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ESTIMATES OF MULTILINEAR SINGULAR INTEGRAL OPERATORS AND MEAN OSCILLATION

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Abstract. We prove the boundedness properties for some multilinear operators related to certain integral operators from Lebesgue spaces to Orlicz spaces. The operators include Calderón–Zygmund singular integral operator, Littlewood–Paley operator and Marcinkiewicz operator.

1. Introduction

We are going to consider some integral operators as follows. Let m be a positive integer and A be a function on R^n. We denote

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

Definition 1.1. Let $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{x=y\}$ such that

$$Tf(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every bounded and compactly supported function f and $x \notin \operatorname{supp} f$, where K satisfies: $|K(x,y)| \leqslant C|x-y|^{-n}$ and for fixed $0 < \varepsilon \leqslant 1$,

$$|K(y,x) - K(z,x)| \leqslant C|y-z|^\varepsilon |x-z|^{-n-\varepsilon}$$

if $2|y-z| \leqslant |x-z|$. The multilinear operator related to the integral operator T is defined by

$$T^A(f)(x) = \int \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)\,dy.$$

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Definition 1.2. Let $F_t(x,y)$ be defined on $R^n\times R^n\times[0,+\infty)$; we denote

$$F_t(f)(x)=\int_{R^n}F_t(x,y)f(y)\,dy$$

for every bounded and compactly supported function f and

$$F_t^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}F_t(x,y)f(y)\,dy.$$

Let $H=\{h:||h||<\infty\}$ be a Banach space. For each fixed $x\in R^n$, we consider $F_t(f)(x)$ and $F_t^A(f)(x)$ as mappings from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t are defined by $S^A(f)(x)=\|F_t^A(f)(x)\|$, where $\|F_t(x,y)\|\leqslant C|x-y|^{-n}$ and for fixed $\varepsilon>0$

$$\|F_t(y,x)-F_t(z,x)\|\leqslant C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

if $2|y-z|\leqslant|x-z|$. We also define $S(f)(x)=\|F_t(f)(x)\|$.

Note that when $m=0$, T^A and S^A are just the commutators of T and S with A (see [**4**, **7**–**9**, **12**]). While when $m>0$, they are nontrivial generalizations of the commutators. Let T be the Calderón–Zygmund singular integral operator. A classical result of Coifman, Rochberg and Weiss [**4**] states that the commutator $[b,T]=T(bf)-bTf$ (where $b\in\mathrm{BMO}(R^n)$) is bounded on $L^p(R^n)$ for $1<p<\infty$. Chanillo [**1**] proves a similar result when T is replaced by a fractional integral operator. In [8], the boundedness properties for the commutators related to the Calderón–Zygmund singular integral operators from Lebesgue spaces to Orlicz ones are obtained. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [**2**,**3**,**5**]). Our main purpose is to prove the boundedness properties for the multilinear operators T^A and S^A from Lebesgue spaces to Orlicz ones.

Let us introduce some notations. Throughout the paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\,dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)\,dx$. It is well known [**6**] that

$$f^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|\,dy.$$

Let M be the Hardy–Littlewood maximal operator defined by

$$M(f)(x)=\sup_{Q\ni x}|Q|^{-1}\int_Q|f(y)|\,dy.$$

We write $M_pf=(M(f^p))^{1/p}$. For $1\leqslant r<\infty$ and $0\leqslant\beta<n$, let

$$M_{\beta,r}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\beta/n}}\int_Q|f(y)|^r dy\right)^{1/r}.$$

We say that f belongs to $\mathrm{BMO}(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{\mathrm{BMO}} = \|f^{\#}\|_{L^\infty}$. More generally, let φ be a nondecreasing positive function and define $\mathrm{BMO}_\varphi(R^n)$ as the space of all the functions f satisfying

$$\frac{1}{|Q(x,r)|}\int_{Q(x,r)} |f(y) - f_Q|\,dy \leqslant C\varphi(r).$$

For $\beta > 0$, the Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f satisfying

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{x\neq y} |f(x) - f(y)|/|x-y|^\beta < \infty.$$

For an f, let $m_f(t) = |\{x \in R^n : |f(x)| > t\}|$ denote the distribution function of f.

