PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 93 (107) (2013), 29–47 DOI: 10.2298/PIM1307029K

QUADRATIC LEVEL QUASIGROUP EQUATIONS WITH FOUR VARIABLES II: THE LATTICE OF VARIETIES

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Communicated by Siniša Crvenković

Abstract. We consider a class of quasigroup identities (with one operation symbol) of the form $x_1x_2 \cdot x_3x_4 = x_5x_6 \cdot x_7x_8$ and with $x_i \in \{x, y, u, v\}$ $(1 \leqslant i \leqslant 8)$ with each of x, y, u, v occurring exactly twice in the identity. There are 105 such identities. They generate 26 quasigroup varieties. The lattice of these varieties is given.

1. Introduction

In the previous paper Krapež [**2**] we defined the quadratic level quasigroup equations with four variables. They are quadratic equations of the form:

$$x_1x_2 \cdot x_3x_4 = x_5x_6 \cdot x_7x_8 \tag{L2}$$

where $x_i \in \{x, y, u, v\}$ $(1 \leqslant i \leqslant 8)$. The operation $\cdot$ is assumed to be a quasigroup. No division operation occurrs in the equation $(L2)$. There are 105 such equations. The complete list is given in [**2**, equations (4.1)–(4.105)] where all definitions of undefined notions and further references can be found. The general solutions of these 105 equations are given in [**2**]. Since quasigroups are defined as models of identities (in the language $\{\cdot, \backslash, /\}$) and equations are also identities, the sets of solutions to above equations are quasigroup varieties. The equations combine into 19 classes of equivalent equations resulting in 19 quasigroup varieties:

$$x = x \qquad \text{(Quasigroups)} \tag{Q}$$

$$xy = yx \qquad \text{(Commutative quasigroups)} \tag{C}$$

$$xy \cdot uv = vu \cdot yx \qquad \text{(4–palindromic quasigroups)} \tag{B11}$$

2010 *Mathematics Subject Classification:* Primary 20N05; Secondary 08B15, 39B52.

Key words and phrases: quasigroup, quasigroup functional equation, quadratic level quasigroup equation, quasigroup identity, quasigroup variety, lattice of varieties.

Supported by Ministry of Education, Science and Technological Development of Serbia through projects ON 174008 and ON 174026.

(U1)	$xy \cdot yx = e$	(Skew symmetric quasigroups)
(U)	$xx = e$	(Unipotent quasigroups)
(Ub0)	$(U), (b0)$	(Unipotent b0-quasigroups)
(Ub1)	$(U), (b1)$	(Unipotent b1-quasigroups)
(CU)	$(U), (C)$	(Commutative unipotent quasigroups)
(LLU)	$(U), (LL)$	(Unipotent left linear quasigroups)
(RLU)	$(U), (RL)$	(Unipotent right linear quasigroups)
(M)	$xy \cdot uv = xu \cdot yv$	(Medial quasigroups)
(P)	$xy \cdot uv = vy \cdot ux$	(Paramedial quasigroups)
(T1)	$(C), (M)$	(Commutative medial quasigroups)
(D1)	$xy \cdot xu = uv \cdot yv$	
(I)	$xy \cdot yu = xv \cdot vu$	(Intermedial quasigroups)
(E)	$xy \cdot ux = vy \cdot uv$	(Extramedial quasigroups)
(ME)	$xy \cdot ux = vu \cdot yv$	
(PI)	$xy \cdot yu = uv \cdot vx$	
(BT1)	$xy \cdot xu = yv \cdot uv$	

We assume:

(b0) $$ex = xe$$

(b1) $$e \cdot xy = yx \cdot e$$

(LL) $$x(u\backslash y) \cdot z = x(u\backslash u) \cdot (u\backslash yz)$$

(RL) $$x \cdot (y/u)z = (xy/u) \cdot (u/u)z$$

Seven more varieties are defined by the systems of two identities (see Table 1).

$U1B11$	$(U1)$, $(B11)$
$UB11$	(U), $(B11)$
$T11$	(M), (P)
$D11$	(E), (I)
BM	(M), (I)
BP	(P), (E)
$BT11$	(M), (PI)

Table 1. Varieties of quasigroups (two identities)

None of the systems is equivalent to just one identity with four variables. However, every one of the systems is equivalent to a single quadratic identity with eight variables. For example $(T11)$ is equivalent to $(xy \cdot uv)(pq \cdot rs) = (xu \cdot yv)(sq \cdot rp)$.

We get more information on some of these varieties by looking at general solutions of the given (systems of) equations. By [**2**, Theorems 9,10]:

THEOREM 1.1. *Quasigroups* $(S; \cdot, \backslash, /)$ *which belong to the variety LLU (RLU) are representable as*

(LLU) $$xy = Ax - Ay + c$$

(RLU) $$xy = c - Ax + Ay$$

where $+$ *is an arbitrary group on* S*,* A *is an automorphism of* $+$ *and* c *is any element of* S*.*

Similarly, the quasigroups from varieties in Table 2 are linear over an Abelian group $+$ (i.e., $xy = Ax + By + c$; A, B–automorphisms of $+$) and satisfy further conditions depending on the particular variety (see Table 2).

variety	identities	conditions on $+$	conditions on A, B
M	(M)	Abelian group	$AB = BA$
P	(P)	Abelian group	$A^2 = B^2$
E	(E)	Abelian group	$A^2 + B^2 = O$
I	(I)	Abelian group	$AB + BA = O$
ME	(ME)	Abelian group	$AB = BA,\ A^2 + B^2 = O$
PI	(PI)	Abelian group	$A^2 = B^2,\ AB + BA = O$
T11	(M), (P)	Abelian group	$AB = BA,\ A^2 = B^2$
D11	(E), (I)	Abelian group	$A^2 + B^2 = O,\ AB + BA = O$
T1	$(T1)$	Abelian group	$A = B$
D1	$(D1)$	Abelian group	$A + B = O$
BM	(M), (I)	Boolean group	$AB = BA$
BP	(P), (E)	Boolean group	$A^2 = B^2$
BT11	(M), (PI)	Boolean group	$A^2 = B^2,\ AB = BA$
BT1	$(BT1)$	Boolean group	$A = B$

TABLE 2. Representation of quasigroups from T–quasigroup varieties in Q_4

The data from Table 2, suggest relationship between varieties of Abelian group isotopes given in Figure 1 (with variety Q added).

In such graphs it is customary that two nodes V and W (W above V) connected by the line represent a relationship $V \lhd W$ of W being immediately above V. The relation $<$ is the transitive closure of $\lhd$. No connection between V and W means that V and W are incomparable (denoted $V \| W$). We informally say that the graph in Figure 1 is valid in the *strong sense*. At the moment we are far from proving such strong relationship between nodes of the graph in Figure 1. All we can say now is that $V \subseteq W$ for V and W connected by a line (with W above V), while not having a line connecting V and W does not necessarily mean $V \| W$. Therefore the graph in Figure 1 is valid in the *weak sense* only. Assumption is similar for Figures 2 and 3.

