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ON STURDY FRAME OF ABSTRACT ALGEBRAS

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Abstract. We introduce the notion of a sturdy frame of abstract algebras which is a common generalization of a sturdy semilattice of semigroups, the sum of lattice ordered systems, the strong distributive lattice of semirings, the sturdy frame of type $(2, 2)$ algebras and the strong b-lattice of semirings. Also, we give some properties and characterizations of the sturdy frame of abstract algebras. As an application, we study the sturdy distributive lattice of lattice ordered groups.

1. Introduction and preliminaries

The union of algebras of the same type has been studied by many algebraists. The sturdy semilattice of semigroups is introduced by Petrich in [**14**]. It is an important tool to study the structures of semigroups, for example, see [**15**]. Pastijn [**12**] introduces the sum of a lattice ordered system. Ghosh [**3**] and Guo, Sen, and Shum [**19**] introduce the concept of strong distributive lattice of semirings, respectively. By using this concept, Guo, Sen, and Shum [**5, 19**] study structures of idempotent semirings. Zhao, Guo, and Shum [**22**] introduce and study sturdy frame of type $(2, 2)$ algebras. By introducing strong b-lattice of semirings, Sen, Maity, and Shum [**20**] study generalized Clifford semirings. We introduce and study a sturdy frame of abstract algebras which is a common generalization of a sturdy semilattice of semigroups, the sum of lattice ordered systems, the strong distributive lattice of semirings, the sturdy frame of type $(2, 2)$ algebras and the strong b-lattice of semirings.

Throughout this paper, unless otherwise stated, we consider abstract algebras and terms of a fixed type $\mathcal{F}$ without nullary operation symbols. For an algebra $\mathbf{A}$, we shall denote the universe of $\mathbf{A}$ by A. Moreover, we shall write the symbols of mappings on the right and the symbols of operations on the left.

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An algebra $\mathbf{A}$ is called *idempotent* if $f^{\mathbf{A}}(a,\dots,a)=a$ for any n-ary $f\in\mathcal{F}$ and any $a\in A$. Bands, idempotent semirings and lattices are examples of idempotent algebras. By a *frame* $\mathbf{B}$ we mean an idempotent algebra endowed with an upper semilattice order $\leqslant$ satisfying $f^{\mathbf{B}}(b_1,\dots,b_n)\leqslant b_1\vee\cdots\vee b_n$ for any n-ary $f\in\mathcal{F}$ and any $b_1,\dots,b_n\in B$, where $b_1\vee\cdots\vee b_n=\mathrm{lub}\{b_1,\dots,b_n\}$. It is easy to see that semilattices and lattices are frames.

Let $\mathbf{B}$ be a frame and $\{\mathbf{A}_\alpha\mid\alpha\in B\}$ a family of pairwise disjoint algebras, indexed by B. For each pair α,β of elements of B such that $\alpha\leqslant\beta$, let $\varphi_{\alpha,\beta}:\mathbf{A}_\alpha\to\mathbf{A}_\beta$ be a monomorphism, and assume that

(a) $\varphi_{\alpha,\alpha}=1_{A_\alpha}$ for every $\alpha\in B$;
(b) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}=\varphi_{\alpha,\gamma}$ for every $\alpha,\beta,\gamma\in B$ such that $\alpha\leqslant\beta\leqslant\gamma$;
(c) If n-ary $f\in\mathcal{F}$ and $\alpha_1\vee\cdots\vee\alpha_n\leqslant\gamma$ for $\alpha_1,\dots,\alpha_n,\gamma\in B$, then

$$f^{\mathbf{A}_\gamma}(a_1\varphi_{\alpha_1,\gamma},\dots,a_n\varphi_{\alpha_n,\gamma})\in(\mathbf{A}_{f^{\mathbf{B}}(\alpha_1,\dots,\alpha_n)})\varphi_{f^{\mathbf{B}}(\alpha_1,\dots,\alpha_n),\gamma}$$

for any $a_i\in A_{\alpha_i}, 1\leqslant i\leqslant n$.

