PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 94 (108) (2013), 169–180 DOI: 10.2298/PIM1308169B

FIXED POINTS AND D-BRANES

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ABSTRACT. The affine Kac–Moody algebras give rise to rational conformal field theories (RCFTs) called the Wess–Zumino–Witten (WZW) models. An important component of an RCFT is its fusion ring, whose structure constants are given by the associated S-matrix. We apply a fixed point property possessed by the WZW models ("fixed point factorization") to calculate non-negative integer matrix representations of the fusion ring, allowing for the calculation of D-brane charges in string theory.

1. Introduction

To each quantum field theory is associated an S-matrix. This matrix expresses amplitudes and thus is a fundamental component of the theory. In string theory, a particle is a finite curve of length approximately 10^{-33}cm. Modern string theories contain both open (e.g., photon) and closed (e.g. graviton) strings. Topologically, a closed string is a circle S^1 and an open string is the interval $[0, 1]$. As a string evolves through time, it traces out a surface called a worldsheet. For example, an incoming closed string travelling from $t = -\infty$ to $t = 0$ traces out a semi-infinite cylinder. An important discovery, in 1989, was of higher dimensional objects ('membranes') called Dirichlet-branes, or D-branes for short (co-discoverers were Polchinski, Dai, Leigh, and Hořava). Physically, D-branes are the membranes where the endpoints of open strings reside. In 1995, Polchinski proved that a consistent theory requires branes. D-branes are physical entities having tension and charge. During physical processes, these charges are conserved – thus D-brane charges in string theory are analogous to electrical charges in particle physics. Unlike regular electrical charges however, D-brane charges are usually preserved only modulo some integer M.

In this paper, we are specifically interested in the Wess–Zumino–Witten (WZW models). The WZW models are a well-studied class of two dimensional rational conformal field theories (RCFTs). A conformal field theory (CFT) is a quantum field theory whose symmetries include the conformal transformations. Since $\mathbb{R}^2 \cong \mathbb{C}$, in two dimensions, the space of conformal maps is infinitely dimensional. Let $f(z)$ be a holomorphic map such that $f'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. Then f is conformal in a neighbourhood of z_0. A *rational* conformal field theory obeys a

2010 *Mathematics Subject Classification:* Primary: 17B81.

further finiteness condition. The WZW models correspond to strings propagating on compact Lie groups (e.g., $\mathrm{SU}(n)$, the group of $n \times n$ unitary matrices with determinant 1).

Our main interest is in calculating D-brane charges for the WZW models. In a given model, the D-brane charges form a discrete abelian group, the *charge group*. One approach to determining the charge group is through K-theory [**43**, **44**, **48**]; however, while this yields the charge group, it does not give the actual charges themselves. To find the charges, we adopt the conformal field theory approach, discussed in Section 4.

For the simply connected groups (e.g., $\mathrm{SU}(n)$), the D-brane charges have been determined using both methods [**1**, **6**, **9**, **19**, **29**, **42**]. Also see [**20**, **37**] for results on twisted charges and [**17**, **18**] for results on semi-simple groups. The K-theory calculation for the non-simply connected group $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ has been done in [**10**]. Other than this, for the non-simply connected groups, the only way known to this author to find D-brane charges requires knowing the NIM-reps. The NIM-rep is explained in Section 3. It is important to note that not every NIM-rep corresponds to a consistent boundary condition of a CFT – e.g., the tadpoles of SU(2) [**13**].

The rest of this paper is structured as follows: in Section 2, we give the relevant mathematical background on the WZW models, including modular data and simple-currents; Section 3 discusses the NIM-rep and fixed point factorization in detail, and in Section 4, we give some details about the conformal field theory approach to finding D-brane charges. This is a work in progress, and the results of solving the charge equation (4.1) for the non-simply connected groups will follow in [**4**]. We conclude with some questions and further work in this direction.

2. The WZW models

The WZW models have the property that most of their quantities have a natural interpretation in terms of the underlying affine Lie algebra $\mathfrak{g}$. We denote the algebra $\mathfrak{g}$ at level k by $\mathfrak{g}_k$. More precisely, the modes of the current algebra generate an algebraic structure called a vertex operator algebra (VOA), whose representations are precisely the integrable representations of $\mathfrak{g}_k$ (for an introduction to VOAs, see e.g., [**41**]). These representations form a modular tensor category, and the fusion ring of $\mathfrak{g}_k$ (see (2.2)) is the Grothendieck ring of that tensor category. For an introduction to these matters, see e.g., [**12**, **23**, **46**]. In the following sections, we describe the WZW models from a Lie-theoretical point of view. For an introduction to Lie theory, see [**26**, **27**, **39**].