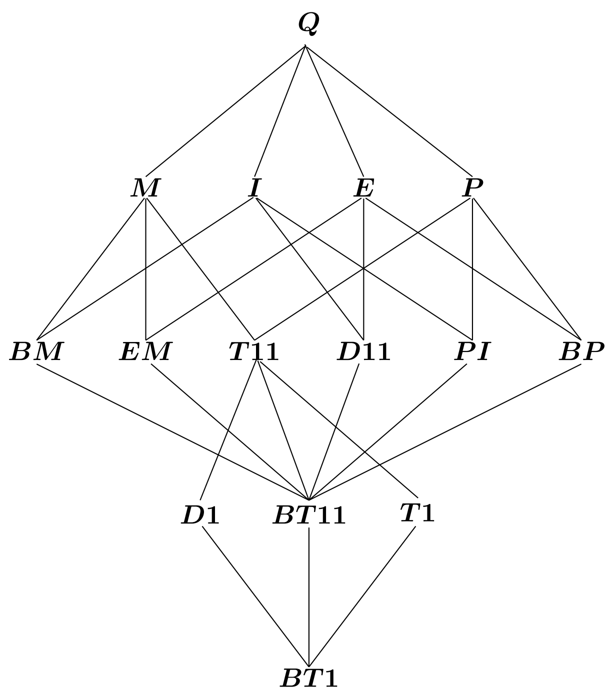

FIGURE 1. Varieties of quasigroups of Abelian group isotopes (with Q added)

The lattice of varieties of quasigroups which are not necessarily group isotopes is given in Figure 2. However, we have to justify relationships (even the weak ones) between varieties in this case.

LEMMA 1.1. *The following relationships hold between varieties of quasigroups which are not necessarily group isotopes.*

1. $CU \subseteq C$
2. $CU \subseteq UB11$
3. $CU \subseteq U1B11$
4. $CU \subseteq Ub0$
5. $C \subseteq B11$
6. $UB11 \subseteq Ub1$
7. $UB11 \subseteq B11$
8. $U1B11 \subseteq U1$
9. $U1B11 \subseteq B11$
10. $Ub0 \subseteq U$
11. $Ub1 \subseteq U$
12. $B11 \subseteq Q$
13. $U1 \subseteq Q$
14. $U \subseteq Q$.

PROOF. 1. By the definition of CU.
2. Assume (C). Then $xy \cdot uv = yx \cdot vu = vu \cdot yx$.
3. Assume (C) and (U). Then (B11) follows by 2. Also $xy \cdot yx = xy \cdot xy = e$ i.e. (U1).
4. $(Ub0)$ is a special case of (CU).

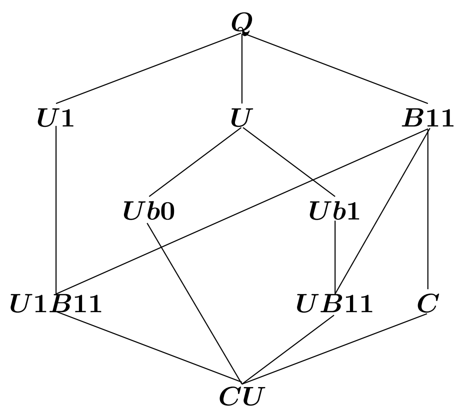

FIGURE 2. Varieties of quasigroups which are not necessarily group isotopes

5. As in 2.
6. Assume (B11). Then $e \cdot xy = zz \cdot xy = yx \cdot zz = yx \cdot e$.
7–14. Trivial. □

There are two varieties which are not included in graphs in Figures 1 and 2: LLU and RLU. They are elements of the subset $\{LLU, LRU, Ub1, D1\}$ with order relations as indicated in Figure 3. We have to justify this claim also.

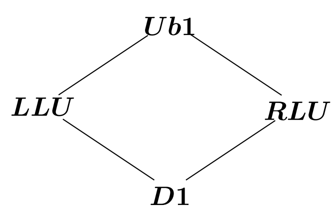

FIGURE 3. An ordered subset of Q_4 containing LLU and RLU

By Theorem 1.1, an operation $\cdot$ in a left linear unipotent quasigroup is of the form $x \cdot y = Ax - Ay + c$ for some group $+$, automorphism A and an element c. By simple checking we prove $(LLU) \Rightarrow (Ub1)$. Similarly, using entry for $D1$ in Table 2, we prove $(D1) \Rightarrow (LLU)$. Therefore, $D1 \subseteq LLU \subseteq Ub1$. By the left–right duality principle for groupoids, we have $D1 \subseteq RLU \subseteq Ub1$ as well.

