PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 97(111) (2015), 57–67 DOI: 10.2298/PIM150220001S

ON LINEAR COMBINATIONS OF CHEBYSHEV POLYNOMIALS

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Abstract. We investigate an infinite sequence of polynomials of the form:

$$a_0T_n(x) + a_1T_{n-1}(x) + \cdots + a_mT_{n-m}(x)$$

where $(a_0, a_1, \ldots, a_m)$ is a fixed m-tuple of real numbers, $a_0, a_m \neq 0$, $T_i(x)$ are Chebyshev polynomials of the first kind, $n = m, m+1, m+2, \ldots$ Here we analyze the structure of the set of zeros of such polynomial, depending on A and its limit points when n tends to infinity. Also the expression of envelope of the polynomial is given. An application in number theory, more precise, in the theory of Pisot and Salem numbers, is presented.

1. Introduction

It is well known [**6**] that the Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by $T_n(x) = \cos n\theta$ where $x = \cos\theta$. Let $A = (a_0, a_1, \ldots, a_m)$ be a $(m+1)$-tuple of real numbers, $a_0, a_m \neq 0$, $m \geqslant 1$. We introduce an infinite sequence of polynomials

$$T_{n,A}(x) = a_0T_n(x) + a_1T_{n-1}(x) + \cdots + a_mT_{n-m}(x) \quad (n \geqslant m).$$

We will refer to $T_{n,A}(x)$ as an A-Chebyshev polynomial. We can naturally extend this definition to the case $m = 0$ and $A = a_0 \neq 0$ by $T_{n,a0}(x) = a_0T_n(x)$. Also, it will be useful to introduce the polynomial $P_A(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$. We will refer to $P_A(t)$ as the characteristic polynomial of the A-Chebyshev polynomial.

2. Roots of A-Chebyshev polynomial

Let $Z_{n,A}$ denote the set of zeros of the A-Chebyshev polynomial. We will analyse the structure of the set $\liminf Z_{n,A}$, when n tends to infinity, depending on A. $\liminf Z_{n,A}$ consists of those elements which are limits of points in $Z_{n,A}$ for all n. That is, $x \in \liminf Z_{n,A}$ if and only if there exists a sequence of points $\{x_k\}$ such that $x_k \in Z_{k,A}$ and $x_k \to x$ as $k \to \infty$.

2010 *Mathematics Subject Classification:* Primary 11B83; Secondary 11R09, 12D10.
Key words and phrases: Chebyshev polynomials, envelope, Pisot numbers, Salem numbers.
Partially supported by Serbian Ministry of Education and Science, Project 174032.
Communicated by Žarko Mijajlović.

Example 2.1. Using our notation, the Chebyshev polynomial $T_n(x)$ is A-Chebyshev polynomial with $A = 1$. It is well known [**5**] that the zeros of $T_n(x)$ are $x_{n,k} = \cos\frac{(n-k+1/2)\pi}{n}$ $(k = 1, 2, \ldots, n)$. It is obviously that $x_{n,1}$ approaches to -1 and $x_{n,n}$ approaches to 1 when $n \to \infty$. Since $x_{n,k}$ are equispaced, it is clear that $\liminf Z_{n,1} = [-1, 1]$.

Example 2.2. What can we say for A-Chebyshev polynomial if $A = (2, -5, 2)$? It is well known [**5**] that $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, $n = 2, 3, \ldots$. So

$$\begin{aligned}
T_{n,(2,-5,2)}(x) &= 2T_n(x) - 5T_{n-1}(x) + 2T_{n-2}(x)\\
&= 4xT_{n-1}(x) - 2T_{n-2}(x) - 5T_{n-1}(x) + 2T_{n-2}(x),\\
T_{n,(2,-5,2)}(x) &= (4x - 5)T_{n-1}(x).
\end{aligned}$$

Now it is obvious that $x = \frac{5}{4}$ is a zero of $T_{n,(2,-5,2)}(x)$, for all $n = 2, 3, \ldots$. So $x = \frac{5}{4} \in Z_{n,(2,-5,2)}$ for all $n = 2, 3, \ldots$. Taking into account the previous example we conclude that $\liminf Z_{n,(2,-5,2)} = [-1, 1] \cup \left\{\frac{5}{4}\right\}$.

Lemma 2.1. *We have* $T_{n,A}(x) = \frac{1}{2}(P_A(w)w^{n-m} + P_A(w^{-1})w^{-(n-m)})$ *where* $w = x + \sqrt{x^2 - 1}$.