Let $A=\bigcup_{\alpha\in B}A_\alpha$, and define an n-ary operation f on A by

$$f^{\mathbf{A}}(a_1,\dots,a_n)=f^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\dots,a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{f^{\mathbf{B}}(\alpha_1,\dots,\alpha_n),\alpha}$$

for any $a_i\in A_{\alpha_i}, 1\leqslant i\leqslant n$, where $\alpha=\alpha_1\vee\cdots\vee\alpha_n$. Then we can check that $\mathbf{A}=\langle A,F\rangle$ is an algebra of type $\mathcal{F}$, denoted by $\mathbf{A}=[\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}]$. We call the constructed algebra $\mathbf{A}=[\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}]$ the *sturdy frame* of algebras $\mathbf{A}_\alpha$.

It is easy to see that the sturdy semilattice of semigroups introduced by Petrich [**14**], the sum of lattice ordered systems introduced by Pastijn [**12**], the strong distributive lattice of semirings introduced by Ghosh [**3**] and Guo, Sen and Shum [**19**], the strong b-lattice of semirings introduced by Sen, Maity and Shum [**20**] and the sturdy frames of type $(2,2)$ algebras introduced by Zhao, Guo and Shum [**22**] are all special cases of the sturdy frame of algebras. Thus the sturdy frame of algebras provides a new tool to investigate the structures of algebras.

If $\mathbf{A}=[\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}]$, then it is easy to see that every $\mathbf{A}_\alpha$ is a subalgebra of $\mathbf{A}$. Also, suppose that the frame $\mathbf{B}$ satisfies the additional condition

$$f^{\mathbf{B}}(\alpha_1,\dots\alpha_n)=\alpha_1\vee\cdots\vee\alpha_n$$

for any n-ary $f\in\mathcal{F}$ and any $\alpha_1,\dots,\alpha_n\in B$. Then the algebra $[\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}]$ coincides with the Płonka sum of the direct systems $\langle\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}\rangle$ [**17**].

For notations and terminologies not given in this paper, the reader is referred to Burris and Sankappanavar [**2**] and Grätzer [**6**] for information concerning universal algebra, to Howie [**8**] and Petrich [**15**] for a background on semigroup theory and to Hebisch and Weinert [**7**] for knowledge on semiring theory, respectively. We shall assume that the reader is familiar with the basic results in these areas.

2. Properties and characterizations of sturdy frame of algebras

In this section we give some properties and characterizations of sturdy frame of algebras.

Proposition 2.1. *Let* $\mathbf{A} = [\mathbf{B}, \leqslant; \mathbf{A}_\alpha, \varphi_{\alpha,\beta}]$ *and* t *an* n*-ary term. Then*

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}$$

for any $a_i \in A_{\alpha_i}$ $(1 \leqslant i \leqslant n)$*, where* $\alpha = \alpha_1 \vee \cdots \vee \alpha_n$*.*

Proof. We prove it by induction on $l(t)$ (the length of t). If $l(t) = 0$, then $t = x_i$ for some i. Further, we have that $t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$ and that

$$t^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha} = a_i\varphi_{\alpha_i,\alpha}\varphi^{-1}_{\alpha_i,\alpha} = a_i.$$

It follows that $t^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}$.