Let ψ be a nondecreasing convex function on R^+ with $\psi(0) = 0$ and let ψ^{-1} be its inverse function. The Orlicz space $L_\psi(R^n)$ is defined by the set of functions f such that $\int \psi(\lambda|f(x)|)\,dx < \infty$ for some $\lambda > 0$. The norm is given by $\|f\|_{L_\psi} = \inf_{\lambda>0} \lambda^{-1}(1 + \int \psi(\lambda|f(x)|)\,dx)$.

We shall prove the following theorems in Section 2.

Theorem 1.1. *Let $1 < p < \infty$ and φ, ψ be two nondecreasing positive functions on R^+ with $\varphi(t) = t^{n/p}\psi^{-1}(t^{-n})$ (or equivalently $\psi^{-1}(t) = t^{1/p}\varphi(t^{-1/n})$. Suppose that ψ is convex, $\psi(0) = 0$, $\psi(2t) \leqslant C\psi(t)$. Let T be the same as in Definition* 1.1 *and bounded on $L^p(R^n)$ for all $1 < p < \infty$. Then $T^A : L^p(R^n) \to L_\psi(R^n)$ is bounded if $D^\alpha A \in \mathrm{BMO}_\varphi(R^n)$ for all α with $|\alpha| = m$.*

Theorem 1.2. *Let $1 < p < \infty$ and φ, ψ be two non-decreasing positive functions on R^+ with $\varphi(t) = t^{n/p}\psi^{-1}(t^{-n})$ (or equivalently $\psi^{-1}(t) = t^{1/p}\varphi(t^{-1/n})$. Suppose that ψ is convex, $\psi(0) = 0$, $\psi(2t) \leqslant C\psi(t)$. Let S be the same as in Definition* 1.2 *and bounded on $L^p(R^n)$ for all $1 < p < \infty$. Then $S^A : L^p(R^n) \to L_\psi(R^n)$ is bounded if $D^\alpha A \in \mathrm{BMO}_\varphi(R^n)$ for all α with $|\alpha| = m$.*

2. Proofs of Theorems

We begin with the following preliminary lemmas.

Lemma 2.1. [**7**] *Let φ be a nondecreasing positive function on R^+ and η be an infinitely differentiable function on R^n with compact support such that $\int \eta(x)\,dx = 1$. Denote $b_t(x) = \int_{R^n} b(x-ty)\eta(y)\,dy$. Then $\|b - b_t\|_{\mathrm{BMO}} \leqslant C\varphi(t)\|b\|_{\mathrm{BMO}_\varphi}$.*

Lemma 2.2. [**7**] *Let $0 < \beta < 1$ and φ be a nondecreasing positive function on R^+ or $\beta = 1$. Then $\|b_t\|_{\mathrm{Lip}_\beta} \leqslant Ct^{-\beta}\varphi(t)\|b\|_{\mathrm{BMO}_\varphi}$.*

Lemma 2.3. [**7**] *Suppose $1 \leqslant p_2 < p < p_1 < \infty$, ρ is a nonincreasing function on R^+, B is a linear operator such that $m_{B(f)}(t^{1/p_1}\rho(t)) \leqslant Ct^{-1}$ if $\|f\|_{L^{p_1}} \leqslant 1$ and $m_{B(f)}(t^{1/p_2}\rho(t)) \leqslant Ct^{-1}$ if $\|f\|_{L^{p_2}} \leqslant 1$. Then $\int_0^\infty m_{B(f)}(t^{1/p}\rho(t))\,dt \leqslant C$ if $\|f\|_{L^p} \leqslant (p/p_1)^{1/p}$.*

Lemma 2.4. [**1**] *Suppose that $1 \leqslant r < p < n/\beta$ and $1/q = 1/p - \beta/n$. Then $\|M_{\beta,r}(f)\|_{L^q} \leqslant C\|f\|_{L^p}$.*

LEMMA 2.5. [**9**] *Suppose that* $1 \leqslant r < \infty$ *and* $b \in \mathrm{Lip}_\beta$. *Then*

$$\|(b-b_Q)f\chi_{2Q}\|_{L^r} \leqslant C|Q|^{1/r}\|b\|_{\mathrm{Lip}_\beta} M_{\beta,r}(f)(x)$$

where χ_{2Q} *is the characteristic function of* $2Q$.