Proof. Starting from the definition of A-Chebyshev polynomial and using the well known formula $T_n(x) = \frac{1}{2}(w^n + w^{-n})$ [**5**] we have

$$\begin{aligned}
T_{n,A}(x) &= \sum_{i=0}^{m} a_i T_{n-i}(x) = \sum_{i=0}^{m} a_i \frac{1}{2}(w^{n-i} + w^{-n+i})\\
&= \frac{1}{2}\left(\sum_{i=0}^{m} a_i w^{n-i} + \sum_{i=0}^{m} a_i w^{-n+i}\right)\\
&= \frac{1}{2}\left(w^{n-m}\sum_{i=0}^{m} a_i w^{m-i} + w^{-n+m}\sum_{i=0}^{m} a_i w^{-m+i}\right)\\
&= \frac{1}{2}\big(w^{n-m}P_A(w) + w^{-n+m}P_A(w^{-1})\big). \qquad \square
\end{aligned}$$

One can calculate that if $w = x + \sqrt{x^2 - 1}$, then $x = \frac{1}{2}(w + w^{-1})$. So, we can deduce from the previous lemma the following

Corollary 2.1. *If there is w such that* $P_A(w) = P_A(w^{-1}) = 0$, *then* $T_{n,A}(x) = 0$ *for* $x = \frac{1}{2}(w + w^{-1})$ *and for all* $n \geqslant m$.

From the previous example we can see that 2 and $\frac{1}{2}$ are the roots of $P_A(x) = 2x^2 - 5x + 2$, therefore $x = \frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4}$ is a zero of $T_{n,A}(x)$ for all $n \geqslant 2$.

For the next corollary we need the following definition: the set T of Salem numbers is the set of real algebraic integers τ greater than 1, such that all its conjugate roots have modules at most 1, at least one of them having modulus equal to 1.

Corollary 2.2. *If* τ *is a Salem number and* $P_A(x)$ *is its minimal polynomial, then* $T_{n,A}(x) = 0$ *for* $x = \frac{1}{2}(\tau + \tau^{-1})$ *and for all* $n \geqslant m$.

The claim is a direct consequence of a well known property of a Salem number [**2**] that $P_A(\tau) = P_A(\tau^{-1}) = 0$.

Theorem 2.1. *If there is a root ω, out of the unit circle, of the polynomial P_A, that is $P_A(\omega) = 0$, $|\omega| > 1$, then for every real number $\varepsilon > 0$, there exists a natural number n_0 such that for all $n > n_0$, there is a root ξ of the A-Chebyshev polynomial $T_{n,A}(x)$ such that $|\xi - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$.*

Proof. It is convenient to use Lemma 2.1 to express $T_{n,A}(x) = \frac{1}{2}P_A(w)w^{n-m} + \frac{1}{2}P_A(w^{-1})w^{-(n-m)}$ where $w = x + \sqrt{x^2-1}$, or equivalently $x = x(w) = \frac{1}{2}(w + w^{-1})$. Since $x(w)$ is continuous for $w > 0$, there is $\delta_1 > 0$ such that if $|w - \omega| < \delta_1$, then $|\frac{1}{2}(w + w^{-1}) - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$. We can take a $\delta_2 < |\omega| - 1$ such that, in the circle $\{z : |z - \omega| \leqslant \delta_2\}$, there is no root of $P_A(w)$ which is different from ω. Let $\delta = \min(\delta_1, \delta_2)$ and $C = \{z : |z - \omega| \leqslant \delta\}$. Since ∂C, the boundary of C is a compact set, $|P_A(w)|$, $|P_A(w^{-1})|$ are continuous on ∂C, there is $w_{\min}$ where $|P_A(w)|$ attains its minimum, and $w_{\max}$ where $|P_A(w^{-1})|$ attains its maximum on ∂C. Since $\frac{1}{2}|P_A(w_{\max}^{-1})|$ is constant and $|\omega| - \delta > 1$, there is n_0 such that $\frac{1}{2}|P_A(w_{\min})|(|\omega| - \delta)^{n_0-m} > \frac{1}{2}|P_A(w_{\max}^{-1})|$. For $n \geqslant n_0$, let us denote $f(w) = \frac{1}{2}P_A(w)w^{n-m}$, $g(w) = \frac{1}{2}P_A(w^{-1})w^{-(n-m)}$. This notation corresponds to Rouché's theorem which we intend to use. We have to prove that $|f(w)| > |g(w)|$ on ∂C. Since $|w| \geqslant |\omega| - \delta > 1$ we have on ∂C

$$\begin{aligned} |f(w)| &= \frac{1}{2}|P_A(w)||w|^{n-m} \geqslant \frac{1}{2}|P_A(w_{\min})|(|\omega| - \delta)^{n_0-m} \\ &> \frac{1}{2}|P_A(w_{\max}^{-1})| \geqslant \frac{1}{2}|P_A(w^{-1})||w|^{-(n-m)}| = |g(w)|. \end{aligned}$$