Suppose that $l(t) \geqslant 1$ and that the result holds for every term w with $l(w) < l(t)$. Then t is the form of $t = f(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n))$, where f is an k-ary operation symbol in $\mathcal{F}$. Since $l(t_i) < l(t)$, we must have that for any i $(1 \leqslant i \leqslant k)$,

$$t_i^{\mathbf{A}}(a_1, \ldots, a_n) = t_i^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_i^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}.$$

Put $\beta = t_1^{\mathbf{B}}(\alpha_1, \ldots, \alpha_n) \vee \cdots \vee t_k^{\mathbf{B}}(\alpha_1, \ldots, \alpha_n)$. We have

$$\begin{aligned}
&t^{\mathbf{A}}(a_1, \ldots, a_n)\varphi_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\\
&\quad= f^{\mathbf{A}}\big(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathbf{A}}(a_1, \ldots, a_n)\big)\varphi_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\\
&\quad= f^{\mathbf{A}}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\big)\varphi_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\\
&\quad= f^{\mathbf{A}_\beta}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}\varphi_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\\
&\quad= f^{\mathbf{A}_\beta}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}\varphi_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\beta}\varphi_{\beta,\alpha}\\
&\quad= f^{\mathbf{A}_\beta}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n,\beta)}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n,\beta)}\big)\varphi_{\beta,\alpha}\\
&\quad= f^{\mathbf{A}_\alpha}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n,\beta)}\varphi_{\beta,\alpha}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n,\beta)}\varphi_{\beta,\alpha}\big)\\
&\quad= f^{\mathbf{A}_\alpha}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_1^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}, \ldots,\\
&\qquad\qquad t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{t_k^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\big)\\
&\quad= f^{\mathbf{A}_\alpha}\big(t_1^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha}), \ldots, t_k^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha})\big)\\
&\quad= t^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \ldots, a_n\varphi_{\alpha_n,\alpha}).
\end{aligned}$$

Consequently, $t^{\mathbf{A}}(a_1,\ldots,a_n) = t^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\ldots,a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{t^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}$. $\square$

In the following we show that the sturdy frame of algebras can be represented as a subdirect product of two algebras.

Theorem 2.1. *Let* $\mathbf{A} = [\mathbf{B},\leqslant;\mathbf{A}_\alpha,\varphi_{\alpha,\beta}]$, $a \in A_\alpha$ *and* $b \in A_\beta$. *Define binary relations* ρ *and* θ *on* A *by*

$$(a,b) \in \rho \Leftrightarrow \alpha = \beta,$$
$$(a,b) \in \theta \Leftrightarrow a\varphi_{\alpha,\alpha\vee\beta} = b\varphi_{\beta,\alpha\vee\beta}.$$

Then ρ *and* θ *are congruences on* $\mathbf{A}$ *and* $\mathbf{A}$ *is a subdirect product of* $\mathbf{B}$ *and* $\mathbf{A}/\theta$. *If each* $\mathbf{A}_\alpha$ *satisfies an identity* $p \approx q$, *so does* $\mathbf{A}/\theta$.

Proof. It is easy to verify that ρ is a congruence on $\mathbf{A}$ and that $\mathbf{A}/\rho$ is isomorphic to $\mathbf{B}$. Also, it is clear that θ is reflexive and symmetric. To show that θ is transitive, let $a \in A_\alpha, b \in A_\beta$ and $c \in A_\gamma$ such that $(a,b) \in \theta$ and $(b,c) \in \theta$. Then $a\varphi_{\alpha,\alpha\vee\beta} = b\varphi_{\beta,\alpha\vee\beta}$ and $b\varphi_{\beta,\beta\vee\gamma} = c\varphi_{\gamma,\beta\vee\gamma}$. Further, we have

$$\begin{aligned}a\varphi_{\alpha,\alpha\vee\beta\vee\gamma} &= a\varphi_{\alpha,\alpha\vee\beta}\varphi_{\alpha\vee\beta,\alpha\vee\beta\vee\gamma} = b\varphi_{\beta,\alpha\vee\beta}\varphi_{\alpha\vee\beta,\alpha\vee\beta\vee\gamma} = b\varphi_{\beta,\alpha\vee\beta\vee\gamma}\\ &= b\varphi_{\beta,\beta\vee\gamma}\varphi_{\beta\vee\gamma,\alpha\vee\beta\vee\gamma} = c\varphi_{\gamma,\beta\vee\gamma}\varphi_{\beta\vee\gamma,\alpha\vee\beta\vee\gamma} = c\varphi_{\gamma,\alpha\vee\beta\vee\gamma}.\end{aligned}$$