2.1. Affine algebras. Let $\overline{\mathfrak{g}}$ be a simple, finite dimensional Lie algebra. We construct the affine algebra $\mathfrak{g} := \overline{\mathfrak{g}}^{(1)}$ as follows. Let $\mathcal{L}(\overline{\mathfrak{g}})$ be the set of all Laurent polynomials in $\overline{\mathfrak{g}}$. That is, $\mathcal{L}(\overline{\mathfrak{g}}) = \{\sum_{n\in\mathbb{Z}} a_n t^n \mid a_n \in \overline{\mathfrak{g}}\}$, where all but finitely many a_n are zero. This is an infinite dimensional Lie algebra with bracket $[at^n, bt^m] = [a, b]t^{n+m}$. To have true, rather than projective, representations, we centrally extend by an element C;[1] i.e., $[C,x]=[x,C]=0$ for all x. This algebra has bracket $[at^n,bt^m]=[ab]t^{n+m}+n\delta_{n+m,0}(a|b)C$, where $(a|b)$ is the Killing form of $\overline{\mathfrak{g}}$. Finally, we extend one more time (non-centrally) by a derivation $\ell_0:=t\frac{d}{dt}$,[2] and denote this algebra by $\overline{\mathfrak{g}}^{(1)}=:\mathfrak{g}$. We call $\overline{\mathfrak{g}}$ the *horizontal subalgebra* of $\mathfrak{g}$. The affine algebras behave similarly to their horizontal subalgebras – they have highest weight representations, and (extended) Dynkin diagrams. The extended Dynkin diagrams are obtained from the Dynkin diagrams of the horizontal subalgebra by the addition of an extra node, the 0 node. The superscript '1' in the notation of the affine algebra means 'nontwisted'. The above construction can be twisted by an automorphism of the Dynkin diagram of $\overline{\mathfrak{g}}$, yielding affine algebras $A_2^{(2)}$, $A_{2r-1}^{(2)}$, $A_{2r}^{(2)}$, $D_{r+1}^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$, where the superscript denotes the order of the automorphism.[3] Though there are subtle differences, these twisted algebras behave mostly analogously to the nontwisted ones.

The algebra $\mathfrak{g}$ constructed above has irreducible highest weight modules, indexed by highest weights $(\lambda_0;\lambda_1,\ldots,\lambda_r)$. In any highest weight representation, the central element C is mapped to a scalar multiple k of the identity – we call k the *level* of the representation. We are interested in the integrable highest weight modules, i.e., those for which $\lambda_i\in\mathbb{Z}_{\geqslant 0}$ for all i; they are indexed by the set

$$P_+^k(\mathfrak{g})=\left\{\lambda=(\lambda_0;\lambda_1,\ldots,\lambda_r)\in\mathbb{Z}_{\geqslant 0}^{r+1}\;\Big|\;\sum_{i=0}^{r}a_i^\vee\lambda_i=k\right\}\tag{2.1}$$

of integrable highest weights. In (2.1), the λ_i are the Dynkin labels, r is the rank of the underlying horizontal subalgebra, and $a_i^\vee$ are the colabels (The zeroth colabel $a_0^\vee$ is always 1). For example, for $\mathfrak{g}=A_r^{(1)}$, the Lie algebra corresponding to $\mathrm{SU}(r+1)$, $a_i^\vee=1$ for all $0\leqslant i\leqslant r$. Physically, the set P_+^k indexes the primaries of the theory, with the vacuum primary (the state of lowest energy) corresponding to the weight $k\Lambda_0=(k;0,\ldots,0)=:0$.

2.2. Modular data. In a rational CFT, the chiral algebra $\mathcal{V}$ has finitely many irreducible modules M.[4] Their characters are the one-point functions $\mathrm{ch}_A(\tau,u)=\mathrm{tr}_A\,o(u)e^{2\pi i(L_0-c/24)}$, where c is the central charge, L_0 is the energy operator, τ, the modular parameter, is in the upper half plane, and $o(u)$ is the zero-mode. We usually specialize to the variable τ to obtain $\mathrm{ch}_A(\tau)=\mathrm{tr}_A\,e^{2\pi i(L_0-c/24)}$, though we lose linear independence of the characters by doing so. These characters satisfy the modularity property

$$\mathrm{ch}_A(-1/\tau)=\sum_B S_{AB}\,\mathrm{ch}_B(\tau),\quad \mathrm{ch}_A(\tau+1)=\sum_B T_{AB}\,\mathrm{ch}_B(\tau)$$

where the sum is over all irreducible modules B. The S and T matrix are called the *modular data* of the theory and generate an $\mathrm{SL}_2(\mathbb{Z})$ representation $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\mapsto S$

[1] Up to isomorphism, there is a unique nontrivial one dimensional central extension.