The conditions in Rouché's theorem are thus satisfied. Consequently, since $f(w)$ has root ω, we conclude that $f(w)+g(w)$ has a root, say ω_1, inside the circle C. Clearly, since $|\omega_1 - \omega| < \delta_1$, if we denote $\xi = \frac{1}{2}(\omega_1 + \omega_1^{-1})$, we conclude $|\xi - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$. Finally, we conclude that

$$T_{n,A}(\xi) = \frac{1}{2}P_A(\omega_1)\omega_1^{n-m} + \frac{1}{2}P_A(\omega_1^{-1})\omega_1^{-(n-m)} = f(\omega_1) + g(\omega_1) = 0. \qquad \square$$

Theorem 2.2. *If $x \in [-1, 1]$, then for every real number $\varepsilon > 0$, there exists a natural number n_0 such that for all $n > n_0$, there is a root ξ of the A-Chebyshev polynomial $T_{n,A}(x)$ such that $|x - \xi| < \varepsilon$.*

Proof. Directly from the definitions of Chebyshev and A-Chebyshev polynomials, we can show that

$$T_{n,A}(x) = a_0 \cos n\theta + a_1 \cos(n-1)\theta + \cdots + a_m \cos(n-m)\theta, \quad n \geqslant m, \tag{2.1}$$

when $x = \cos\theta$. Since

$$\begin{aligned} a_k \cos(n-k)\theta &= a_k \cos(n-m+m-k)\theta \\ &= a_k(\cos(n-m)\theta\cos(m-k)\theta - \sin(n-m)\theta\sin(m-k)\theta) \end{aligned}$$

the equation $T_{n,A}(x) = 0$ is equivalent with

$$\cos(n-m)\theta \sum_{k=0}^{m} a_k \cos(m-k)\theta = \sin(n-m)\theta \sum_{k=0}^{m} a_k \sin(m-k)\theta.$$

Finally we get

$$\tan(n-m)\theta = \frac{\sum_{k=0}^{m} a_k \cos(m-k)\theta}{\sum_{k=0}^{m} a_k \sin(m-k)\theta}.$$

The function on the right-hand side, denote it $R(\theta)$, does not depend on n. The graph of $\tan(n-m)\theta$ consists of parallel equispaced tangents branches. So if we take $n := 2n-m$, we double $n-m$ and get a new graph which is actually the union of the old one with branches settled in the middle of each pair of neighbouring branches of the old graph. We conclude that all roots of $\tan(n-m)\theta = R(\theta)$, remain to be roots of $\tan 2(n-m)\theta = R(\theta)$, and new roots interlace with old. Finally, changing variables $\theta = \arccos x$ will preserve order and denseness of the roots. $\square$

3. Envelope of an A-Chebyshev polynomial

Let us observe the Chebishev polynomial $T_n(x)$ again. It is well known that, for any n, the graph of the polynomial oscillates between -1 and 1 when $x \in [-1, 1]$. As n increases, we have more and more oscillations. Something like that we have in the case of an A-Chebyshev polynomial.

Example 3.1. Let $A = (1, 0, 1)$, so

$$T_{n,A}(x) = T_n(x) + T_{n-2}(x) = 2xT_{n-1}(x) - T_{n-2}(x) + T_{n-2}(x) = 2xT_{n-1}(x).$$

Now it is obvious that $T_{n,A}(x)$ oscillates between the lines $y = \pm 2x$, for $x \in [-1, 1]$. We will refer to these lines as an envelope of the A-Chebyshev polynomial.

Using (2.1) we can study the following

Example 3.2. Let $A = (1, 0, -1)$, so

$$\begin{aligned} T_{n,A}(x) &= \cos n\theta - \cos((n-2)\theta) = -2\sin((n-1)\theta)\sin\theta \\ &= -2\sin((n-1)\theta)\sqrt{1-\cos^2\theta} = -2\sin((n-1)\theta)\sqrt{1-x^2}. \end{aligned}$$

Now it is obvious that $T_{n,A}(x)$ oscillates between the upper and lower halves of the ellipse $y = \pm 2\sqrt{1-x^2}$, for $x \in [-1, 1]$. These halves constitute the envelope of the A-Chebyshev polynomial in this case.