It follows that

$$a\varphi_{\alpha,\alpha\vee\gamma} = a\varphi_{\alpha,\alpha\vee\beta\vee\gamma}\varphi^{-1}_{\alpha\vee\gamma,\alpha\vee\beta\vee\gamma} = c\varphi_{\gamma,\alpha\vee\beta\vee\gamma}\varphi^{-1}_{\alpha\vee\gamma,\alpha\vee\beta\vee\gamma} = c\varphi_{\gamma,\alpha\vee\gamma}$$

and so $(a,c) \in \theta$. This shows that θ is transitive. Thus θ is an equivalence on A.

Let $a_i \in A_{\alpha_i}$ and $b_i \in A_{\beta_i}$ such that $(a_i,b_i) \in \theta$, $1 \leqslant i \leqslant n$. Then $a_i\varphi_{\alpha_i,\alpha_i\vee\beta_i} = b_i\varphi_{\beta_i,\alpha_i\vee\beta_i}$. Put $\alpha = \alpha_1 \vee \cdots \vee \alpha_n$ and $\beta = \beta_1 \vee \cdots \vee \beta_n$. We have

$$a_i\varphi_{\alpha_i,\alpha_i\vee\beta_i}\varphi_{\alpha_i\vee\beta_i,\alpha\vee\beta} = b_i\varphi_{\beta_i,\alpha_i\vee\beta_i}\varphi_{\alpha_i\vee\beta_i,\alpha\vee\beta}.$$

This implies that $a_i\varphi_{\alpha_i,\alpha\vee\beta} = b_i\varphi_{\beta_i,\alpha\vee\beta}$.

Choose $\gamma = f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n) \vee f^{\mathbf{B}}(\beta_1,\ldots,\beta_n)$. We have

$$\begin{aligned}
&f^{\mathbf{A}}(a_1,\ldots,a_n)\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\gamma}\\
&\quad= f^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\ldots,a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\gamma}\\
&\quad= f^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\ldots,a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\ldots,a_n\varphi_{\alpha_n,\alpha})\varphi^{-1}_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha}\varphi_{\alpha,\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha},\ldots,a_n\varphi_{\alpha_n,\alpha})\varphi_{\alpha,\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_{\alpha\vee\beta}}(a_1\varphi_{\alpha_1,\alpha}\varphi_{\alpha,\alpha\vee\beta},\ldots,a_n\varphi_{\alpha_n,\alpha}\varphi_{\alpha,\alpha\vee\beta})\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_{\alpha\vee\beta}}(a_1\varphi_{\alpha_1,\alpha\vee\beta},\ldots,a_n\varphi_{\alpha_n,\alpha\vee\beta})\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_{\alpha\vee\beta}}(b_1\varphi_{\beta_1,\alpha\vee\beta},\ldots,b_n\varphi_{\beta_n,\alpha\vee\beta})\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_{\alpha\vee\beta}}(b_1\varphi_{\beta_1,\beta}\varphi_{\beta,\alpha\vee\beta},\ldots,b_n\varphi_{\beta_n,\beta}\varphi_{\beta,\alpha\vee\beta})\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= f^{\mathbf{A}_\beta}(b_1\varphi_{\beta_1,\beta},\ldots,b_n\varphi_{\beta_n,\beta})\varphi_{\beta,\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}\\
&\quad= (f^{\mathbf{A}_\beta}(b_1\varphi_{\beta_1,\beta},\ldots,b_n\varphi_{\beta_n,\beta})\varphi^{-1}_{f^{\mathbf{B}}(\beta_1,\ldots,\beta_n),\beta})\varphi_{f^{\mathbf{B}}(\beta_1,\ldots,\beta_n),\beta}\varphi_{\beta,\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}
\end{aligned}$$