LEMMA 2.6. [**3**] *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *with* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A;x,y)| \leqslant C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}$ *is the cube centered on* x *and having side length* $5\sqrt{n}|x-y|$.

To prove the theorems of the paper, we need the following

KEY LEMMA 1. *Let* $0 < \beta < n$, T *and* S *be the same as in Definitions* 1.1 *and* 1.2. *Suppose that* $Q = Q(x_0, d)$ *is a cube with* $\mathrm{supp}\, f \subset (2Q)^c$ *and* $x, \tilde{x} \in Q$.

(a) *If* $D^\alpha A \in \mathrm{BMO}(R^n)$ *for all* α *with* $|\alpha| = m$, *then*

$$|T^A(f)(x) - T^A(f)(x_0)| \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_r(f)(\tilde{x}) \text{ for any } r > 1;$$

(b) *If* $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ *for all* α *with* $|\alpha| = m$, *then*

$$|T^A(f)(x) - T^A(f)(x_0)| \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{Lip}_\beta} M_{\beta,1}(f)(\tilde{x});$$

(c) *If* $D^\alpha A \in \mathrm{BMO}(R^n)$ *for all* α *with* $|\alpha| = m$, *then*

$$\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_r(f)(\tilde{x}) \text{ for any } r > 1;$$

(d) *If* $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ *for all* α *with* $|\alpha| = m$, *then*

$$\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{Lip}_\beta} M_{\beta,1}(f)(\tilde{x}).$$

PROOF. Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$, then

$$R_{m+1}(A;x,y) = R_{m+1}(\tilde{A};x,y) \text{ and } D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_Q \text{ for } |\alpha| = m.$$

Suppose that $\mathrm{supp}\, f \subset (2Q)^c$ and $x, \tilde{x} \in Q = Q(x_0, d)$. Note that $|x_0 - y| \approx |x - y|$ for $y \in (2Q)^c$. We write

$$\begin{aligned} T^A(f)(x) - T^A(f)(x_0) &= \int_{R^n}\left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right] R_m(\tilde{A};x,y)f(y)\,dy \\ &\quad + \int_{R^n} \frac{K(x_0,y)f(y)}{|x_0-y|^m}[R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y)]\,dy \\ - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}&\left(\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x_0,y)(x_0-y)^\alpha}{|x_0-y|^m}\right) D^\alpha \tilde{A}(y)f(y)\,dy \\ &:= I + II + III. \end{aligned}$$

(a) By Lemma 2.6 and the following inequality (see [**10**]), for $b \in \mathrm{BMO}(R^n)$,

$$|b_{Q_1} - b_{Q_2}| \leqslant C \log(|Q_2|/|Q_1|)\|b\|_{\mathrm{BMO}} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^k Q$ with $k \geqslant 1$,

$$\begin{aligned}|R_m(\tilde{A}; x, y)| &\leqslant C|x-y|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{\mathrm{BMO}} + |(D^\alpha A)_Q - (D^\alpha A)_{Q(x,y)}|) \\ &\leqslant Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}},\end{aligned}$$

thus

$$\begin{aligned}|I| &\leqslant C \int_{R^n \setminus 2Q} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}} \right) |R_m(\tilde{A}; x, y)||f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^k Q} k \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \sum_{k=1}^\infty k(2^{-k} + 2^{-k\varepsilon}) \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M(f)(\tilde{x}).\end{aligned}$$