[2] This extension is taken so that the simple roots are linearly independent.

[3] We use the notation of [**39**] for the twisted algebras.

[4] We assume the the theory is non-heterotic, i.e., that the two chiral algebras are isomorphic.

and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \mapsto T$.[5] For an introduction to modular data, see [**36**]. The entries of S and T lie in a cyclotomic field and S and T satisfy the following properties:

- T is diagonal and of finite order,
- S is unitary and symmetric (i.e., $SS^* = I$),
- $(ST)^3 = S^2 =: C$, an order two permutation matrix called *charge-conjugation*.

For the WZW models, the modules M are labelled by the set P_+^k of level k highest weights λ, and the characters coincide with the characters of the underlying affine algebra, specialized to τ.

The coefficients defined by *Verlinde's formula*

$$N_{\lambda\mu}^{\nu} = \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}^*}{S_{0\alpha}} \tag{2.2}$$

are nonnegative integers. They are structure constants for a commutative, associative ring called the *fusion ring*. That is, the ring multiplication is given by $x_\lambda * x_\mu = \sum_\nu N_{\lambda\mu}^{\nu} x_\nu$, where the sum is over P_+^k. For each $\lambda \in P_+^k$, the *fusion matrix* is the matrix indexed by P_+^k and given by entries $(N_\lambda)_{\mu\nu} := N_{\lambda\mu}^{\nu}$.

The Kac–Peterson formula gives an explicit formula for the S-matrix [**40**]:

$$S_{\lambda\mu} = \kappa^{-r/2} s \sum_{w \in \overline{W}} (\det w) \exp\left[-2\pi i \frac{w(\overline{\lambda+\rho}) \cdot (\overline{\mu+\rho})}{\kappa}\right], \tag{2.3}$$

where $\overline{W}$ is the $\overline{\mathfrak{g}}$ Weyl group, $\overline{\rho} = (1, \ldots, 1)$ is the Weyl vector, and κ and s are constants depending on the rank r and level k.

A simple example of modular data is for $A_1^{(1)}$. The group manifolds SU(2) and SO(3)=SU(2)/$\mathbb{Z}_2$ are based on this algebra. The S and T matrices are

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin\left[\pi \frac{(\lambda+1)(\mu+1)}{k+2}\right],$$
$$T_{\lambda\lambda} = \exp\left[-\frac{\pi i}{4}\right] \exp\left[\pi i \frac{(\lambda+1)^2}{2(k+2)}\right],$$

where $0 \leqslant \lambda,\, \mu \leqslant k$, and the fusion coefficients are

$$N_{\lambda\mu}^{\nu} = \begin{cases} 1, & \text{if } \nu \equiv_2 \lambda + \mu \text{ and } |\lambda - \mu| \leqslant \nu \leqslant \min\{\lambda+\mu,\, 2k-\lambda-\mu\} \\ 0, & \text{otherwise} \end{cases}$$

2.3. Simple-currents. The S-matrix has a strictly positive column and satisfies the inequality

$$S_{\lambda 0} \geqslant S_{00} > 0.$$

Equality occurs for weights called *simple-currents*. These weights are given by some permutation J of the vacuum–we also refer to the permutation J as a simple-current. In all cases except $E_8^{(1)}$, level 2, simple-currents correspond to extended diagram automorphisms [**21**]. This is realized by labelling each node of the extended diagram with a Dynkin label of λ– the automorphism of the graph permutes the

[5] The group is $\mathrm{SL}_2(\mathbb{Z})$ rather than $\mathrm{PSL}_2(\mathbb{Z})$ because of the u variable in the original definition of character.