With the same technique we can find the envelope in the next

Example 3.3. Let $A = (1, -1)$, so

$$\begin{aligned} T_{n,A}(x) &= \cos n\theta - \cos((n-1)\theta) = -2\sin((n-\tfrac{1}{2})\theta)\sin\tfrac{\theta}{2} \\ &= -2\sin((n-\tfrac{1}{2})\theta)\sqrt{(1-\cos\theta)/2} = -\sqrt{2}\sin((n-\tfrac{1}{2})\theta)\sqrt{1-x}. \end{aligned}$$

Now it is obvious that the envelope of the A-Chebyshev polynomial is a parabola $y = \pm\sqrt{2}\sqrt{1-x}$, for $x \in [-1, 1]$.

Using previous examples, we can formulate the characteristics that an envelope of the A-Chebyshev polynomial must have.

(Env1) The envelope depends only on A. If A is fixed, it is unique for $T_{n,A}(x)$, $n \in \mathbb{N}$.
(Env2) The envelope is a non negative function.
(Env3) The A-Chebyshev polynomial is not greater in modulus than the envelope, $x \in [-1, 1]$.
(Env4) The envelope is a smooth function except at its zeros.
(Env5) If the envelope and the A-Chebyshev polynomial have equal positive value in x, they have also equal the first derivative in x.

We define the envelope of an A-Chebyshev polynomial as a function satisfying (Env1)–(Env5). It is naturally to ask how we can find the envelope for an A-Chebyshev polynomial. The next lemma will be useful.

LEMMA 3.1. *Let $R(t)$, $I(t)$ be real differentiable functions of real argument, with $R'(t)$, $I'(t)$ continuous, $E(t) = \sqrt{R^2(t) + I^2(t)}$, $t \in \mathbb{R}$. Then*

(i) $|R(t)| \leqslant E(t)$,
(ii) $|R(t)| = E(t)$ *if and only if* $I(t) = 0$,
(iii) *if* $I(t) = 0$ *and* $R(t) > 0$, *then* $R(t) = E(t)$ *and* $R'(t) = E'(t)$.

PROOF. The first and the second statements are straightforward. To demonstrate that $R'(t) = E'(t)$, we need to determinate $E'(t) = \frac{2R(t)R'(t)+2I(t)I'(t)}{2\sqrt{R^2(t)+I^2(t)}}$. Using $I(t) = 0$, $R(t) > 0$ we get the claim. $\square$

THEOREM 3.1. *The envelope $E_A(x)$ for an A-Chebyshev polynomial $T_{n,A}(x)$ is the square root of the modulus of*

$$\sum_{i=0}^{m} a_i^2 + 2\sum_{i=0}^{m-1} a_i a_{i+1} T_1(x) + 2\sum_{i=0}^{m-2} a_i a_{i+2} T_2(x) + \cdots$$
$$\cdots + 2\sum_{i=0}^{m-k} a_i a_{i+k} T_k(x) + \cdots + 2a_0 a_m T_m(x),$$

or in more compact form

$$E_A(x) = \left| \sum_{i=0}^{m} \sum_{k=0}^{m} a_i a_k T_{|i-k|}(x) \right|^{1/2}.$$

PROOF. Let $z_A(t) = a_0 \cos(nt) + a_1 \cos((n-1)t) + \cdots + a_m \cos((n-m)t) + i(a_0 \sin(nt) + a_1 \sin((n-1)t) + \cdots + a_m \sin((n-m)t))$ be an auxiliary function on $t \in \mathbb{R}$. We can see that $T_{n,A}(x) = Re(z_A(t))$ so $|T_{n,A}(x)|^2 \leqslant |z_A(t)|^2$, $x = \cos(t)$. We will show that $E_A(x) = |z(t)|$.

$$|z_A(t)|^2 = \left(\sum_{k=0}^{m} a_k \cos((n-k)t)\right)^2 + \left(\sum_{k=0}^{m} a_k \sin((n-k)t)\right)^2$$

$$= \sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k \cos((n-i)t)\cos((n-k)t)$$
$$+ \sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k \sin((n-i)t)\sin((n-k)t)$$
$$= \sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k (\cos((n-i)t)\cos((n-k)t) + \sin((n-i)t)\sin((n-k)t))$$
$$= \sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k \cos((i-k)t).$$

Substituting $x = \cos(t)$ in $\cos((i-k)t)$ we get $T_{|i-k|}(x)$. Since $E_A(x)$ does not depend on n, (Env1) is fulfilled. (Env2), (Env3), (Env4) are straightforward. Using the previous lemma if $R(t) = \mathrm{Re}(z(t))$, $I(t) = \mathrm{Im}(z(t))$, we can easily obtain (Env5). □

It is useful to calculate the envelope of the A-Chebyshev polynomial for $m = 1, 2, 3, 4$. Actually, we give the calculation of the square of the envelope, to avoid cumbersome square roots. Using the previous formula we get