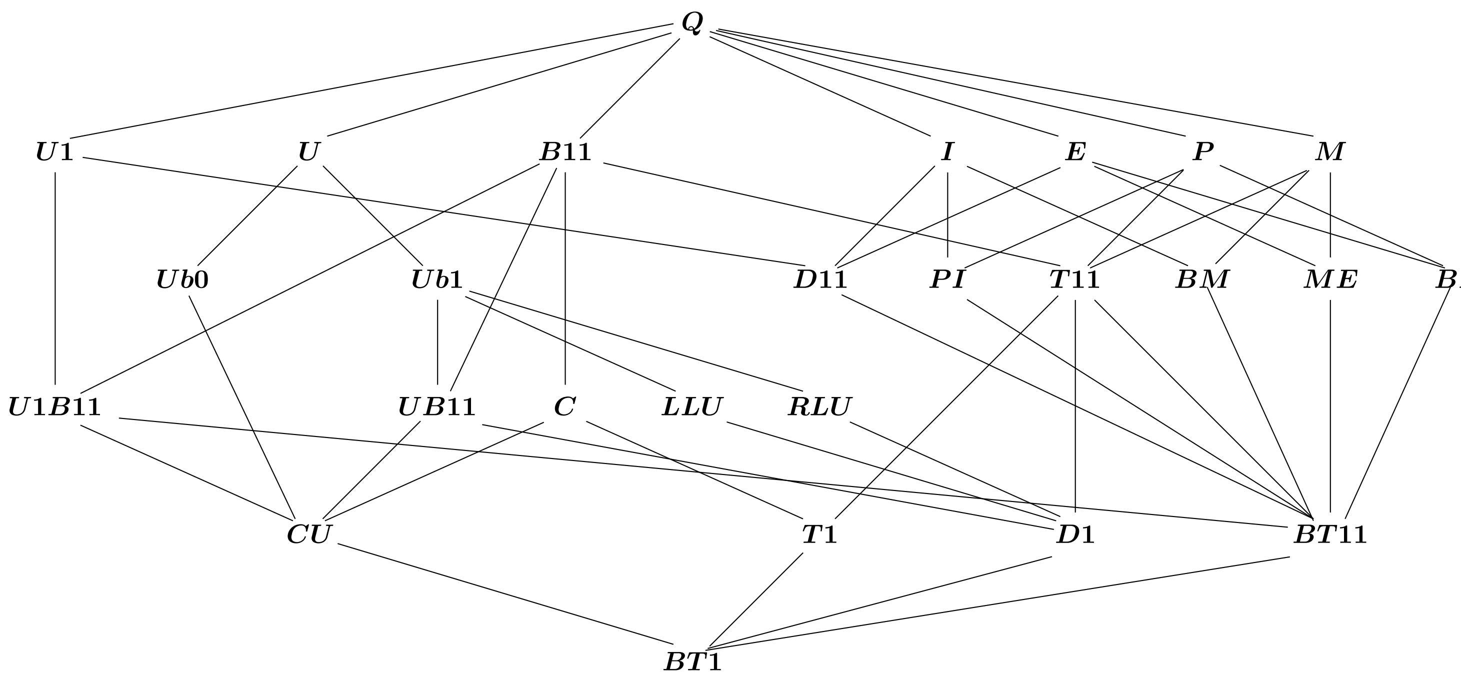

FIGURE 4. The lattice of all varieties from Q_4

There are several more relationships with which we have to deal separately.

LEMMA 1.2. *The following relationships hold between the indicated varieties.*

1. $T11 \subseteq B11$
2. $D11 \subseteq U1$
3. $BT11 \subseteq U1B11$
4. $T1 \subseteq C$
5. $D1 \subseteq UB11$
6. $BT1 \subseteq CU$
7. $M \subseteq Q$
8. $P \subseteq Q$
9. $E \subseteq Q$
10. $I \subseteq Q$

The proof of 1 to 6 uses appropriate entries from Table 2 and requires simple checking only. Relations 7 to 10 are obvious.

Our ultimate goal is to prove that the ordered set[1] $(Q_4; \subseteq)$ of 26 varieties of quasigroups axiomatized by one or more of 105 quadratic level identities is the lattice given in Figure 4.

2. The partition of Q_4

In order to make the proof of the stated result easier, we divide the set Q_4 into sets of varieties of quasigroups which consist of:

- Boolean group isotopes: $BT1, BT11, BM, BP$
- Abelian group isotopes but not neccessarily Boolean group isotopes: $M, P, E, I, ME, PI, T11, D11, T1, D1$
- Group isotopes but not neccessarily Abelian group isotopes: LLU, RLU
- Not neccessarily group isotopes: $Q, C, B11, U1, U1B11, U, Ub0, Ub1, UB11, CU$.

The partition is based on the results of Sections 5–9 of [**2**], but we should emphasize again that the 26 varieties above are not yet proven to be different one from the other. However, by the well known result of quasigroup theory, that if a loop is isotopic to a group, then they are isomorphic, it follows that the four classes above are pairwise disjoint.

Another, independent partition of Q_4 is:

- varieties with unipotent quasigroups only: $U, Ub0, Ub1, UB11, CU, LLU$, RLU, $D1$, $BT1$
- varieties which contain quasigroups which are not neccessarily unipotent: $Q, C, B11, U1, U1B11, M, P, E, I, ME, PI, T11, D11, T1, BM, BP, BT11$.

The above partition is justified by:

LEMMA 2.1. *All the quasigroups from varieties U, $Ub0$, $Ub1$, $UB11$, CU, LLU, RLU, $D1$, $BT1$ are unipotent.*

Every one of the varieties Q, C, $B11$, $U1$, $U1B11$, M, P, E, I, ME, PI, $T11$, $D11$, $T1$, BM, BP, $BT11$ contains a quasigroup which is not unipotent.

PROOF. The proof of the first part of the lemma follows from [**2**, Theorems 6.1–6.4, 7.1, 7.2, 8.3, 9.1] and the definition of $(UB11)$. The second part is from the following two examples. □

[1] To avoid foundational issues, we work within a given universal set.

EXAMPLE 2.1. Let $(\mathbb{C}_2^4;+)$ be the fourth power of the two–element (Boolean) group $(\mathbb{C}_2;+)$. For

$$A=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\qquad\text{and}\qquad B=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

define operation $\oplus$ on $\mathbb{C}_2^4$ by $x\oplus y=Ax+By$. We can easily see that $A^2=B^2=\mathrm{Id}$ as well as $AB=BA$ and therefore $(\mathbb{C}_2^4;\oplus)$ is a model of $(BT11)$ (also of (Q), $(U1)$, $(B11)$, $(U1B11)$, (M), (P), (E), (I), (ME), (PI), $(T11)$, $(D11)$, (BM) and (BP)) but not of (U).

EXAMPLE 2.2. Let $(\mathbb{R};+)$ be the additive group of reals. Then it is an Abelian group isotope $(A=B=1)$ and therefore a model of $(T1)$ and (C), but because $x+x=2x\neq 0$ it is not a model of (U).

The meet of these two partitions is a partition related to an equivalence on Q_4 which we denote by $\sim$.

DEFINITION 2.1. The system $({Q_4}_{/\sim};\leqslant)$ is defined by:

$$\begin{array}{rcl}\mathcal{Q}&=&\{Q,C,B11,U1,U1B11\}\\ \mathcal{U}&=&\{U,Ub0,Ub1,UB11,CU\}\\ \mathcal{G}&=&\{LLU,RLU\}\\ \mathcal{A}&=&\{M,P,E,I,ME,PI,T11,D11,T1\}\\ \mathcal{D}&=&\{D1\}\\ \mathcal{B}&=&\{BM,BP,BT11\}\\ \mathcal{Z}&=&\{BT1\}\\ \mathcal{V}\leqslant\mathcal{W}&\text{iff}&\bigcup\mathcal{V}\subseteq\bigcup\mathcal{W}.\end{array}$$

The lattice of these classes is given in Figure 5.