Dynkin labels yielding $J\lambda$. The set of all simple-currents forms an abelian group $\mathcal{J}$, isomorphic to the centre of the universal cover of the corresponding Lie group. For example, take $A_r^{(1)}$, the affine algebra for the models $\mathrm{SU}(r+1)/H$, where H is any subgroup of $\mathbb{Z}_{r+1}$, the centre of $\mathrm{SU}(r+1)$. Its graph is an $(r+1)$-gon. Let $\lambda \in P_+^k$ and label each node consecutively by the Dynkin labels of λ. Let J be the rotation of $2\pi/(r+1)$. Then J acts on λ by the rule $(\lambda_0; \lambda_1, \dots, \lambda_r) \mapsto (\lambda_r; \lambda_0, \dots, \lambda_{r-1})$. This simple-current generates the group $\mathcal{J}$, isomorphic (as it must be) to $\mathbb{Z}_{r+1}$. For $B_r^{(1)}$, $C_r^{(1)}$, $A_{2r-1}^{(2)}$, and $D_{r+1}^{(2)}$, there is one nontrivial diagram automophism, of order two, so $\mathcal{J} \cong \mathbb{Z}_2$. For $D_r^{(1)}$, $\mathcal{J} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if r is even and $\mathbb{Z}_4$ if r is odd. If r is even, the generators of the simple-current group are J_v which exchanges the zeroth and first nodes, and $J_s : (\lambda_0; \dots, \lambda_r) \mapsto (\lambda_r; \dots, \lambda_0)$, and if r is odd, $\mathcal{J}$ is generated by the order four simple-current $J_v : (\lambda_0; \dots, \lambda_r) \mapsto (\lambda_{r-1}; \lambda_r, \lambda_{r-2}, \dots, \lambda_0)$.

2.4. Modular invariants. The *modular invariant partition function* of a WZW model is the function

$$\mathcal{Z}(\tau) = \sum_{\lambda,\mu} M_{\lambda\mu} \chi_\lambda(\tau) \chi_\mu^*(\tau) \tag{2.4}$$

where the sum is over all $\lambda, \mu \in P_+^k$, and where the $M_{\lambda\mu} \in \mathbb{Z}_{\geqslant 0}$ are the multiplicities in the decomposition $\mathcal{H} = \oplus M_{\lambda\mu} \lambda \otimes \mu$ of state-space $\mathcal{H}$ into $\mathcal{V} \otimes \mathcal{V}'$-modules, which are labelled by P_+^k.[6] The function (2.4) satisfies $\mathcal{Z}(A.\tau) = \mathcal{Z}(\tau)$ for any $A \in \mathrm{SL}_2(\mathbb{Z})$, where $A.\tau = \frac{a\tau+b}{c\tau+d}$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and the numbers $M_{\lambda\mu}$ are nonnegative integers with $M_{00} = 1$. The matrix M of coefficients is called a *modular invariant*. It satisfies the following three properties

- $M_{\lambda\mu} \in \mathbb{Z}_{\geqslant 0}$ for all $\lambda, \mu \in P_+^k$ (*positivity*)
- $M_{00} = 1$ (*uniqueness of vacuum*)
- $MS = SM$ and $MT = TM$ (*modular invariance*).

Modular invariants fall into series–the $\mathcal{A}$-series (this includes the identity and charge-conjugation $C := S^2$), the $\mathcal{D}$-series (simple-current extensions–see (2.5)) and the $\mathcal{E}$-series (exceptionals that do not fit into the previous two categories–these occur at low levels). This notation is used because the modular invariants in the first classification, for $A_1^{(1)}$ fell into an A-D-E pattern [**11**] (see [**34**] for another approach to this classification, which has been generalized to other cases).[7] Classifying modular invariants is an important step in classifying RCFTS. A precise mathematical formulation of this problem (simultaneously the classification of NIM-reps) as the classification of Frobenius algebras in their representation categories is given in [**15**]. Also see the review [**23**].

[6] Physically, state-space is the set of all possible states of the theory– it carries a representation of $\mathcal{V} \otimes \mathcal{V}'$, where $\mathcal{V}$ and $\mathcal{V}'$ are the vertex operator algebras of holomorphic and antiholomorphic fields resp. (a typical quantum field is neither holomorphic nor antiholomorphic, but a sesquilinear combination of both). A vertex operator algebra is the natural algebraic structure formed by the holomorphic (or antiholomorphic) fields.

[7] Because the exceptionals must be classified separately each time, complete modular invariant classifications are rare and difficult–see e.g., [**32, 33**] for the $A_2^{(1)}$ classification.