$$= f^{\mathbf{A}}(b_1, \dots, b_n)\varphi_{f^{\mathbf{B}}(\beta_1,\dots,\beta_n),\beta}\varphi_{\beta,\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta}$$
$$= f^{\mathbf{A}}(b_1, \dots, b_n)\varphi_{f^{\mathbf{B}}(\beta_1,\dots,\beta_n),\alpha\vee\beta}\varphi^{-1}_{\gamma,\alpha\vee\beta} = f^{\mathbf{A}}(b_1, \dots, b_n)\varphi_{f^{\mathbf{B}}(\beta_1,\dots,\beta_n),\gamma}.$$

Thus $(f^{\mathbf{A}}(a_1, \dots, a_n), f^{\mathbf{A}}(b_1, \dots, b_n)) \in \theta$ and so θ is a congruence on $\mathbf{A}$. Notice that $\rho \cap \theta = \Delta$, where Δ is the equality relation. Hence $\mathbf{A}$ is a subdirect product of $\mathbf{B}$ and $\mathbf{A}/\theta$.

Let $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ be an identity. In the following we shall show that $\mathbf{A}$ satisfies $p \approx q$ if each $\mathbf{A}_\alpha$ satisfies $p \approx q$. In fact, we have

$$p^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = p^{\mathbf{A}/\theta}(a_1\varphi_{\alpha_1,\alpha}/\theta, \dots, a_n\varphi_{\alpha_n,\alpha}/\theta)$$
$$= p^{\mathbf{A}}(a_1\varphi_{\alpha_1,\alpha}, \dots, a_n\varphi_{\alpha_n,\alpha})/\theta = p^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \dots, a_n\varphi_{\alpha_n,\alpha})/\theta.$$

Similarly, $q^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = q^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \dots, a_n\varphi_{\alpha_n,\alpha})/\theta$. Since $\mathbf{A}_\alpha$ satisfies $p \approx q$, it follows that $p^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \dots, a_n\varphi_{\alpha_n,\alpha}) = q^{\mathbf{A}_\alpha}(a_1\varphi_{\alpha_1,\alpha}, \dots, a_n\varphi_{\alpha_n,\alpha})$. Thus

$$p^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = q^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta)$$

and so $\mathbf{A}/\theta$ satisfies $p \approx q$. □

Theorem 2.1 generalizes and enriches Lemma I.8.11 in [**14**], Lemma 2.6 in [**20**] and Theorem 2.2 in [**22**], respectively. If $\mathbf{A} = [\mathbf{B}, \leqslant; \mathbf{A}_\alpha, \varphi_{\alpha,\beta}]$, then it is easy to verify that $\mathbf{A}/\theta$ in which θ is defined in Theorem 2.1 is a direct limit of the family $\{\mathbf{A}_\alpha \mid \alpha \in B\}$ [**6**]. By Theorem 2.1, we can immediately have the following result, which generalizes and enriches Theorem 1.2 in [**3**], Lemma 3.2 in [**19**], Theorem 2.4 in [**20**] and Theorem 2.3 in [**22**], respectively.

Corollary 2.1. *Let* $\mathbf{A} = [\mathbf{B}, \leqslant; \mathbf{A}_\alpha, \varphi_{\alpha,\beta}]$ *and* $p \approx q$ *an identity. Then the following statements are equivalent:*

(i) $\mathbf{B}$ *and each algebra* $\mathbf{A}_\alpha$ *satisfy* $p \approx q$*;*
(ii) $\mathbf{B}$ *and* $\mathbf{A}/\theta$ *satisfy* $p \approx q$*;*
(iii) $\mathbf{A}$ *satisfies* $p \approx q$*.*

A variety is said to be a *frame variety* if every member in it can become a frame under some upper semilattice order. The variety of semilattices and the variety of lattices are examples of a frame variety. Every member in a variety V will be called a V-algebra. For a variety V and an algebra $\mathbf{A}$ there exists the smallest congruence ρ on $\mathbf{A}$ such that $\mathbf{A}/\rho$ is a V-algebra. This congruence will be called the *least* V*-congruence* on $\mathbf{A}$. The following theorem characterizes a sturdy frame of algebras by subdirect product decomposition.