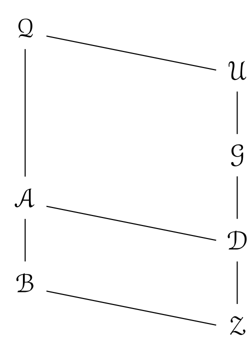

FIGURE 5. The lattice of $\sim$–classes in Q_4

THEOREM 2.1. *The function $f : Q_4 \longrightarrow Q_{4/\sim}$ ($f(V) = V^{\sim}$) is an order preserving surjection but is not a lattice homomorphism.*

PROOF. The first part of the statement is obvious (check Figures 4 and 5). To see that f is not a homomorphism take M and I. Then $f(M \cap I) = f(BM) = \mathcal{B}$ while $f(M) \wedge f(I) = \mathcal{A} \wedge \mathcal{A} = \mathcal{A}$. $\square$

If V, W are varieties from classes $\mathcal{V}, \mathcal{W}$ respectively, in general there are four possibilities for the relationship between V and W: $V = W$, $V \subset W$, $V \supset W$ and $V \parallel W$. However, if $\mathcal{V} < \mathcal{W}$, then only two possibilities remain: either $V \subset W$ or $V \| W$. This is the reason for the introduction of the equivalence $\sim$ and its classes. For the reference, we formulate the above and two similar results as a separate Lemma and use it extensively in the rest of the paper.

LEMMA 2.2. *Let V, W be varieties from classes $\mathcal{V}, \mathcal{W}$ respectively.*

- *If $\mathcal{V} < \mathcal{W}$ then either $V \subset W$ or $V \| W$.*
- *If $V \wedge W \notin \mathcal{V} \cup \mathcal{W}$ then $V \| W$.*
- *In particular, if $\mathcal{V} \| \mathcal{W}$ then $V \| W$.*

3. The main result

Some parts of Q_4 are either well known or trivial. For example, the variety Q is the greatest and the variety $BT1$ is the smallest element. The first fact is obvious. The second fact follows from the easily verifiable property of any unipotent quasigroup, linear over a Boolean group, that it satisfies all 105 quadratic level equations with four variables. Namely, every quasigroup from $BT1$ is of the form $xy = Ax + Ay + c$, where $+$ is a Boolean group, A is an automorphism of $+$ and $c \in S$. By [**2**, Lemma 8.1], any equation (L2) reduces to $AA(x_1 + \cdots + x_4) = AA(x_5 + \cdots + x_8)$. This is equivalent to $\sum_{i=1}^{8} x_i = 0$ which is always true since every variable appears exactly twice in the sum. Therefore:

LEMMA 3.1. *For every variety V from Q_4 we have $BT1 \subseteq V$.*

But we want to prove that $BT1 \neq V$ for any variety V from Q_4 (except $BT1$ itself). This is obvious as the variety $BT1$ is the single element in the class $\mathcal{Z}$.

Another part of the lattice $(Q_4; \subseteq)$ that is known, is the lattice of all varieties of quasigroups defined by balanced identities with four variables, given implicitly in Förg–Rob, Krapež [**1**] and reproduced from Krapež [**2**] as Figure 6 here.

We proceed by proving the rest of the relationships among varieties from Q_4.

Because of $\mathcal{D} \| \mathcal{B}$, we have:

LEMMA 3.2. *The variety $D1$ is incomparable to all varieties from $\mathcal{B}$ i.e.,*

	BT11	*BM*	*BP*
D1	$\|$	$\|$	$\|$

If we force the requirement that $+$ is Boolean on (M), (P), $(T11)$, we get (BM), (BP), $(BT11)$ respectively (see Table 2). The order $\subseteq$ on corresponding varieties is inherited, but we need to prove that BM, BP, $BT11$ are all different

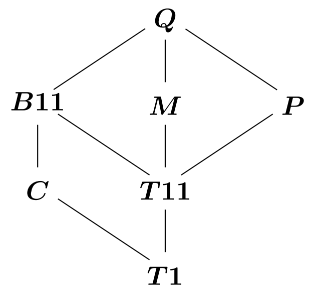

FIGURE 6. Varieties of quasigroups defined by balanced identities

one to the other. The following examples prove that $BT11 \subset BM$, $BT11 \subset BP$ and $BM \| BP$.

EXAMPLE 3.1. Let $\mathbb{V} = (V;+)$ with $V = \{0,1,2,3\}$ be a four-group, let $A = (123), B = (132)$ and let $(V;\oplus)$ be a quasigroup defined by $x \oplus y = Ax + By$. Then, because $AB = \mathrm{Id} = BA$ and $A^2 = B \neq A = B^2$, the quasigroup $(V;\oplus)$ is a model of (BM) but of neither $(BT11)$ nor (BP). This proves $BT11 \subset BM$ and $BM \nsubseteq BP$.

EXAMPLE 3.2. Let $\mathbb{V}$ be as in Example 3.1, let $A = (12)$, $B = (13)$ and let $(V;\oplus)$ be defined by $x \oplus y = Ax + By$. Then, because $A^2 = \mathrm{Id} = B^2$ and $AB = (123) \neq (132) = BA$, the quasigroup $(V;\oplus)$ is a model of (BP) but not of (BM), which proves both $BT11 \subset BP$ and $BP \nsubseteq BM$.

Therefore we proved:

LEMMA 3.3. *The relationship between varieties from $\mathcal{B}$ is given by the following table:*

	BT11	*BM*	*BP*
BT11	$=$	$\subset$	$\subset$
BM	$\supset$	$=$	$\|$
BP	$\supset$	$\|$	$=$

We now give some examples which will be needed later.

EXAMPLE 3.3. Let $(\mathbb{R};-)$ be the groupoid of reals under subtraction. Then $(\mathbb{R};-)$ is an Abelian group isotope $(Ax = x, Bx = -x)$ and satisfies $A + B = 0$, $A^2 = 1 = B^2$, $AB = -1 = BA$ and therefore $(\mathbb{R};-)$ is a model of $(D1)$, (P), (M) and $(T11)$. However, $A = 1 \neq -1 = B$, $A^2 + B^2 = 2 \neq 0$, $AB + BA = -2 \neq 0$ which implies that $(\mathbb{R};-)$ is a model of neither $(T1)$, $(D11)$, (ME), (E), (I) nor (PI).

EXAMPLE 3.4. Let $(\mathbb{R}^2;+)$ be the additive group of pairs of reals and $x \oplus y = Ax + By$, where $A = \left[\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 4 & 5 \\ -5 & -4 \end{smallmatrix}\right]$. Then $(\mathbb{R}^2;\oplus)$ is an Abelian group isotope that satisfies $A^2 + B^2 = O$, $AB + BA = O$ and therefore $(\mathbb{R}^2;\oplus)$ is a model of (E), (I) and $(D11)$. Also, $A \neq B$, $A+B \neq O$, $A^2 \neq B^2$, $AB \neq BA$ and therefore $(\mathbb{R}^2; \oplus)$ is not a model of $(T1)$, $(D1)$, (M), (P), $(T11)$, (ME), (PI).