Every (non-heterotic) RCFT has a modular invariant and a NIM-rep [**14**] (we describe the NIM-rep in Section 3 below). Our interest is in the simple-current modular invariants, as they correspond to non-simply connected Lie groups. Let G be a compact simply connected Lie group corresponding to the affine algebra $\mathfrak{g}_k$ (this denotes the algebra $\mathfrak{g}$ at level k). Let J be an order n simple-current for $\mathfrak{g}_k$, and define the matrix [**45**]

$$M[J]_{\lambda\mu} = \sum_{i=1}^{\mathrm{ord}(J)} \delta_{J^i\lambda,\mu}\delta^{\mathbb{Z}}(Q_J(\lambda) + ir_J) \tag{2.5}$$

where $\delta^{\mathbb{Z}}(x) = 1$ if $x \in \mathbb{Z}$ and 0 otherwise, and $Q_J(\lambda)$ and r_J are rational numbers. The matrix $M[J]$ is a modular invariant if and only if $T_{J0,J0}T^*_{00}$ is an nth root of unity. In this case, it corresponds to the model with group $G/\langle J\rangle$. For example, for $A_1^{(1)}$ (with Lie group SU(2)) there is one order two simple-current $J(\lambda_0;\lambda_1) = (\lambda_1;\lambda_0)$. At level 4, the $\mathcal{D}$-series modular invariant (2.5) for J is

$$\mathcal{D}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group $\langle J\rangle$ is isomorphic to $\mathbb{Z}_2$. This modular invariant corresponds to the model with Lie group SU(2)/$\mathbb{Z}_2$=SO(3).

Because of the similarity of (2.3) to the Weyl character formula (see e.g., [**39**]), we have the relationship

$$\chi_\lambda(\mu) := ch_{\overline{\lambda}}\left(-2\pi i\,\frac{(\overline{\mu+\rho})}{\kappa}\right) = \frac{S_{\lambda\mu}}{S_{0\mu}} \tag{2.6}$$

between the ratios of the S-matrix and characters at elements of finite order of the underlying horizontal subalgebra $\overline{\mathfrak{g}}$. Equation (2.6) is the key to Theorem 3.1 below.

3. The NIM-rep

In this section, we discuss the NIM-rep and a property of the S-matrix – fixed point factorization – that allows for the calculation of NIM-rep coefficients. This in turn allows for the calculation of D-brane charges.

3.1. Description of the NIM-rep. A *NIM-rep* [**2**] $\mathcal{N}$ is a nonnegative integer representation of the fusion ring. More precisely, to each $\lambda \in P_+^k$, associate a nonnegative integer matrix $\mathcal{N}_\lambda$ such that $\mathcal{N}_\lambda\mathcal{N}_\mu = \sum_\nu N_{\lambda\mu}^\nu\mathcal{N}_\nu$, with $\mathcal{N}_0 = I$, the identity matrix, and $\mathcal{N}_{C\lambda} = \mathcal{N}_\lambda^t$, where ^{t} denotes transpose. Two NIM-reps $\mathcal{N}$ and $\mathcal{N}'$ are called *equivalent* if there is a permutation matrix P such that $\mathcal{N}'_\lambda = P^{-1}\mathcal{N}_\lambda P$ for all $\lambda \in P_+^k$. The easiest example of a NIM-rep is the assignment $\lambda \mapsto N_\lambda$, the fusion matrices. These are indexed by P_+^k, but a general NIM-rep matrix $\mathcal{N}_\lambda$ is indexed by *boundary states*–these were given a Lie theoretic interpretation in [**28**,**30**] and will be described below.

In a given NIM-rep, all matrices are normal and commute, so are simultaneously diagonalized by a unitary matrix Ψ. The eigenvalues of $\mathcal{N}_\lambda$ are $S_{\lambda\mu}/S_{0\mu}$, where μ lies in a multi-set, with multiplicities as eigenvalues. We call the multi-set of all such μ the *exponents* of the NIM-rep and denote it by $\mathcal{E}(\mathcal{N})$. We then have the Verlinde-like formula (recall eqn. (2.2))

$$\mathcal{N}_{\lambda x}^{y} = \sum_{\mu} \frac{\Psi_{x\mu} S_{\lambda\mu} \Psi^{\dagger}_{y\mu}}{S_{0\mu}}, \tag{3.1}$$

where the sum is over all exponents of the NIM-rep, x and y are boundary states, and $\dagger$ denotes complex conjugate transpose.