Theorem 2.2. *Let* $\mathbf{A}$ *be an algebra, let* V *be a variety and* W *a frame variety. Assume that* τ_1 *is the least* V*-congruence on* $\mathbf{A}$ *and that* τ_2 *is the least* W*-congruence on* $\mathbf{A}$*. Then the following statements are equivalent:*

(i) $\mathbf{A}$ *is the subdirect product of* $\mathbf{A}/\tau_1$ *and* $\mathbf{A}/\tau_2$*;*
(ii) $\mathbf{A}$ *can be expressed as the sturdy frame* $[\mathbf{B}, \leqslant; \mathbf{A}_\alpha, \varphi_{\alpha,\beta}]$ *of* V*-algebras* $\mathbf{A}_\alpha$ *on frame* $\mathbf{B}$ *in* W*;*
(iii) $\mathbf{A}$ *is a subdirect product of a* V*-algebra and a* W*-algebra.*

Proof. (i) ⇒ (ii). By hypothesis, it follows that $\mathbf{A}/\tau_2$ can become a frame under the upper semilattice order $\leqslant$. We shall denote $\mathbf{A}/\tau_2$ by $\mathbf{B}$ and $\mathbf{A}/\tau_1$ by $\mathbf{C}$, respectively. For any $\alpha \in B$, let $\mathbf{A}_\alpha$ denote the algebra whose universe is $\{\alpha\} \times C \cap A$. It is easy to see that $\mathbf{A}_\alpha$ belongs to V. Also, it is routine to verify that A is the union of all A_α, every $\mathbf{A}_\alpha$ is a subalgebra of $\mathbf{A}$ and that $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\alpha, \beta, \gamma \in B$ such that $\alpha \leqslant \beta \leqslant \gamma$. Define a mapping $\varphi_{\alpha,\beta} : A_\alpha \to A_\beta$ by $(\alpha, c)\varphi_{\alpha,\beta} = (\beta, c)$ $((\alpha, c) \in A_\alpha)$. It is clear that $\varphi_{\alpha,\beta}$ is injective and that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. Moreover, for any $(\alpha, c_i) \in A_\alpha$ $(1 \leqslant i \leqslant n)$, we have

$$\begin{aligned} f^{\mathbf{A}_\alpha}((\alpha, c_1), \ldots, (\alpha, c_n))\varphi_{\alpha,\beta} &= (f^{\mathbf{B}}(\alpha, \ldots, \alpha), f^{\mathbf{C}}(c_1, \ldots, c_n))\varphi_{\alpha,\beta} \\ &= (\alpha, f^{\mathbf{C}}(c_1, \ldots, c_n))\varphi_{\alpha,\beta} = (\beta, f^{\mathbf{C}}(c_1, \ldots, c_n)) \\ &= (f^{\mathbf{B}}(\beta, \ldots, \beta), f^{\mathbf{C}}(c_1, \ldots, c_n)) = f^{\mathbf{A}_\beta}((\beta, c_1), \ldots, (\beta, c_n)) \\ &= f^{\mathbf{A}_\beta}((\alpha, c_1)\varphi_{\alpha,\beta}, \ldots, (\alpha, c_n)\varphi_{\alpha,\beta}). \end{aligned}$$

This shows that $\varphi_{\alpha,\beta}$ is a monomorphism.