EXAMPLE 3.5. Let $(\mathbb{C}; +)$ be the additive group of complex numbers and $x \oplus y = x + iy$. Then $(\mathbb{C}; \oplus)$ is an Abelian group isotope that satisfies $A^2+B^2 = 0$, $AB = BA$ and therefore $(\mathbb{C}; \oplus)$ is a model of (M), (E) and (ME). Likewise, from $A \neq B$, $A + B \neq 0$, $A^2 \neq B^2$, $AB + BA \neq 0$ it follows that $(\mathbb{C}; \oplus)$ is not a model of $(T1)$, $(D1)$, (I), (P), $(T11)$, $(D11)$, (PI).

EXAMPLE 3.6. Let $(\mathbb{Q}; +)$ be the additive group of quaternions and $x \oplus y = ix + jy$. Then $(\mathbb{Q}; \oplus)$ is an Abelian group isotope that satisfies $A^2 = B^2$, $AB + BA = 0$ and therefore $(\mathbb{Q}; \oplus)$ is a model of (P), (I) and (PI). On the other hand $A \neq B$, $A + B \neq 0$, $A^2 + B^2 \neq 0$, $AB \neq BA$ and therefore $(\mathbb{Q}; \oplus)$ is not a model of $(T1)$, $(D1)$, (M), (E), $(T11)$, $(D11)$, (ME).

LEMMA 3.4. *The relationship between varieties from $\mathcal{D}$ and $\mathcal{A}$ is given by the following table:*

	T1	*T11*	*D11*	*ME*	*PI*	*M*	*P*	*E*	*I*
D1	$\parallel$	$\subset$	$\parallel$	$\parallel$	$\parallel$	$\subset$	$\subset$	$\parallel$	$\parallel$

PROOF. 1. $D1 \not\subseteq T1$ by Example 3.3. $T1 \not\subseteq D1$ by Example 2.2.
2. We have $D1 \subseteq T11$. Since $D1$ and $T11$ belong to different classes $\mathcal{D}$ and $\mathcal{A}$, they are different too.
3. $D1 \not\subseteq D11$ by Example 3.3; $D11 \not\subseteq D1$ by Example 3.4.
4. $D1 \not\subseteq ME$ by Example 3.3; $ME \not\subseteq D1$ by Example 3.5.
5. $D1 \not\subseteq PI$ by Example 3.3; $PI \not\subseteq D1$ by Example 3.6.
6. $D1 \subset M$ and $D1 \subset P$ follow by the transitivity of $\subset$.
7. $D1 \not\subseteq E$ by Example 3.3; $E \not\subseteq D1$ by Example 3.5.
8. $D1 \not\subseteq I$ by Example 3.3; $I \not\subseteq D1$ by Example 3.6. $\square$

LEMMA 3.5. *The relationship between varieties from $\mathcal{B}$ and $\mathcal{A}$ is given by the following table:*

	T1	*T11*	*D11*	*ME*	*PI*	*M*	*P*	*E*	*I*
BT11	$\parallel$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
BM	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\subset$	$\parallel$	$\parallel$	$\subset$
BP	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\subset$	$\subset$	$\parallel$

PROOF. 1. Since $BT11$ consists of Boolean group isotopes and $T1$ of T-quasigroups such that $A = B$, we conclude that $BT11 \cap T1 = BT1$. As $BT1$ belongs to the class $\mathcal{Z}$, it is different from both $BT11$ and $T1$. Therefore $BT11 \| T1$.
2. That $BT11$ is strictly smaller than all other elements of $\mathcal{A}$ follow from the fact that $BT11$ does not belong to $\mathcal{A}$.
3. Similarly, BM does not belong to $\mathcal{A}$ and therefore $BM \subset M$ and $BM \subset I$.
4. $BM \cap T1 = BT1$ which belongs to $\mathcal{Z}$ and consequently $BM \| T1$.
5. $BM \cap T1 \subseteq BT11 \cap T1 = BT1$ and $BM \| T1$.
6. The meet of BM and any of $T11$, $D11$, ME, PI, P, E is $BT11$ and so BM is incomparable to any of them.
7. The proof for entries of BP is analogous to 3–6. $\square$

If we force $+$ to be Boolean on (I), (E), $(D11)$ we get (BM), (BP), $(BT11)$ respectively. The order is preserved and since the later varieties are different, the mapping from I, E, $D11$ to BM, BP, $BT11$ is surjective. Therefore:

LEMMA 3.6. *We have $D11 \subset I$, $D11 \subset E$, and $I \| E$.*

The same schema we can apply to M, E, ME and conclude:

LEMMA 3.7. *The following relationships are true: $ME \subset M$, $ME \subset E$, and $M \| E$.*

Again, applying the scheme to I, P, and PI we get:

LEMMA 3.8. *The relationships $PI \subset I$, $PI \subset P$, and $I \| P$ hold.*

LEMMA 3.9. *The relationship between varieties from $\mathcal{A}$ is given by the following table:*

	T1	*T11*	*D11*	*ME*	*PI*	*M*	*P*	*E*	*I*
T1	$=$	$\subset$	$\|$	$\|$	$\|$	$\subset$	$\subset$	$\|$	$\|$
T11	$\supset$	$=$	$\|$	$\|$	$\|$	$\subset$	$\subset$	$\|$	$\|$
D11	$\|$	$\|$	$=$	$\|$	$\|$	$\|$	$\|$	$\subset$	$\subset$
ME	$\|$	$\|$	$\|$	$=$	$\|$	$\subset$	$\|$	$\subset$	$\|$
PI	$\|$	$\|$	$\|$	$\|$	$=$	$\|$	$\subset$	$\|$	$\subset$
M	$\supset$	$\supset$	$\|$	$\supset$	$\|$	$=$	$\|$	$\|$	$\|$
P	$\supset$	$\supset$	$\|$	$\|$	$\supset$	$\|$	$=$	$\|$	$\|$
E	$\|$	$\|$	$\supset$	$\supset$	$\|$	$\|$	$\|$	$=$	$\|$
I	$\|$	$\|$	$\supset$	$\|$	$\supset$	$\|$	$\|$	$\|$	$=$

PROOF. 1. From 6 we see that $T1 \subset T11, T1 \subset M$ and $T1 \subset P$.
2. By Example 2.2, $(\mathbb{R}, +)$ is the model of $(T1)$ but none of: $(D11)$, (ME), (PI), (E), (I). This proves that $T1$ is not a subset of any of $D11$, ME, PI, E, I.

Following Example 3.4, $(\mathbb{R}^2, \oplus)$ is a model of $(D11)$, (E), (I) but not of $(T1)$. This proves $T1 \| D11$, $T1 \| E$ and $T1 \| I$.

Following Example 3.5, $(\mathbb{C}, \oplus)$ is a model of (ME) but not of $(T1)$. This proves $ME \not\subset T1$ and consequently $T1 \| ME$.