Consider a simple-current J of order n and its associated modular invariant $M[J]$. Let $\lambda \in P_+^k$. Define the *order* of λ, written $\mathrm{ord}(\lambda)$, to be the order of the stabilizer of λ in $\langle J \rangle$. By $[\lambda]$ we mean the J-orbit $[\lambda] = \{J^a\lambda \mid 0 \leqslant a \leqslant n-1\}$. Then the set of boundary states for the model is the set of all pairs $[[\lambda], j]$ such that $1 \leqslant j \leqslant \mathrm{ord}(\lambda)$. The exponents of a modular invariant M are elements of the multi-set $\mathcal{E}(M)$ consisting of all λ with $M_{\lambda\lambda} \neq 0$, appearing with multiplicity $M_{\lambda\lambda}$. We will associate this with the set $\{(\lambda, i) \mid 1 \leqslant i \leqslant M_{\lambda\lambda}\}$. If $\mathcal{E}(M) = \mathcal{E}(\mathcal{N})$, then we say that $\mathcal{N}$ and M correspond to each other. Recall the NIM-rep $\mathcal{N}_\lambda = N_\lambda$ for all $\lambda \in P_+^k$. In this case, the boundary states coincide with the set P_+^k, each weight occurring with multiplicity 1.

3.2. Fixed points and NIM-reps. Here, we describe fixed point factorization in detail. Fixed point factorization has been used successfully in the case of $\mathrm{SU}(n)$ to find D-brane charges for non-simply connected groups [**30**, **31**]. As mentioned in the introduction, fixed point factorization refers to a simplification of the S-matrix at entries where at least one index is a fixed point of a simple-current. More precisely, the S-matrix ratio $\chi_{\Lambda_\ell}(\varphi)$ (see (2.6)) reduces to some simple polynomial in S-matrix ratios for a smaller rank algebra (also at the fundamental weights). This is remarkable because a priori, there would have been no reason to expect the modular data of the two theories to be linked. The theorem below, adapted from Theorem 3.1 of [**3**], makes this precise. By a fixed point, we mean a weight φ such that $J\varphi = \varphi$, and by a truncated fixed point $\tilde{\varphi}$, we mean the part of φ that contains the non-redundant Dynkin labels. These are given for each algebra in [**3**], and below for the examples relevant to us. For example, the extended diagram for $A_5^{(1)}$ is a regular hexagon. Consider the simple-current rotation by $2\pi/3$ radians–it has fixed points $\varphi = (\varphi_0; \varphi_1, \varphi_0, \varphi_1, \varphi_0, \varphi_1)$ with $\tilde{\varphi} = (\varphi_0; \varphi_1)$.

THEOREM 3.1. *Let $X_r^{(n)}$ be one of the algebras in the first column of Table 1 or 2, with fixed level k, and let $\chi_\lambda(\mu)$ be defined as in* (2.6). *Let J be a simple-current, φ a fixed point of J, and $\tilde{\varphi}$ the truncated fixed point associated to φ. Then*

(1) *For any fundamental weight Λ_i of $X_r^{(n)}$, the character $\chi_{\Lambda_i}(\varphi)$ can be expressed as a linear combination, with all coefficients in the set $\{-1, 0, 1\}$, in the characters $\chi'_{\Lambda'_j}(\tilde{\varphi})$, $0 \leqslant j \leqslant s$, for the algebra $X_s'^{(m)}$ given in the third column, where the level is given in the fourth column;*

(2) *For any* $\lambda \in P_+^k(X_r^{(n)})$*, there exists a polynomial* P_λ *such that* $\chi_\lambda(\varphi) = P_\lambda(\chi'_{\Lambda'_1}(\tilde{\varphi}), \ldots, \chi'_{\Lambda'_s}(\tilde{\varphi}))$.

Explicit formulas for part (1) are given in [**38**] (the A-series) and [**3**] (all other nonexceptional algebras); we show two below. Part (2) follows from Part (1), equation (2.6), and the classical result that the character ring of $\overline{\mathfrak{g}}$ is generated by the characters at the fundamental weights (see e.g., [**5**, Chapitre IV–VI]).

