln f)X. \tag{3.4}$$

On the other hand, from (2.4) and the fact that ξ is tangent to $N_\perp$, we have $\bar{\nabla}_X \xi = \beta X$. Using (2.5), we obtain

$$\nabla_X \xi = \beta X, \quad h(X, \xi) = 0. \tag{3.5}$$

By equations (3.4) and (3.5), it follows that $\xi \ln f = \beta$. Now, for any $X \in TN_\top$ and $Z \in TN_\perp$, we have $(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z$. Using (2.3), (2.6), (2.8), (2.9) and by the orthogonality of two distributions, we derive

$$\beta\eta(Z)\phi X = -A_{FZ}X + \nabla^\perp_X FZ - P\nabla_X Z - F\nabla_X Z - Bh(X, Z) - Ch(X, Z).$$

Equating the tangential components, we get

$$-\beta\eta(Z)\phi X = A_{FZ}X + P\nabla_X Z + Bh(X, Z).$$

Taking the product with ϕX and using (2.2) and (3.1), we derive

$$\begin{aligned} -\beta\eta(Z)\|X\|^2 &= g(A_{FZ}X, \phi X) + (Z \ln f)g(PX, \phi X) + g(Bh(X, Z), \phi X) \\ &= g(h(X, \phi X), FZ) + (Z \ln f)g(\phi X, \phi X) + g(\phi h(X, Z), \phi X). \end{aligned}$$

Then from (2.2), we obtain

$$g(h(X, \phi X), FZ) = -\{\beta\eta(Z) + Z \ln f\}\|X\|^2. \tag{3.6}$$

$\square$

THEOREM 3.2. *Let $M = N_\perp \times_f N_\top$ be a warped product CR-submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that ξ is tangent to $N_\perp$. If $h(X, \phi X) \in \mu$ the invariant normal subbundle of M, then $Z \ln f = -\beta\eta(Z)$ for all $X \in TN_\top$ and $Z \in TN_\perp$.*

PROOF. The assertion follows from formula (3.6) by using the given fact. $\square$

For the warped product of the type $N_\top \times_f N_\perp$, we have the following theorem.

THEOREM 3.3. *There does not exist a warped product CR-submanifold $M = N_\top \times_f N_\perp$ of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that ξ is tangent to $N_\perp$.*

PROOF. As $\xi \in TN_\perp$, then by formula (3.1), we have

$$\nabla_X \xi = (X \ln f)\xi, \tag{3.7}$$

for any $X \in TN_\top$. Whereas from (2.4), (2.5) and the fact that $\xi \in TN_\perp$, we have

$$\nabla_X \xi = \beta X, \quad h(X, \xi) = 0 \tag{3.8}$$

From (3.7) and (3.8), it follows that $X \ln f = 0$, for all $X \in TN_\top$, and this means that f is constant on $N_\top$. This proves the theorem. □

The remaing case, when $\xi \in TN_\top$ is dealt with the following two theorems.

THEOREM 3.4. *Let $M = N_\top \times_f N_\perp$ be a warped product CR-submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ such that ξ is tangent to $N_\top$. Then $(\bar{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_\top$ and $Z \in TN_\perp$.*

PROOF. For any $X \in TN_\top$ and $Z \in TN_\perp$, we have

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X Z, Z) = g(\nabla_X Z, Z).$$

Using (3.1), we get

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2. \tag{3.9}$$

On the other hand, for any $X \in TN_\top$ and $Z \in TN_\perp$, we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi\bar{\nabla}_X Z.$$

Using (2.3) and the fact that ξ is tangent to $N_\top$, the left-hand side of the above equation is identically zero, then we get

$$\phi\bar{\nabla}_X Z = \bar{\nabla}_X \phi Z. \tag{3.10}$$

Taking the product with ϕZ in (3.10) and making use of formula (2.6), we obtain

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\nabla^\perp_X FZ, FZ).$$

Then from (2.11), we derive $g(\phi\bar{\nabla}_X Z, \phi Z) = g((\bar{\nabla}_X F)Z, FZ) + g(F\nabla_X Z, FZ)$. Using (3.1), we get

$$\begin{aligned} g(\phi\bar{\nabla}_X Z, \phi Z) &= (X \ln f)g(FZ, FZ) + g((\bar{\nabla}_X F)Z, FZ) \\ &= (X \ln f)g(\phi Z, \phi Z) + g((\bar{\nabla}_X F)Z, FZ). \end{aligned}$$

Therefore by (2.2), we obtain

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2 + g((\bar{\nabla}_X F)Z, FZ). \tag{3.11}$$