Finally, following Example 3.6, $(\mathbb{Q}, \oplus)$ is the model of (PI) but not of $(T1)$, which proves $PI \not\subset T1$ and therefore $T1 \| PI$.
3. Analogously, using the same models but with $T11$ instead of $T1$, we can prove incomparability of $T11$ to all of $(D11)$, (ME), (PI), (I), (E).
4. Following Lemma 3.6, we have $D11 \subset E$ and $D11 \subset I$.
5. Following Example 3.4, $(\mathbb{R}^2; \oplus)$ is a model of $(D11)$ but of neither (ME) nor (PI), (M), (P).

Following Example 3.5, $(\mathbb{C}; \oplus)$ is a model of (ME) and (M) but not of $(D11)$. Therefore $D11 \| ME$ and $D11 \| M$.

Following Example 3.6, $(\mathbb{Q}; \oplus)$ is a model of (PI) and (P) but not of $(D11)$. Consequently, $D11 \| PI$ and $D11 \| P$.
6. To prove that neither of ME, M, E is a subset of any of PI, P, I use Example 3.5. To prove that neither of PI, P, I is a subset of any of ME, M, E use Example 3.6.
7. $PI \subset P$ and $PI \subset I$ follow from Lemma 3.8.

8. Example 2.2 gives us the model of M and P but of neither E nor I. Example 3.4 gives us the model of E and I but of neither M nor P.
9. The rest of the relations from Table follows from the symmetry of $\|$ and the duality of $\subset$ and $\supset$. $\square$

Lemma 3.10. *For any variety V from any of the classes $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{Z}$ we have $V \subset Q$.*

Proof. We already concluded that Q is the greatest variety in Q_4. As it does not belong to $\mathcal{A}$, we have $M \subset Q$, $P \subset Q$, $E \subset Q$, $I \subset Q$. The rest of relations follow from the transitivity of $\subset$. $\square$

Collected together, Lemmas 3.1–3.10 prove:

Theorem 3.1. *The relationships given in Figure 1 are valid in the strong sense.*

We aim to prove the same result for Figure 2. For that, we need more examples.

Example 3.7. Let $(S; \circ)$ be a quasigroup with the Cayley table for the operation $\circ$ given in Table 3. It is a model of $(Ub0)$ (with $e = 0$) but not of (C) because elements 1 and 2 do not commute.

Similarly, since $0 \circ (1 \circ 2) \neq (2 \circ 1) \circ 0$, $(Ub1)$ is not true either.

Example 3.8. Let $(S; \circ)$ be a quasigroup with the Cayley table for the operation $\circ$ given in Table 4. It is a model of (U) but not of $(b0)$.

$\circ$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

Table 3. A model of $(Ub0)$ but not of (C)

$\circ$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Table 4. A model of (U) but not of $(b0)$

Example 3.9. Let a multiplicative group S_3 be given and let us define an operation $/$ by $x/y = xy^{-1}$. Then the quasigroup $(S; /)$ is a model of $(Ub1)$ (with $e = 0$), but not of $(B11)$ because $(1/0)/(0/5) \neq (5/0)/(0/1)$.

Lemma 3.11. *The relationship between varieties from $\mathcal{U}$ is given by the following table:*

	CU	*UB11*	*Ub1*	*Ub0*	*U*
CU	$=$	$\subset$	$\subset$	$\subset$	$\subset$
UB11	$\supset$	$=$	$\subset$	$\|$	$\subset$
Ub1	$\supset$	$\supset$	$=$	$\|$	$\subset$
Ub0	$\supset$	$\|$	$\|$	$=$	$\subset$
U	$\supset$	$\supset$	$\supset$	$\supset$	$=$

PROOF. 1. We have $CU \subseteq UB11$. If we force the operation $\cdot$ to be a T–quasigroup in $(CU), (UB11)$, we get $(BT1), (D1)$ respectively. Since $BT1$ and $D1$ are different, the same must be true for CU and $UB11$. Therefore, $CU \subset UB11$.

Also, $CU \subseteq Ub0$. Using model $(S; \circ)$ from Example 3.7 we prove $CU \subset Ub0$.
2. $UB11 \subseteq Ub1$. By Example 3.9, $UB11 \neq Ub1$.

Take a quasigroup from $UB11 \cup Ub0$. If we apply unipotency in $(B11)$ (with $y = x$), we get $e \cdot uv = vu \cdot e$ and (using $(b0)$) $e \cdot uv = e \cdot vu$. Commutativity follows. Therefore, such quasigroup belongs to CU which is different from both $UB11$ and $Ub0$ proving $UB11 \| Ub0$.
3. Taking $Ub1$ instead of $UB11$, we prove $Ub1 \| Ub0$.
4. According to Example 3.8 $Ub0 \subset U$.
5. The rest of the relations are either trivial or follow by the transitivity of $\subset$, or else by duality of $\subset$ and $\supset$. □

LEMMA 3.12. *The relationship between varieties from* $\mathcal{U}$ *and* $\mathcal{Q}$ *is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
CU	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
UB11	$\|$	$\|$	$\subset$	$\|$	$\subset$
Ub1	$\|$	$\|$	$\|$	$\|$	$\subset$
Ub0	$\|$	$\|$	$\|$	$\|$	$\subset$
U	$\|$	$\|$	$\|$	$\|$	$\subset$

PROOF. 1. We have $CU \subseteq U1B11$. Since CU and $U1B11$ belong to $\mathcal{U}$ and $\mathcal{Q}$ respectively they must be different, so $CU \subset U1B11$.
2. It is easy to see that the meet of $UB11$ with any of $C, U1B11, U1$ is CU which is different from any of them and consequently $UB11 \| C, UB11 \| U1B11, UB11 \| U1$.
3. The meet of $Ub1$ and any of $C, U1B11, U1$ is CU. Therefore, $Ub1 \| C, Ub1 \| U1B11$ and $Ub1 \| U1$.

The meet of $Ub1$ and $B11$ is $UB11$ which is different from both, so $Ub1 \| B11$.
4. The meet of $Ub0$ and any of C, $U1B11$, $B11$, $U1$ is CU and consequently $Ub0 \| C$, $Ub0 \| U1B11$, $Ub0 \| B11$, $Ub0 \| U1$.
5. The case of U is analogous to 4.
6. The rest of the relations are trivial. □

LEMMA 3.13. *The relationship between varieties from* $\mathcal{Q}$ *is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
C	$=$	$\|$	$\subset$	$\|$	$\subset$
U1B11	$\|$	$=$	$\subset$	$\subset$	$\subset$
B11	$\supset$	$\supset$	$=$	$\|$	$\subset$
U1	$\|$	$\supset$	$\|$	$=$	$\subset$
Q	$\supset$	$\supset$	$\supset$	$\supset$	$=$

PROOF. Take the class $\mathcal{Q} = \{Q, U1, U1B11, B11, C\}$ and add assumption that all operations from all varieties are T-quasigroups. We get five varieties of quasigroups: the variety T of T-quasigroups (which is not a member of Q_4), $D11$,

$BT11$, $T11$ and $T1$. Moreover, this mapping is an order isomorphism. The relationships between elements of $\mathcal{Q}$ are determined by the relationships of their images in $\mathcal{A} \cup \mathcal{B} \cup \{Q\}$ (replacing T by Q). □

Therefore we have:

THEOREM 3.2. *The relationships given in Figure 2 are valid in the strong sense.*

The following Lemmas reveal relationships between varieties from $\mathcal{Z}, \mathcal{D}, \mathcal{B}, \mathcal{A}$ on one side and varieties from $\mathcal{U}, \mathcal{Q}$ on the other.