The A-series fixed point factorization is as follows: $\mathfrak{g}{=}A_r^{(1)}$. Suppose $d \mid (r+1)$, and fix a level k, divisible by $\frac{r+1}{d}$. The simple-current J^d, of order $(r+1)/d$, has fixed points $\varphi = (\varphi_0; \ldots, \varphi_{d-1}, \ldots, \varphi_0, \ldots, \varphi_{d-1})$. Let $\widetilde{\varphi} = (\varphi_0; \ldots, \varphi_{d-1})$. Then

$$\chi_{\Lambda_\ell}(\varphi) = \chi'_{\Lambda'_{\ell d/(r+1)}}(\widetilde{\varphi})$$

if $\frac{r+1}{d} \mid \ell$, where primes denote $A_{d-1}^{(1)}$ level $kd/(r+1)$ quantities. If $\frac{r+1}{d} \nmid \ell$, then $\chi_{\Lambda_\ell}(\varphi) = 0$. This is a very clean example of how the character $\chi_{\Lambda_\ell}(\varphi)$ reduces to that of a smaller rank algebra. The paper [**38**] also gives a formula for general λ. However, the existence of such a nice formula for general λ turned out to be special to the A-series and did not generalize to the other cases.

To illustrate the most complicated scenario, consider the algebra $C_r^{(1)}$, where r is even, and let J be the order two simple-current. Fixed points of J are $\varphi = (\varphi_0; \ldots, \varphi_{r/2}, \ldots, \varphi_0)$. Let $\widetilde{\varphi} = (\varphi_{r/2}; \ldots, \varphi_0)$. Then

$$\chi_{\Lambda_{2m}}(\varphi) = (-1)^m \sum_{\ell=0}^{m} \chi'_{\Lambda'_\ell}(\widetilde{\varphi}),$$

where primes denote $A_{2(\frac{r}{2})}^{(2)}$ level k quantities, and $\chi_{\Lambda_\ell}(\varphi) = 0$ if ℓ is odd.

The formulas are similar in the other cases.[8] Tables 1 and 2 below show the smaller rank algebras involved in each case. We call these the 'fixed point factorization' (FPF) algebras. It is interesting to note that the fixed point factorization algebras are the orbit Lie algebras of [**24**] (the orbit Lie algebra $\breve{g}$ of a symmetrizable Kac–Moody algebra $\mathfrak{g}$ is obtained by a matrix-folding, or diagram-folding technique).

Recall the NIM-rep coefficients $\mathcal{N}_{\lambda,[[\nu],i]}^{[[\kappa],j]}$ of (3.1). If at least one of ν, κ is not a fixed point, then these reduce to fusions of $\mathfrak{g}_k$ (we give an example below), which can be calculated easily–for example using the Kac–Walton formula [**39**,**47**]. However, when both ν and κ are fixed points, the NIM-rep coefficients are more difficult to calculate, and the only way known to this author is to use the formulas given by fixed point factorization. In this case, the NIM-rep also reduces to fusions, but of both the original algebra and the fixed point factorization algebra.

[8]The algebras $E_6^{(1)}$ and $E_7^{(1)}$ have nontrivial simple-currents; the current work suggests that fixed point factorization formulas exist for these algebras as well, but these have only to be worked out.

TABLE 1. Fixed point factorization algebras for the nontwisted algebras

$X_r^{(1)}$, level k	Simple-current	FPF algebra	Level
$A_r^{(1)}$	J^d	$A_{d-1}^{(1)}$	$\frac{kd}{r+1}$
$B_r^{(1)}$	J	$A_{2(r-1)}^{(2)}$	k
$C_r^{(1)}$, r odd	J	$C_{\frac{r-1}{2}}^{(1)}$	$\frac{k}{2}$
$C_r^{(1)}$, r even	J	$A_{2(\frac{r}{2})}^{(2)}$	k
$D_r^{(1)}$	J_v	$C_{r-2}^{(1)}$	$\frac{k}{2}$
$D_r^{(1)}$, r odd	J_s	$C_{\frac{r-3}{2}}^{(1)}$	$\frac{k}{4}$
$D_r^{(1)}$, r even	J_s	$B_{\frac{r}{2}}^{(1)}$	$\frac{k}{2}$

TABLE 2. Fixed point factorization algebras for the twisted algebras

$X_r^{(1)}$, level k	Simple-current	FPF algebra	Level
$A_{2r-1}^{(2)}$	J	$C_{r-1}^{(1)}$	$\frac{k}{2}$
$D_{r+1}^{(2)}$, r odd	J	$A_{2(\frac{r-1}{2})}^{(2)}$	$\frac{k}{2}$
$D_{r+1}^{(2)}$, r even	J	$D_{\frac{r}{2}+1}^{(2)}$	$\frac{k}{2}$