For II, by the formula (see [**3**]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A}; x, x_0)(x-y)^\eta$$

and Lemma 2.6, we get

$$\begin{aligned}|II| &\leqslant C \int_{R^n \setminus 2Q} \frac{|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)|}{|x_0-y|^{m+n}} |f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M(f)(\tilde{x}).\end{aligned}$$

For III, similar to the estimates of I, we obtain, for any $r > 1$ with $1/r + 1/r' = 1$,

$$\begin{aligned}|III| &\leqslant C \int_{R^n \setminus 2Q} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |D^\alpha A(y) - (D^\alpha A)_Q||f(y)|\,dy \\ &\leqslant C \sum_{|\alpha|=m} \sum_{k=1}^\infty (2^{-k} + 2^{-k\varepsilon}) \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|^r dy \right)^{1/r} \\ &\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha A(x) - (D^\alpha A)_Q|^{r'} dx \right)^{1/r'}\end{aligned}$$

$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x}).$$

Thus

$$|T^A(f)(x)-T^A(f)(x_0)|\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x}).$$

(b) By Lemma 2.6 and the following inequality, for $b\in \mathrm{Lip}_\beta$,

$$|b(x)-b_Q|\leqslant \frac{1}{|Q|}\int_Q \|b\|_{\mathrm{Lip}_\beta}|x-y|^\beta dy\leqslant \|b\|_{\mathrm{Lip}_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_m(\tilde{A};x,y)|\leqslant \sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}(|x-y|+d)^{m+\beta},$$

then

$$\begin{aligned}
|I|&\leqslant C\int_{R^n\setminus 2Q}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)|R_m(\tilde{A};x,y)||f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\beta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\beta}}\right)|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\sum_{k=1}^\infty(2^{-k}+2^{-k\varepsilon})\frac{1}{|2^{k+1}Q|^{1-\beta/n}}\int_{2^{k+1}Q}|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}M_{\beta,1}(f)(\tilde{x}),\\
|II|&\leqslant C\int_{R^n\setminus 2Q}\frac{|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n}}|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\frac{|x-x_0|}{|x_0-y|^{n+1-\beta}}|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}M_{\beta,1}(f)(\tilde{x}),\\
|III|&\leqslant C\int_{R^n\setminus 2Q}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\beta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\beta}}\right)|D^\alpha A(y)-(D^\alpha A)_Q||f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\sum_{k=1}^\infty(2^{-k}+2^{-k\varepsilon})\frac{1}{|2^{k+1}Q|^{1-\beta/n}}\int_{2^{k+1}Q}|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}M_{\beta,1}(f)(\tilde{x}).
\end{aligned}$$

Thus

$$|T^A(f)(x)-T^A(f)(x_0)|\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}M_{\beta,1}(f)(\tilde{x}).$$

The same argument as in the proof of (a) and (b) will give the proof of (c) and (d), we omit the details. □

Now we are in the position to prove our theorems.

PROOF OF THEOREM 1.1. We prove it in several steps. First, we prove

$$(T^A(f))^{\#} \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_r(f) \tag{1}$$

for any $1 < r < \infty$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{R^n \setminus 2Q}$,

$$\begin{aligned} T^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)\,dy \\ &= \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f_2(y)\,dy + \int_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} K(x,y) f_1(y)\,dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y)\,dy; \end{aligned}$$

then

$$\begin{aligned} \frac{1}{|Q|}\int_Q \left|T^A(f)(x) - T^A(f_2)(x_0)\right| dx &\leqslant \frac{1}{|Q|}\int_Q \left| T\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1\right)(x)\right| dx \\ &\quad + \frac{1}{|Q|}\int_Q \sum_{|\alpha|=m} \frac{1}{\alpha!} \left| T\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(x)\right| dx \\ &\quad + \frac{1}{|Q|}\int_Q \left|T^A(f_2)(x) - T^A(f_2)(x_0)\right| dx := I_1 + I_2 + I_3. \end{aligned}$$