LEMMA 3.14. *The relationship between varieties from $\mathcal{D}$ and $\mathcal{U}$ is given by the following table:*

	CU	*UB11*	*Ub1*	*Ub0*	*U*
D1	$\|$	$\subset$	$\subset$	$\|$	$\subset$

PROOF. A $D1$-quasigroup is of the form $xy = Ax - Ay + e$. Applying this to CU and $Ub0$ we get $BT1$ which is different from all the three, proving $D1\|CU$ and $D1\|Ub0$.

According to Lemma 1.1 $D1 \subseteq UB11$. Since $D1$ and $UB11$ belong to different classes, they are different. Therefore $D1 \subset UB11$. According to Lemma 3.13 and the transitivity of $\subset$, we have $D1 \subset Ub1$ and $D1 \subset U$. □

LEMMA 3.15. *The relationship between varieties from $\mathcal{D}$ and $\mathcal{Q}$ is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
D1	$\|$	$\|$	$\subset$	$\|$	$\subset$

PROOF. The meet of $D1$ and any of $C, U1B11, U1$ is $BT1$. Therefore, $D1$ is incomparable to any of $C, U1B11, U1$.

Trivially, $D1 \subset UB11 \subset B11 \subset Q$. □

LEMMA 3.16. *For a $V \in \mathcal{B}, W \in \mathcal{U}$ we have $V\|W$.*

PROOF. Follows from $\mathcal{B}\|\mathcal{U}$. □

LEMMA 3.17. *The relationship between varieties from $\mathcal{B}$ and $\mathcal{Q}$ is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
BT11	$\|$	$\subset$	$\subset$	$\subset$	$\subset$
BM	$\|$	$\|$	$\|$	$\|$	$\subset$
BP	$\|$	$\|$	$\|$	$\|$	$\subset$

PROOF. 1. The meet of $BT11$ and C is $BT1$ which is different from both and so $BT11\|C$.

According to Lemma 1.2 $BT11 \subseteq U1B11$ but, as they belong to different classes, they must be different. Consequently, $BT11 \subset U1B11$. From transitivity, $BT11 \subset B11, BT11 \subset U1$ and $BT11 \subset Q$.

2. The meet of BM and C is $BT1$. Therefore, $BM\|C$.

The meet of BM and any of $U1, B11, U1B11$ is $BT11$. Consequently BM is incomparable to any of them.

The relation $BM \subset Q$ is trivially true.

3. The case with BP instead of BM is analogous. $\square$

Lemma 3.18. *For a* $V \in \mathcal{A}, W \in \mathcal{U}$ *we have* $V \| W$.

Proof. Follows from $\mathcal{A} \| \mathcal{U}$. $\square$

Lemma 3.19. *The relationship between varieties from* $\mathcal{A}$ *and* $\mathcal{Q}$ *is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
T1	$\subset$	$\|$	$\subset$	$\|$	$\subset$
T11	$\|$	$\|$	$\subset$	$\|$	$\subset$
D11	$\|$	$\|$	$\|$	$\subset$	$\subset$
ME	$\|$	$\|$	$\|$	$\|$	$\subset$
PI	$\|$	$\|$	$\|$	$\|$	$\subset$
M	$\|$	$\|$	$\|$	$\|$	$\subset$
P	$\|$	$\|$	$\|$	$\|$	$\subset$
E	$\|$	$\|$	$\|$	$\|$	$\subset$
I	$\|$	$\|$	$\|$	$\|$	$\subset$

Proof. 1. All $T1$-quasigroups are commutative since $A = B$. From transitivity we have $T1 \subset B11$ and $T1 \subset Q$.

$T1 \cap U1 = BT1$ and consequently $T1 \| U1$. $T1 \cap U1B11 \subseteq T1 \cap U1 = BT1$. Therefore, $T1 \| U1B11$ as well.

2. According to Lemma 1.2 $T11 \subseteq B11$. Since they belong to different classes $\mathcal{A}$ and $\mathcal{Q}$ respectively, they must be different.

As $T11 \cap C = T1$ we have $T11 \| C$.

$T11 \cap U1B11 = T11 \cap U1 = BT11$ and consequently $T11 \| U1B11$ and $T11 \| U1$.

3. According to Lemma 1.2 $D11 \subseteq U1$. Since they belong to different classes $\mathcal{A}$ and $\mathcal{Q}$ respectively, they must be different.

$D11 \cap C = BT1$ and $D11 \cap U1B11 = D11 \cap B11 = BT11$, so $D11 \| C$, $D11 \| U1B11$, $D11 \| B11$.

4. $ME \cap C = BT1$ and $ME \cap U1B11 = ME \cap B11 = ME \cap U1 = BT11$; therefore $ME \| C, ME \| U1B11, ME \| B11, ME \| U1$.

5. $PI \cap C = BT1$ and $PI \cap U1B11 = PI \cap B11 = PI \cap U1 = BT11$; therefore $PI \| C, PI \| U1B11, PI \| B11, PI \| U1$.

6. The meets of M and $C, U1B11, B11, U1$ are $T1, BT11, T11$ and $BT11$ respectively. This proves incomparability of M to any of $C, U1B11, B11, U1$.

7. Incomparability of E and I to $C, U1B11, B11, U1$ is proven similarly. $\square$

Finally, we have to determine the relationship of LLU and RLU to each other and to all other varieties from Q_4.

Lemma 3.20. *The varieties* LLU *and* RLU *are incomparable.*

Proof. For a quasigroup $\cdot$ from LLU, $x \cdot y = Ax - Ay + c$ for appropriate $A, +$ and c. If we apply this to an identity which determines RLU, for example $xx \cdot yz = uy \cdot uz$, we get commutativity and consequently $D1$. As $D1 \in \mathcal{D}$ and is therefore different from both LLU, RLU, this implies $LLU \| RLU$. $\square$

LEMMA 3.21. *We have* $D1 \subset LLU, D1 \subset RLU$.