(m =1) $a_0^2 + a_1^2 + 2a_0a_1x$;

($m = 2$) $a_0^2 + a_1^2 + a_2^2 + 2(a_0a_1 + a_1a_2)x + 2a_0a_2(2x^2 - 1) =$
$a_0^2 + a_1^2 + a_2^2 - 2a_0a_2 + (2a_0a_1 + 2a_1a_2)x + 4a_0a_2x^2$;

(m =3) $a_0^2 + a_1^2 + a_2^2 + a_3^2 + 2(a_0a_1 + a_1a_2 + a_2a_3)x$
$+2(a_0a_2 + a_1a_3)(2x^2 - 1) + 2a_0a_3(4x^3 - 3x) =$
$a_0^2 + a_1^2 + a_2^2 + a_3^2 - 2a_0a_2 - 2a_1a_3 + (2a_0a_1 + 2a_1a_2 + 2a_2a_3 - 6a_0a_3)x +$
$(4a_0a_2 + 4a_1a_3)x^2 + 8a_0a_3x^3$;

($m = 4$) $a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2(a_0a_1 + a_1a_2 + a_2a_3 + a_3a_4)x + 2(a_0a_2 + a_1a_3 +$
$a_2a_4)(2x^2 - 1) + 2(a_0a_3 + a_1a_4)(4x^3 - 3x) + 2a_0a_4(8x^4 - 8x^2 + 1) =$
$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_0a_2 - 2a_1a_3 - 2a_2a_4 + (2a_0a_1 + 2a_1a_2 + 2a_2a_3 +$
$a_3a_4 - 6a_0a_3 - 6a_1a_4)x + (4a_0a_2 + 4a_1a_3 + 4a_2a_4 - 16a_0a_4)x^2 + (8a_0a_3 +$
$8a_1a_4)x^3 + 16a_0a_4x^4$.

Remark 3.1. There is a relation with the theory of signal processing. The analytic signal $z(t)$ can be expressed in terms of complex polar coordinates, $z(t) = f(t) + i\hat{f}(t) = A(t)e^{i\phi(t)}$ where $A(t) = (f^2(t) + \hat{f}^2(t))^{1/2}$, and $\phi(t) = \arctan\frac{\hat{f}(t)}{f(t)}$. These functions are respectively called the amplitude envelope and instantaneous phase of the signal, $\hat{f}(t)$ is the Hilbert transform of $f(t)$.

4. The relation between the envelope and the characteristic polynomial

Until now we have used the envelope to describe the graph of the A-Chebyshev polynomial if x is of modulus not greater than 1. If $|x| > 1$ we have preferred the characteristic polynomial $P_A(x)$. It is natural to ask, if there is any relation between the envelope and the characteristic polynomial of the A-Chebyshev polynomial. The following theorem shows that the answer is affirmative.

THEOREM 4.1. *The envelope of the A-Chebyshev polynomial is the function*

$$E_A(x) = \Big|P_A\big(x+\sqrt{x^2-1}\big)P_A\big(x-\sqrt{x^2-1}\big)\Big|^{1/2}.$$

PROOF. We shall start from the compact form of the envelope given in Theorem 3.1 and use well known [**5**] formula $T_n(x) = \frac{1}{2}(w^n + w^{-n})$ where $w = x + \sqrt{x^2-1}$.

$$\begin{aligned} E_A(x) &= \left|\sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k T_{|i-k|}(x)\right|^{1/2} = \left|\sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k \frac{1}{2}(w^{i-k} + w^{-(i-k)})\right|^{1/2} \\ &= \left|\sum_{K=0}^{m}\sum_{I=0}^{m} \frac{1}{2} a_K a_I w^{K-I} + \sum_{i=0}^{m}\sum_{k=0}^{m} \frac{1}{2} a_i a_k w^{k-i}\right|^{1/2} \end{aligned}$$

(Here we renamed i with K and k with I in the first double sum. Now we shall switch the order of summing in the first double sum and apply obvious $a_K a_I = a_I a_K$.)

$$\begin{aligned} &= \left|\sum_{I=0}^{m}\sum_{K=0}^{m} \frac{1}{2} a_I a_K w^{K-I} + \sum_{i=0}^{m}\sum_{k=0}^{m} \frac{1}{2} a_i a_k w^{k-i}\right|^{1/2} \\ &= \left|2\sum_{i=0}^{m}\sum_{k=0}^{m} \frac{1}{2} a_i a_k w^{k-i}\right|^{1/2} = \left|\sum_{i=0}^{m}\sum_{k=0}^{m} a_i a_k w^{m-i} w^{-m+k}\right|^{1/2} \\ &= \left|\sum_{i=0}^{m} a_i w^{m-i} \sum_{k=0}^{m} a_k w^{-m+k}\right|^{1/2} \\ &= \big|P_A(w)P_A(w^{-1})\big|^{1/2} = \Big|P_A\big(x+\sqrt{x^2-1}\big)P_A\big(x-\sqrt{x^2-1}\big)\Big|^{1/2}. \quad \square \end{aligned}$$