Now, for I_1, if $x \in Q$ and $y \in 2Q$, using Lemma 2.6, we get

$$R_m(\tilde{A};x,y) \leqslant C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}},$$

thus, by the L^r boundedness of T for any $1 < r < \infty$ and Holder' inequality, we obtain

$$\begin{aligned} I_1 &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \frac{1}{|Q|}\int_Q |T(f_1)(x)|\,dx \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|T(f_1)\|_{L^r} |Q|^{-1/r} \\ &\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f_1\|_{L^r} |Q|^{-1/r} \leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_r(f)(\tilde{x}). \end{aligned}$$

For I_2, for any $q > 1$, $l > 1$ and denoting $r = ql$, by L^q-boundedness of T, we gain

$$\begin{aligned} I_2 &\leqslant \frac{C}{|Q|}\int_Q \left| T\left(\sum_{|\alpha|=m} (D^\alpha A - (D^\alpha A)_Q) f_1\right)(x)\right| dx \\ &\leqslant C \sum_{|\alpha|=m} \left(\frac{1}{|Q|}\int_Q |T((D^\alpha A - (D^\alpha A)_Q) f_1)(x)|^q dx\right)^{1/q} \end{aligned}$$

$$\leqslant C|Q|^{-1/q}\sum_{|\alpha|=m}\|(D^\alpha A-(D^\alpha A)_Q)f_1\|_{L^q}$$

$$\leqslant C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q|D^\alpha A(y)-(D^\alpha A)_Q|^{ql'}dy\right)^{1/(ql')}\left(\frac{1}{|Q|}\int_Q|f(y)|^{ql}dy\right)^{1/(ql)}$$

$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x}).$$

For I_3, by using Key Lemma, we have $I_3\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x})$. We now put these estimates together, and taking the supremum over all Q such that $\tilde{x}\in Q$, we obtain $(T^A(f))^{\#}(\tilde{x})\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x})$; Thus, taking r such that $1<r<p$, we obtain

$$\|T^A(f)\|_{L^p}\leqslant C\|(T^A(f))^{\#}\|_{L^p}\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|M_r(f)\|_{L^p}\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{L^p}. \tag{2}$$

Secondly, we prove that, for $D^\alpha A\in \mathrm{Lip}_\beta(R^n)$ with $|\alpha|=m$

$$(T^A(f))^{\#}\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}(M_{\beta,r}((f))+M_{\beta,1}(f)) \tag{3}$$

for any $1\leqslant r<n/\beta$. In fact, by Lemma 2.6, we have, for $x\in Q$ and $y\in 2Q$

$$|R_m(\tilde{A};x,y)|\leqslant C|x-y|^m\sum_{|\alpha|=m}\sup_{z\in 2Q}|D^\alpha A(z)-(D^\alpha A)_Q|\leqslant C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}$$

and by Lemma 2.5, we have

$$\|(D^\alpha A-(D^\alpha A)_{2Q})f\chi_{2Q}\|_{L^r}\leqslant C|Q|^{1/r}\|D^\alpha A\|_{\mathrm{Lip}_\beta}M_{\beta,r}(f)(x).$$

Similarly to the proof of (1), we obtain

$$\frac{1}{|Q|}\int_Q\left|T^A(f)(x)-T^A(f)(x_0)\right|dx\leqslant\frac{1}{|Q|}\int_Q\left|T\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|dx$$
$$+\frac{1}{|Q|}\int_Q\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|T\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right|dx$$
$$+\frac{1}{|Q|}\int_Q\left|T^A(f_2)(x)-T^A(f_2)(x_0)\right|dx$$
$$\leqslant\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\frac{C}{|Q|^{1/r-\beta/n}}\left(\int_Q|T(f_1)(x)|^r dx\right)^{1/r}$$
$$+\sum_{|\alpha|=m}\left(\frac{C}{|Q|}\int_Q|T(D^\alpha\tilde{A}f\chi_{2Q})(x)|^r dx\right)^{1/r}$$

$$+\frac{1}{|Q|}\int_Q \left|T^A(f_2)(x)-T^A(f_2)(x_0)\right|dx$$
$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\frac{1}{|Q|^{1/r-\beta/n}}\|f_1\|_{L^r}$$
$$+C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_{R^n}|(D^\alpha A(x)-(D^\alpha A)_Q)f(x)\chi_{2Q}(x)|^r dx\right)^{1/r}$$
$$+\frac{1}{|Q|}\int_Q \left|T^A(f_2)(x)-T^A(f_2)(x_0)\right|dx$$
$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}(M_{\beta,r}(f)(\tilde{x})+M_{\beta,1}(f)(\tilde{x}));$$