PROOF. We have $D1 \subseteq LLU$ and $D1 \subseteq RLU$. Since $D1$ belongs to $\mathcal{D}$, while LLU, RLU belong to $\mathcal{G}$, we infer $D1 \subset LLU$ and $D1 \subset RLU$. □

LEMMA 3.22. *For a* $V \in \mathcal{B}, W \in \mathcal{G}$ *we have* $V \| W$.

PROOF. Follows from $\mathcal{B} \| \mathcal{G}$. □

LEMMA 3.23. *For a* $V \in \mathcal{A}, W \in \mathcal{G}$ *we have* $V \| W$.

PROOF. Follows from $\mathcal{A} \| \mathcal{G}$. □

LEMMA 3.24. *The relationship between varieties from* $\mathcal{G}$ *and* $\mathcal{U}$ *is given by the following table:*

	CU	*UB11*	*Ub1*	*Ub0*	*U*
LLU	$\|$	$\|$	$\subset$	$\|$	$\subset$
RLU	$\|$	$\|$	$\subset$	$\|$	$\subset$

PROOF. We have $LLU \subseteq Ub1 \subset U$. Since LLU belongs to $\mathcal{G}$ and $Ub1$ belongs to $\mathcal{U}$, it follows that $LLU \subset Ub1$ and $LLU \subset U$.

The meet of LLU and any of $CU, UB11, Ub0$ is $BT1$. Therefore, $LLU \| CU$, $LLU \| UB11$, $LLU \| Ub0$.

The relationships for RLU follow from the left-right duality for groupoids. □

LEMMA 3.25. *The relationship between varieties* $\mathcal{G}$ *and* $\mathcal{Q}$ *is given by the following table:*

	C	*U1B11*	*B11*	*U1*	*Q*
LLU	$\|$	$\|$	$\|$	$\|$	$\subset$
RLU	$\|$	$\|$	$\|$	$\|$	$\subset$

PROOF. The meet of LLU and any of $C, U1B11, U1$ is $BT1$. Consequently, $LLU \| C$, $LLU \| U1B11, LLU \| U1$.

From $LLU \cap B11 = D1$ it follows that $LLU \| B11$.

The relationships for RLU follow from the left–right duality for groupoids. □

Using the symmetry of $\|$ and duality of $\subset$ and $\supset$, we can complete the proof of the main theorem of the paper.

THEOREM 3.3. *The relationships given in Figure 4 are valid in the strong sense.*

4. Conclusions

In [**2**], we explicitly promised to prove in this paper:

(1) That 19 varieties: $Q, C, B11, U1, U, Ub0, Ub1, CU, LLU, RLU, M, P, E, I, ME, PI, T1, D1, BT1$ are mutually distinct.

(2) That each of the seven varieties $U1B11, UB11, T11, D11, BM, BP$ and $BT11$ (also mutually distinct, as well as different from above 19 varieties) can be axiomatized by two level identities with four variables (in the variety of quasigroups), cannot be axiomatized by a single level identity with four variables but can be axiomatized by the single level identity with eight variables.

(3) That the conjunction of any subset of 105 identities gives one of the above 26 varieties.
(4) That the ordering 'being a subset' on the set Q_4 of the above varieties is a lattice ordering. However, this lattice is not a sublattice of the lattice of all varieties of quasigroups.

Two more promisees were given elsewhere in [**2**]:

(5) That the proof of the independence of (U) and $(U1)$ will be given.
(6) That the diagram of the lattice Q_4 will be given.

We can fulfill these promisses now.

PROOF. (1) The proof is spread throughout Section 3.
(2) The seven varieties are defined in Table 1 by two level identities with four variables. The equivalence of these systems to some level identities with eight variables is hinted in the text on page 30. In Table 5 we give the correspondence of these varieties and some of the identities which define them.

variety	defining identity
$U1B11$	$(xy \cdot yx)(pq \cdot rs) = (uv \cdot vu)(sr \cdot qp)$
$UB11$	$(xx \cdot yy)(pq \cdot rs) = (uu \cdot vv)(sr \cdot qp)$
$T11$	$(xy \cdot uv)(pq \cdot rs) = (xu \cdot yv)(sq \cdot rp)$
$D11$	$(xy \cdot ux)(pq \cdot qr) = (vy \cdot uv)(ps \cdot sr)$
BM	$(xy \cdot uv)(pq \cdot qr) = (xu \cdot yv)(ps \cdot sr)$
BP	$(xy \cdot uv)(pq \cdot rp) = (vy \cdot ux)(sq \cdot rs)$
$BT11$	$(xy \cdot uv)(pq \cdot qr) = (xu \cdot yv)(rs \cdot sp)$

TABLE 5. Varieties of quasigroups–one identity with eight variables

As none of these systems is equivalent to above 19 identities, the varieties cannot be axiomatized by a single level identity with four variables. The proof that each of the seven varieties is different from any other in Q_4 is also spread throughout Section 3.
(3) Follows from the induction and the closeness of Q_4 under the meet operation.
(4) The lattice property can be verified in Figure 4 directly. The join of M and P in Q_4 is Q. In the lattice of all varieties of quasigroups, the join of M and P must be a subvariety of the variety T of all T-quasigroups (as both M and P are T–quasigroups), but the variety Q is not a T–quasigroup.
(5) On account of Lemma 3.12 $U \| U1$. Independence follows.
(6) On account of Theorem 3.3, the lattice Q_4 is given in Figure 4. □

5. Problems

The following problems suggest themselves:

PROBLEM 1. Solve (systems of) quasigroup level equations with eight variables. Give the lattice Q_8 of varieties determined by the corresponding identities.

PROBLEM 2. Solve (systems of) quasigroup level equations with 2^n variables for a given n. Describe the lattice Q_{2^n} of varieties determined by the corresponding identities.

PROBLEM 3. Solve (systems of) quasigroup level equations of any length. Describe the lattice Q_∞ of varieties determined by corresponding identities.

The methods of this and other papers from the reference list of [**2**] are sufficiently strong to solve these problems. The real problem lays in finding the method to handle the combinatorial explosion borne by the growth of n. For example, the number of quadratic level equations with eight variables is 2 027 025.

We can always classify varieties in $Q_{2^n}(Q_\infty)$ as we did in Section 2. There is a possibility that there is a new class of varieties with all quasigroups being group isotopes, but such that every variety contains a non–unipotent quasigroup. Let us call this $\sim$-class $\mathcal{H}$.

PROBLEM 4. Is there a (nonempty) $\mathcal{H}$ in $Q_{2^n}(Q_\infty)$? If there is, what is the minimal n such that $\mathcal{H} \neq \emptyset$?

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(Received 08 05 2012)