Graphs of A-Chebyshev polynomials $T_{14,(1,0,0,1)}(x)$, $T_{44,(1,0,0,1)}(x)$ are showed on Figure 1 together with their common envelope $E(x) = \sqrt{|2+6x-8x^3|}$.

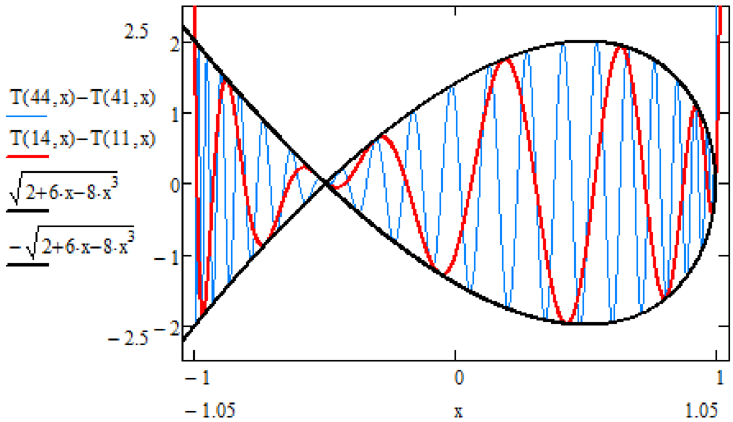

FIGURE 1. Graphs of A-Chebyshev polynomials $T_{14,(1,0,0,1)}(x)$, $T_{44,(1,0,0,1)}(x)$ and their common envelope $\pm|2+6x-8x^3|^{1/2}$.

5. A-Chebyshev polynomial of the second kind

It is well known [**6**] that the Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial in x of degree n, defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \text{ when } x = \cos\theta.$$

Let $A = (a_0, a_1, \dots, a_m)$ be a (m+1)-tuple of real numbers, $a_0, a_m \neq 0$, $m \geqslant 1$. We introduce an infinite sequence of polynomials

$$U_{n,A}(x) = a_0 U_n(x) + a_1 U_{n-1}(x) + \cdots + a_m U_{n-m}(x) \ \ (n \geqslant m).$$

We will refer to $U_{n,A}(x)$ as an A-Chebyshev polynomial of the second kind. We can naturally extend this definition in the case $m = 0$ and $A = a_0 \neq 0$:

$$U_{n,a0}(x) = a_0 U_n(x).$$

We will refer to the polynomial

$$P_A(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

as the characteristic polynomial of the A-Chebyshev polynomial.

Lemma 5.1. $U_{n,A}(x) = \frac{1}{w-w^{-1}}\big(w^{n+1-m}P_A(w) - w^{-n-1+m}P_A(w^{-1})\big)$ *where* $w = x + \sqrt{x^2-1}$.

Proof. Starting from the definition of A-Chebyshev polynomial and using well known [**5**] formula $U_n(x) = \frac{w^{n+1}-w^{-n-1}}{w-w^{-1}}$ we have

$$\begin{aligned}
U_{n,A}(x) &= \sum_{i=0}^{m} a_i U_{n-i}(x) = \sum_{i=0}^{m} a_i \frac{w^{n+1-i} - w^{-n-1+i}}{w - w^{-1}} \\
&= \frac{1}{w-w^{-1}}\left(\sum_{i=0}^{m} a_i w^{n+1-i} - \sum_{i=0}^{m} a_i w^{-n-1+i}\right) \\
&= \frac{1}{w-w^{-1}}\left(w^{n+1-m}\sum_{i=0}^{m} a_i w^{m-i} - w^{-n-1+m}\sum_{i=0}^{m} a_i w^{-m+i}\right) \\
&= \frac{1}{w-w^{-1}}\big(w^{n+1-m}P_A(w) - w^{-n-1+m}P_A(w^{-1})\big). \qquad \square
\end{aligned}$$

Theorem 5.1. *If there is a root ω, out of the unit circle, of the polynomial P_A, that is $P_A(\omega) = 0, |\omega| > 1$, then for every real number $\varepsilon > 0$, there exists a natural number n_0 such that for all $n > n_0$, there is a root ξ of the A-Chebyshev polynomial of the second kind $U_{n,A}(x)$ such that $|\xi - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$.*