Thus, taking $1\leqslant r<p<n/\beta$, $1/q=1/p-\beta/n$ and by Lemma 2.4, we obtain

$$\begin{aligned}\|T^A(f)\|_{L^q}&\leqslant C\|(T^A(f))^\#\|_{L^q}\\ &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}(\|M_{\beta,r}(f)\|_{L^q}+\|M_{\beta,1}(f)\|_{L^q})\\ &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.\end{aligned}\tag{4}$$

Now we verify that T^A satisfies the conditions of Lemma 2.3. In fact, for any $1<p_i<n/\beta$, $1/q_i=1/p_i-\beta/n$ $(i=1,2)$ and $\|f\|_{L^{p_i}}\leqslant 1$, note that $T^A(f)(x)=T^{A-A_s}(f)(x)+T^{A_s}(f)(x)$ and $D^\alpha(A_s)=(D^\alpha A)_s$, by (2) and Lemma 2.1, we obtain

$$\begin{aligned}\|T^{A-A_s}(f)\|_{L^{p_i}}&\leqslant C\sum_{|\alpha|=m}\|D^\alpha(A-A_s)\|_{\mathrm{BMO}}\|f\|_{L^{p_i}}\\ &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A-(D^\alpha A)_s\|_{\mathrm{BMO}}\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}_\varphi}\varphi(s),\end{aligned}$$

and by (4) and Lemma 2.2, we obtain

$$\|T^{A_s}(f)\|_{L^{q_i}}\leqslant C\sum_{|\alpha|=m}\|(D^\alpha A)_s\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p_i}}\leqslant Cs^{-\beta}\varphi(s)\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}_\varphi}.$$

Thus, for $s=t^{-1/n}$,

$$\begin{aligned}m_{T^A(f)}(t^{1/p_i}\varphi(t^{-1/n}))&\leqslant m_{T^{A-A_s}(f)}(t^{1/p_i}\varphi(t^{-1/n})/2)+m_{T^{A_s}(f)}(t^{1/p_i}\varphi(t^{-1/n})/2)\\ &\leqslant C\left[\left(\frac{\varphi(s)}{t^{1/p_i}\varphi(s)}\right)^{p_i}+\left(\frac{s^{-\beta}\varphi(s)}{t^{1/p_i}\varphi(s)}\right)^{q_i}\right]=Ct^{-1}.\end{aligned}$$

Taking $1<p_2<p<p_1<n/\beta$ and by Lemma 2.3, we get, for $\|f\|_{L^p}\leqslant(p/p_1)^{1/p}$,

$$\int_{R^n}\psi(|T^A(f)(x)|)\,dx=\int_0^\infty m_{T^A(f)}(\psi^{-1}(t))\,dt\leqslant C,$$

and thus, $\|T^A(f)\|_{L_\psi}\leqslant C$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let Q, $\tilde{A}(x)$, f_1 and f_2 be the same as the proof of Theorem 1.1, we write

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} F_t(x,y) f(y)\,dy \\ &= \int_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} F_t(x,y) f(y)\,dy + \int_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} F_t(x,y) f_1(y)\,dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{F_t(x,y)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y)\,dy, \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{|Q|}\int_Q \left|S^A(f)(x) - S^A(f_2)(x_0)\right| dx = \frac{1}{|Q|}\int_Q \left|\|F_t^A(f)(x)\| - \|F_t^A(f_2)(x_0)\|\right| dx \\ &\leqslant \frac{1}{|Q|}\int_Q \left\| F_t\Big(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1\Big)(x)\right\| dx \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!}\frac{1}{|Q|}\int_Q \left\| F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(x)\right\| dx \\ &\quad + \frac{1}{|Q|}\int_Q \|F_t^A(f_2)(x) - F_t^A(f_2)(x_0)\|\,dx. \end{aligned}$$

By using the same argument as in the proof of Theorem 1.1 will give the proof of Theorem 1.2, so we omit the details. □

3. Applications

In this section we shall apply Theorems 1.1 and 1.2 to some particular operators such as the Calderón–Zygmund singular integral operator, Littlewood–Paley operator, and Marcinkiewicz operator.