If $\alpha_1 \vee \cdots \vee \alpha_n \leqslant \gamma$, then

$$\begin{aligned} f^{\mathbf{A}_\gamma}((\alpha_1, c_1)\varphi_{\alpha_1,\gamma}, \ldots, (\alpha_n, c_n)\varphi_{\alpha_n,\gamma}) &= f^{\mathbf{A}_\gamma}((\gamma, c_1), \ldots, (\gamma, c_n)) \\ &= (f^{\mathbf{B}}(\gamma, \ldots, \gamma), f^{\mathbf{C}}(c_1, \ldots, c_n)) = (\gamma, f^{\mathbf{C}}(c_1, \ldots, c_n)) \\ &= (f^{\mathbf{B}}(\alpha_1, \ldots, \alpha_n), f^{\mathbf{C}}(c_1, \ldots, c_n))\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\gamma} \\ &\in (\mathbf{A}_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n)})\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\gamma}. \end{aligned}$$

Put $\alpha = \alpha_1 \vee \cdots \vee \alpha_n$. We have

$$\begin{aligned} & f^{\mathbf{A}}((\alpha_1, c_1), \ldots, (\alpha_n, c_n))\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha} \\ &\quad = f^{\mathbf{B}\times\mathbf{C}}((\alpha_1, c_1), \ldots, (\alpha_n, c_n))\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha} \\ &\quad = (f^{\mathbf{B}}(\alpha_1, \ldots, \alpha_n), f^{\mathbf{C}}(c_1, \ldots, c_n))\varphi_{f^{\mathbf{B}}(\alpha_1,\ldots,\alpha_n),\alpha} \\ &\quad = (\alpha, f^{\mathbf{C}}(c_1, \ldots, c_n)) = (f^{\mathbf{B}}(\alpha, \ldots, \alpha), f^{\mathbf{C}}(c_1, \ldots, c_n)) \\ &\quad = f^{\mathbf{A}_\alpha}((\alpha, c_1), \ldots, (\alpha, c_n)) = f^{\mathbf{A}_\alpha}((\alpha_1, c_1)\varphi_{\alpha_1,\alpha}, \ldots, (\alpha_n, c_n)\varphi_{\alpha_n,\alpha}). \end{aligned}$$

Thus $\mathbf{A}$ can be expressed as the sturdy frame $[\mathbf{B}, \leqslant; \mathbf{A}_\alpha, \varphi_{\alpha,\beta}]$ of algebras $\mathbf{A}_\alpha$ in V on frame $\mathbf{B}$ in W.

(ii) ⇒ (iii). This follows from Theorem 2.1 immediately.

(iii) ⇒ (i). Assume that $\mathbf{A}$ is a subdirect product of a V-algebra and a W-algebra. Then there exist congruences $\tilde{\tau}_1$ and $\tilde{\tau}_2$ on $\mathbf{A}$ such that $\mathbf{A}/\tilde{\tau}_1 \in V$, $\mathbf{A}/\tilde{\tau}_2 \in W$ and $\tilde{\tau}_1 \cap \tilde{\tau}_2 = \Delta$. Since $\tau_1 \subseteq \tilde{\tau}_1$ and $\tau_2 \subseteq \tilde{\tau}_2$, it follows that $\tau_1 \cap \tau_2 = \Delta$. Thus $\mathbf{A}$ is the subdirect product of $\mathbf{A}/\tau_1$ and $\mathbf{B}/\tau_2$. □

As a corollary, we have the following corresponding result for semigroups, which generalizes some results obtained by Petrich and Reilly [**16**].

Corollary 2.2. *Let S be a semigroup and V a semigroup variety. Then S is a sturdy semilattice of semigroups in V if and only if S is a subdirect product of a semilattice and a semigroup in V.*

By a semiring we mean an algebra $(S,+,\cdot)$ with two binary operations $+$ and $\cdot$ such that both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y+z) \approx xy + xz, \quad (y+z)x \approx yx + zx.$$