Proof. It is convenient to use the previous lemma to express $U_{n,A}(x) = \frac{1}{w-w^{-1}}\big(w^{n+1-m}P_A(w) - w^{-n-1+m}P_A(w^{-1})\big)$ where $w = x + \sqrt{x^2-1}$ or equivalently $x = x(w) = \frac{1}{2}(w + w^{-1})$. Since $x(w)$ is continuous for $w > 0$, there is $\delta_1 > 0$ such that if $|w - \omega| < \delta_1$, then $|\frac{1}{2}(w + w^{-1}) - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$. We can take an $\delta_2 < |\omega| - 1$ such that, in the circle $\{z : |z - \omega| \leqslant \delta_2\}$, there is no root of $P_A(w)$ which is different from ω. Let $\delta = \min(\delta_1, \delta_2)$ and $C = \{z : |z - \omega| \leqslant \delta\}$. Since ∂C, the boundary of C, is a compact set, $|P_A(w)|$, $|P_A(w^{-1})|$ are continuous on ∂C, there is $w_{\min}$ where $|P_A(w)|$ gets its minimum and $w_{\max}$ where $|P_A(w^{-1})|$ gets its maximum on ∂C. Since $\frac{1}{w-w^{-1}}|P_A(w_{\max}^{-1})|$ is constant and $|\omega| - \delta > 1$, there is n_0 such that

$$\frac{1}{w-w^{-1}}|P_A(w_{\min})|(|\omega|-\delta)^{n_0+1-m} > \frac{1}{w-w^{-1}}|P_A(w_{\max}^{-1})|.$$

For $n \geqslant n_0$ let us denote

$$f(w) = \frac{1}{w-w^{-1}} w^{n+1-m} P_A(w), \quad g(w) = \frac{-1}{w-w^{-1}} w^{-n-1+m} P_A(w^{-1}).$$

This notation corresponds to Rouché's theorem which we intend to use. We have to prove that $|f(w)| > |g(w)|$ on ∂C. Since $|w| \geqslant |\omega| - \delta > 1$ we have on ∂C:

$$\begin{aligned}|f(w)| &= \frac{1}{w-w^{-1}}|P_A(w)||w|^{n+1-m} \geqslant \frac{1}{w-w^{-1}}|P_A(w_{\min})|(|\omega|-\delta)^{n_0+1-m}\\ &> \frac{1}{w-w^{-1}}|P_A(w_{\max}^{-1})| \geqslant \frac{1}{w-w^{-1}}|P_A(w^{-1})||w|^{-(n+1-m)}| = |g(w)|.\end{aligned}$$

The conditions in Rouché's theorem are thus satisfied. Consequently, since $f(w)$ has root ω, we conclude that $f(w) + g(w)$ has a root, let it be ω_1, inside the circle C. Clearly, since $|\omega_1 - \omega| < \delta_1$, if we denote $\xi = \frac{1}{2}(\omega_1 + \omega_1^{-1})$, we conclude $|\xi - \frac{1}{2}(\omega + \omega^{-1})| < \varepsilon$. Finally, we conclude that

$$\begin{aligned}U_{n,A}(\xi) &= \frac{1}{w-w^{-1}} P_A(\omega_1)\omega_1^{n+1-m} - \frac{1}{w-w^{-1}} P_A(\omega_1^{-1})\omega_1^{-(n+1-m)}\\ &= f(\omega_1) + g(\omega_1) = 0. \qquad \square\end{aligned}$$

6. An application in number theory

Recall that $q > 1$ is a Pisot number if q is an algebraic integer, whose other conjugates are of modulus strictly less than 1. Salem proved that every Pisot number is a limit point of the set T of Salem numbers. Let $Q(x) = x^m P(\frac{1}{x}) = a_0 + a_1 x + \cdots + a_m x^m$ be the reciprocal polynomial of the polynomial $P(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$. Salem showed that if $P(x)$ is the minimal polynomial of a Pisot number q, then $R_k(x) = x^k P(x) + Q(x)$ is a polynomial with a root τ_k that is a Salem number, and the limit of the sequence τ_k is q, $k \to \infty$. There is a relation between the Salem sequence $R_k(x)$ and the A-Chebyshev polynomials $T_{n,A}(x)$, the characteristic polynomial of which is $P(x)$. We have seen that

$$T_{n,A}(\tfrac{1}{2}(w + w^{-1})) = T_{n,A}(x) = \sum_{i=0}^{m} a_i T_{n-i}(x) = \sum_{i=0}^{m} a_i \tfrac{1}{2}(w^{n-i} + w^{-n+i}).$$