3.1. Calderón–Zygmund singular integral operator. Let T be the Calderón–Zygmund operator [**4**, **6**, **10**], i.e., the multilinear operator related to T is defined by

$$T^A(f)(x) = \int \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)\,dy.$$

Then it is easily to verify that Key Lemma holds for T^A, thus T satisfies the conditions in Theorem 1.1. So, the conclusion of Theorem 1.1 holds for T^A.

3.2. Littlewood–Paley operator. Let $\varepsilon > 0$ and ψ be a fixed functions satisfying

(1) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1)}$,

(2) $|\psi(x+y) - \psi(x)| \leqslant C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The multilinear Littlewood-Paley operator is defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)\,dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We write that $F_t(f) = \psi_t * f$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood–Paley g function [**11**];

Let $H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$ be space. Then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be regarded as a mapping from $[0,+\infty)$ to H, and it is clear that $g_\psi(f)(x) = \|F_t(f)(x)\|$ and $g_\psi^A(f)(x) = \|F_t^A(f)(x)\|$. It has been known that g_ψ is bounded on $L^p(R^n)$ for all $1 < p < \infty$. Thus it is only to verify that Key Lemma holds for g_ψ^A. In fact, we write, for a cube $Q = Q(x_0,d)$ with $\operatorname{supp} f \subset (2Q)^c$, $x, \tilde{x} \in Q = Q(x_0,d)$,

$$\begin{aligned}
F_t^A(f)(x) - F_t^A(f)(x_0) &= \int_{R^n}\left(\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m}\right)R_m(\tilde{A};x,y)f(y)\,dy \\
&\quad + \int_{R^n}\frac{\psi_t(x_0-y)}{|x_0-y|^m}(R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y))f(y)\,dy \\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left(\frac{(x-y)^\alpha\psi_t(x-y)}{|x-y|^m} - \frac{(x_0-y)^\alpha\psi_t(x_0-y)}{|x_0-y|^m}\right)D^\alpha\tilde{A}(y)f(y)\,dy \\
&:= J_1 + J_2 + J_3;
\end{aligned}$$

By the condition of ψ and Minkowski's inequality, we obtain, for any $r > 1$,

$$\begin{aligned}
\|J_1\| &\leqslant C\int_{R^n}\frac{|R_m(\tilde{A};x,y)||f(y)|}{|x_0-y|^m} \\
&\quad\times\left[\int_0^\infty\left(\frac{t|x-x_0|}{|x_0-y|(t+|x_0-y|)^{n+1}} + \frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon}}\right)^2\frac{dt}{t}\right]^{1/2}dy \\
&\leqslant C\int_{(2Q)^c}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)|R_m(\tilde{A};x,y)||f(y)|\,dy \\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M(f)(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
\|J_2\| &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^\infty\int_{2^{k+1}\setminus 2^kQ}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy \\
&\leqslant C\|D^\alpha A\|_{\mathrm{BMO}}M(f)(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
\|J_3\| &\leqslant C\sum_{|\alpha|=m}\sum_{k=1}^\infty\int_{2^{k+1}\setminus 2^kQ}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|D^\alpha\tilde{A}(y)||f(y)|\,dy \\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M_r(f)(\tilde{x}).
\end{aligned}$$

From the above estimates, we see that Theorem 1.2 holds for g_ψ^A.