Thus (3.9) and (3.11) imply

$$g((\bar{\nabla}_X F)Z, FZ) = 0. \tag{3.12}$$

Also, as $N_\top$ is an invariant submanifold then $\phi W \in TN_\top$ for any $W \in TN_\top$, thus on using (2.11) and the fact that the product of tangential component with normal is zero, we obtain

$$g((\bar{\nabla}_X F)Z, \phi W) = 0. \tag{3.13}$$

Hence from (3.12) and (3.13), it follows that $(\bar{\nabla}_X F)Z \in \mu$, for all $X \in TN_\top$ and $Z \in TN_\perp$. Thus, the proof is complete. □

THEOREM 3.5. *A proper CR-submanifold M of a Lorentzian β-Kenmotsu manifold $\bar{M}$ is locally a CR-warped product if and only if the shape operator of M satisfies*

$$A_{\phi Z}X = (\phi X \mu)Z,\ \ X \in D \oplus \langle\xi\rangle,\ \ Z \in D^\perp \tag{3.14}$$

for some function μ on M satisfying $V(\mu) = 0$, for each $V \in D^\perp$.

PROOF. Let $M = N_\top \times_f N_\perp$ be a CR-warped product submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ with $\xi \in TN_\top$, then for any $X \in TN_\top$ and $Z, W \in TN_\perp$, we have

$$\begin{aligned} g(A_{\phi Z}X, W) = g(h(X,W), \phi Z) &= g(\bar{\nabla}_W X, \phi Z) = g(\phi\bar{\nabla}_W X, Z) \\ &= g(\bar{\nabla}_W \phi X, Z) - g((\bar{\nabla}_W \phi)X, Z). \end{aligned}$$

Using (2.3), (3.1) and the fact that ξ is tangent to $N_\top$, the above equation yields

$$g(A_{\phi Z}X, W) = (\phi X \ln f) g(Z, W). \tag{3.15}$$

Further, we have $g(h(X,Y), FZ) = g(\bar{\nabla}_X Y, \phi Z) = g(\phi\bar{\nabla}_X Y, Z) = -g(\phi Y, \bar{\nabla}_X Z)$, for each $X, Y \in TN_\top$ and $Z \in TN_\perp$. Using (3.1), we obtain $g(h(X,Y), FZ) = 0$. Taking into account this fact in (3.15), we obtain (3.14).

Conversely, suppose that M is a proper CR-submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$ satisfying (3.14), then for any $X, Y \in D \oplus \langle\xi\rangle$,

$$g(h(X,Y), \phi Z) = g(A_{\phi Z}X, Y) = 0.$$

This implies that $g(\bar{\nabla}_X \phi Y, Z) = 0$, that is, $g(\nabla_X Y, Z) = 0$. This means $D \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. Now, for any $Z, W \in D^\perp$ and $X \in D \oplus \langle\xi\rangle$, we have

$$\begin{aligned} g(\nabla_Z W, \phi X) = g(\bar{\nabla}_Z W, \phi X) &= g(\phi\bar{\nabla}_Z W, X) \\ &= g(\bar{\nabla}_Z \phi W, X) - g((\bar{\nabla}_Z \phi)W, X). \end{aligned}$$

Then, using (2.3) and (2.6), we obtain $g(\nabla_Z W, \phi X) = -g(A_{\phi W}Z, X)$. Thus from (2.7), we arrive at $g(\nabla_Z W, \phi X) = -g(h(Z,X), \phi W)$. Again using (2.7) and (3.14), we obtain

$$g(\nabla_Z W, \phi X) = -g(A_{\phi W}X, Z) = -(\phi X \mu) g(Z, W). \tag{3.16}$$

Let $N_\perp$ be a leaf of $D^\perp$ and $h^\perp$ be the second fundamental form of the immersion of $N_\perp$ into M. Then for any $Z,\ W \in D^\perp$, we have

$$g(h^\perp(Z,W), \phi X) = g(\nabla_Z W, \phi X). \tag{3.17}$$

Hence, from (3.16) and (3.17), we conclude that

$$g(h^{\perp}(Z,W), \phi X) = -(\phi X\mu)g(Z,W).$$

This means that integral manifold $N_{\perp}$ of $D^{\perp}$ is totally umbilical in M. Since $V(\mu) = 0$ for each $V \in D^{\perp}$, which implies that the integral manifold of $\mathcal{D}^{\perp}$ is an extrinsic sphere in M, this means that the curvature vector field is nonzero and parallel along $N_{\perp}$. Hence by virtue of a result in [7], M is locally a warped product $N_{\top} \times_f N_{\perp}$, where $N_{\top}$ and $N_{\perp}$ denote the integral manifolds of the distributions $D \oplus \langle \xi \rangle$ and $D^{\perp}$, respectively and f is the warping function. $\square$

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