Now we can show that $2w^n T_{n,A}(\frac{1}{2}(w + w^{-1})) = R_{2n-m}(w)$. Really, we obtain

$$\begin{aligned}2w^n T_{n,A}(\tfrac{1}{2}(w + w^{-1})) &= \sum_{i=0}^{m} a_i (w^{2n-i} + w^i)\\ &= w^{2n-m} \sum_{i=0}^{m} a_i (w^{m-i}) + \sum_{i=0}^{m} a_i (w^i) = w^{2n-m} P(w) + Q(w).\end{aligned}$$

The question is what will happen if we use A-Chebyshev polynomial of the second kind instead of $T_{n,A}(x)$. We will demonstrate that one more sequence of Salem numbers, which converges to the Pisot number q, appears. It is obvious that

$$U_{n,A}(\tfrac{1}{2}(w+w^{-1})) = U_{n,A}(x) = \sum_{i=0}^{m} a_i U_{n-i}(x) = \sum_{i=0}^{m} \frac{a_i}{w-w^{-1}}(w^{n+1-i} - w^{-n-1+i})$$
$$= \sum_{i=0}^{m} a_i(w^{n-i} + w^{n-i-2} + w^{n-i-4} + \cdots + w^{-n+i+2} + w^{-n+i}).$$

We claim that $S_{2n}(w) = w^n U_{n,A}(\frac{1}{2}(w+w^{-1}))$ is a polynomial of degree $2n$ with a root τ_{2n} that is a Salem number, and the limit of the sequence τ_{2n} is q, $n \to \infty$. Using the previous theorem it is clear that there is a root τ_{2n} of $S_{2n}(w)$ such that $\tau_{2n} \to q$. It is obvious that $S_{2n}(w)$ is a reciprocal polynomial. It remains to be proved that all other roots of $S_{2n}(w)$ are in the unit circle. We shall apply the method Salem (communicated by Hirschman) used to prove the same property of his sequence $R_k(x)$ [**7**]. Using Lemma 5.1 we have

$$\begin{aligned} S_{2n}(w) &= w^n U_{n,A}(\tfrac{1}{2}(w+w^{-1})) \\ &= \frac{w^n}{w-w^{-1}}(w^{n+1-m}P_A(w) - w^{-n-1+m}P_A(w^{-1})) \\ &= \frac{w^{n+1}}{w^2-1}(w^{n+1-m}P_A(w) - w^{-n-1+m}P_A(w^{-1})) \\ &= \frac{1}{w^2-1}(w^{2n+2-m}P_A(w) - w^m P_A(w^{-1})) \\ &= \frac{1}{w^2-1}(w^{2n+2-m}P_A(w) - Q(w)). \end{aligned}$$

We denote by ϵ a positive number and consider the equation

$$(1+\epsilon)w^{2n+2-m}P_A(w) - Q(w) = 0.$$

Since for $|w| = 1$, we have $|P(w)| = |Q(w)|$, it follows by Rouché's theorem that inside the circle $|w| = 1$ the number of roots of the last equation is equal to the number of roots of $w^{2n+2-m}P(w)$, that is, $(2n+2-m)+m-1$. As $\epsilon \to 0$, these roots vary continuously. Hence, for $\epsilon = 0$ we have $2n+1$ roots with modulus $\leqslant 1$. It is obvious that two roots are $1, -1$ so the fraction can be reduced with w^2-1. Finally we conclude that at most one root of $S_{2n}(w)$ is outside the unit circle.

Example 6.1. Let q be the golden ratio, the greater root of $P(x) = x^2 - x - 1$. Then $R_k(w) = w^{k+2} - w^{k+1} - w^k - w^2 - w + 1$ is the Salem's sequence. Our sequence of Salem numbers τ_{2m} which converge to q consists of the greatest in modulus roots of the polynomials

$$S_{2n}(w) = \frac{1}{w^2-1}(w^{2n+2-m}P_A(w) - Q(w)) = \frac{1}{w^2-1}(w^{2n}(w^2 - w - 1) + w^2 + w - 1)$$
$$= (w^{2n} + w^{2n-2} + w^{2n-4} + \cdots + w^2 + 1) - (w^{2n-1} + w^{2n-3} + w^{2n-5} + \cdots + w^3 + w)$$
$$- (w^{2n-2} + w^{2n-4} + w^{2n-6} + \cdots + w^4 + w^2).$$

Finally

$$S_{2n}(w) = w^{2n} - (w^{2n-1} + w^{2n-3} + w^{2n-5} \cdots + w^3 + w) + 1.$$

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(Received 18 03 2014)
(Revised 25 02 2015)