3.3. Marcinkiewicz integral operator. Let Ω be homogeneous of degree zero on R^n and $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$ for $0 < \gamma \leqslant 1$, i.e., there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leqslant M|x-y|^\gamma$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy,$$

and we write

$$F_t(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy.$$

We also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz integral [**12**].

Let $H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$ be a space. Then it is clear that $\mu_\Omega(f)(x) = \|F_t(f)(x)\|$ and $\mu_\Omega^A(f)(x) = \|F_t^A(f)(x)\|$. Now, it is only to verify that Key Lemma holds for μ_Ω^A. In fact, for a cube $Q = Q(x_0, d)$ with $\operatorname{supp} f \subset (2Q)^c$, $x, \tilde{x} \in Q = Q(x_0, d)$ and $r > 1$, we have

$$\begin{aligned}
&\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \\
&\quad \leqslant \left(\int_0^\infty \left| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{m+n-1}} f(y)\,dy \right.\right. \\
&\qquad\qquad \left.\left. - \int_{|x_0-y|\leqslant t} \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}} f(y)\,dy \right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + \sum_{|\alpha|=m} \left(\int_0^\infty \left| \int_{|x-y|\leqslant t} \left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \int_{|x_0-y|\leqslant t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}} \right) D^\alpha \tilde{A}(y) f(y)\,dy \right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad \leqslant \left(\int_0^\infty \left[\int_{|x-y|\leqslant t,\ |x_0-y|>t} \frac{|\Omega(x-y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} |f(y)|\,dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left[\int_{|x-y|>t,\ |x_0-y|\leqslant t} \frac{|\Omega(x_0-y)||R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n-1}} |f(y)|\,dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\qquad + \left(\int_0^\infty \left[\int_{|x-y|\leqslant t, |x_0-y|\leqslant t} \left| \frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{m+n-1}} \right.\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad - \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}}\Big||f(y)|\,dy\Big]^2\frac{dt}{t^3}\Bigg)^{1/2}\\
&+\sum_{|\alpha|=m}\Bigg(\int_0^\infty\Bigg|\int_{|x-y|\leqslant t}\Bigg(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}}\\
&\qquad\qquad - \int_{|x_0-y|\leqslant t}\frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\Bigg)D^\alpha\tilde{A}(y)f(y)\,dy\Bigg|^2\frac{dt}{t^3}\Bigg)^{1/2}\\
&:= L_1+L_2+L_3+L_4
\end{aligned}
$$

and

$$
\begin{aligned}
L_1 &\leqslant C\int_{R^n}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}\Bigg(\int_{|x-y|\leqslant t<|x_0-y|}\frac{dt}{t^3}\Bigg)^{1/2}dy\\
&\leqslant C\int_{R^n}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}\Bigg(\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\Bigg)^{1/2}dy\\
&\leqslant C\int_{(2Q)^c}\frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M(f)(\tilde{x}).
\end{aligned}
$$

Similarly, we have $L_2\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M(f)(\tilde{x})$.

For L_3, by the following inequality [**12**]:

$$
\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right|\leqslant C\left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),
$$

we gain

$$
\begin{aligned}
L_3 &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\int_{(2Q)^c}\left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right)\\
&\qquad\qquad\times\Bigg(\int_{|x_0-y|\leqslant t,|x-y|\leqslant t}\frac{dt}{t^3}\Bigg)^{1/2}|f(y)|\,dy\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^\infty k(2^{-k}+2^{-\gamma k})M(f)(x)\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}M(f)(\tilde{x});
\end{aligned}
$$

Similarly to the cases of L_1, L_2 and L_3, we obtain for L_4

$$
\begin{aligned}
L_4 \leqslant C\sum_{|\alpha|=m}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}&\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma}}\right)\\
&\times|D^\alpha\tilde{A}(y)||f(y)|\,dy
\end{aligned}
$$

$$\leqslant C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^{\alpha}\tilde{A}(y)||f(y)|\,dy$$
$$\leqslant C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{\mathrm{BMO}}M_r(f)(\tilde{x}).$$

Thus, Theorem 1.2 holds for μ_{Ω}^{A}.

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