For example, for $C_r^{(1)}$, r even, we have the NIM-rep coefficients

$$\mathcal{N}_{\lambda[\nu]}^{[\kappa]} = N_{\lambda\nu}^{\kappa} + N_{\lambda\nu}^{J\kappa}, \qquad \mathcal{N}_{\lambda[\nu]}^{([\kappa],j)} = N_{\lambda\nu}^{\kappa}$$

$$\mathcal{N}_{\Lambda_n(\varphi,i)}^{(\psi,j)} = \begin{cases} \frac{1}{2}\left(N_{\Lambda_{2m}\varphi}^{\psi} + (-1)^{i+j+m}\sum_{\ell=0}^{m} \widetilde{N}_{\widetilde{\Lambda_\ell}\widetilde{\varphi}}^{\widetilde{\psi}}\right) & \text{if } n = 2m \\ \frac{1}{2}N_{\Lambda_n\varphi}^{\psi} & \text{if } n = 2m+1 \end{cases}$$

where tildes indicate $A_{2(\frac{r}{2})}^{(2)}$ level k quantities. Interestingly, our formulas above look similar to formulas found in [**7**] for the fusion rules of orbifolds involving twist fields (see e.g., eqn. (2.55)).[9]

4. Charges of WZW D-branes

In this section, we introduce the conformal field theory description of D-brane charges.

[9]We thank one of the anonymous referees for pointing this out to us.

Consider a WZW model that corresponds to the modular invariant M. The number of maximally symmetric, untwisted D-branes (our interest in this paper) is the trace of M; they are indexed by the boundary states from Section 3.1. Denote by q_x the charge of the D-brane corresponding to the boundary state x. These satisfy the *charge equation* [**19**]

$$\dim(\lambda) q_x = \sum_y \mathcal{N}_{\lambda x}^y q_y \tag{4.1}$$

where $\dim(\lambda)$ is the Weyl dimension of λ in $\overline{\mathfrak{g}}$; $\mathcal{N}_{\lambda x}^y$ are the NIM-rep coefficients, and the sum is over all boundary states.

The simply connected group G corresponds to the charge-conjugation modular invariant C. In this case, the D-branes are parametrized by P_+^k; the NIM-rep coefficients are precisely the fusion coefficients, and modulo an integer $M(\mathfrak{g}, k)$, (4.1) has the unique solution $q_\lambda = \dim\lambda$. The integer $M(\mathfrak{g}, k)$ has been found for all algebras and levels [**1, 6, 19**], and the charge groups are then $\mathbb{Z}_{M(\mathfrak{g},k)}$ [**43, 44, 48**].

In the case of the non-simply connected groups, not as much is known. The K-theory calculation for SO(3)=SU(2)/$\mathbb{Z}_2$ was done in [**10**]. This corresponds to the $A_1^{(1)}$ modular invariant for the order two simple-current. The paper [**30**] adopts the CFT approach, finding the charge groups to be $\mathbb{Z}_2 \times \mathbb{Z}_2$ if $4|k$ and $\mathbb{Z}_4$ if $4 \nmid k$. On the other hand, the charge group for SU(2) is $\mathbb{Z}_{k+2}$, which grows with the level k, showing that the behaviour of the non-simply connected groups may be quite different from the simply connected ones. The charge groups for SO(3) were also studied, using the CFT approach, in [**16**], where different groups were found, corresponding to another supersymmetric CFT. Using the NIM-rep coefficients for $A_r^{(1)}$, obtained via fixed point factorization, the charges and charge groups were found for the non-simply connected groups SU$(n)/\mathbb{Z}_d$ in [**30, 31**]. Our goal is to use the NIM-rep coefficients found for the remaining algebras [**4**] to solve (4.1) for all WZW models listed in Tables 1 and 2.

5. Summary and future work

In this paper, we described the NIM-reps of WZW models and how fixed point factorization can be used to find them. We then discussed how this will lead to solving the charge equation (4.1). A full treatment of D-brane charges requires finding the actual charges themselves, which in turn requires using the conformal field theory description of D-branes. Thus the only way we know to do this in the non-simply connected case is to use Theorem 3.1. Once this is completed, we are interested in comparing our results with K-theory calculations. Further work based on fixed point factorization is to find more physical applications for, in addition to calculating D-brane charges, find a conceptual explanation for this phenomenon, and explain the link between fixed point factorization and the orbit Lie algebras.

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