A semiring $\mathbf{S}$ is said to be *idempotent* if both $(S,+)$ and $(S,\cdot)$ are bands. The class of all idempotent semirings whose additive reducts are semilattices will be denoted by $\mathbf{Sl}^+$. Let $S \in \mathbf{Sl}^+$. Define an upper semilattice order $\leqslant$ on S by

$$a \leqslant b \Leftrightarrow a + b = b. \tag{2.1}$$

By Lemma 3.3 in [**22**] we have that $(S,\leqslant)$ is a frame. The class of all idempotent semirings for which the two reducts are semilattices will be denoted by $\mathbf{Bi}$. Given $S \in \mathbf{Bi}$. Define an upper semilattice order $\leqslant$ on S by

$$a \leqslant b \Leftrightarrow ab = b. \tag{2.2}$$

It follows from Lemma 4.2 in [**22**] that $(S,\leqslant)$ is a frame. By Theorem 2.2 we can immediately obtain the following result for semirings.

Corollary 2.3. *Let S be a semiring and V a semiring variety. Then S is a sturdy frame of semirings in V on a frame which is described by* (2.1)/(2.2) *if and only if S is a subdirect product of a semiring in* $\mathbf{Sl}^+[\mathbf{Bi}]$ *and a semiring in V.*

It is clear that $\mathbf{Sl}^+$ coincides with the class of all b-lattices, which are introduced in [**20**] and that the variety of all distributive lattices is a subvariety of $\mathbf{Bi}$. Consequently, Corollary 2.3 extends and enriches some results obtained by Bandelt and Petrich [**1**], Ghosh [**3**], Guo, Sen, and Shum [**5, 19**], Sen, Maity, and Shum [**20**] and Shao and Zhao [**21**], respectively.

3. Sturdy distributive lattice of lattice ordered groups

As an application of sturdy frame of algebras, the sturdy distributive lattice of lattice ordered groups will be investigated in this section.

Recall [**8, 9**] that a partially ordered semigroup S is said to be a $\vee$-semilatticed semigroup if there exists the least upper bound $a \vee b$ for each pair of elements $a,\ b \in S$ and if the multiplication distributes over the join operation $\vee$, that is,

$$(\forall a,\ b,\ c \in S)\ a(b \vee c) = ab \vee ac \text{ and } (a \vee b)c = ac \vee bc.$$

In a dual way, we may consider $\wedge$-semilatticed semigroups. A $\vee$-semilatticed semigroup or a $\wedge$-semilatticed semigroup is simply called a semilatticed semigroup. In particular, if a partially ordered semigroup S is both a $\vee$-semilatticed semigroup and a $\wedge$-semilatticed semigroups, then S is called an lattice ordered semigroup. We denote by $(S,\vee,\ \wedge,\ \cdot)$ the lattice ordered semigroup S.

Suppose that $(S,\ \cdot)$ is an inverse semigroup. We denote by $\preceq$ the natural partial order on S. That is to say (Section II.4 in [**15**]),

$$a \preceq b \quad \Leftrightarrow \quad (\exists e, f \in E)\, a = be = fb$$

holds for any $a,\ b \in S$, in which E is the set of idempotents of $(S,\ \cdot)$. It is easy to verify that $(\forall a,b,c \in S)\ a \preceq b \Rightarrow ac \preceq bc,\ ca \preceq cb$. Suppose that $(S,\vee,\cdot)$

is a $\vee$-semilatticed semigroup. We denote by $\mathcal{L}$ ($\mathcal{R}$, $\mathcal{D}$ and $\mathcal{H}$, respectively) denotes Green's $\mathcal{L}$-relation ($\mathcal{R}$-relation, $\mathcal{D}$-relation, $\mathcal{H}$-relation, respectively) on the multiplicative reduct $(S, \cdot)$ of S.

Suppose that $(B, \vee, \cdot)$ is a $\vee$-semilatticed semilattice. If B satisfies the absorption law $x \vee xy \approx x$, then $(B, \vee, \cdot)$ is called a distributive lattice.

Suppose that $(S, \vee, \cdot)$ is a $\vee$-semilatticed inverse semigroup under the partial order $\leqslant$. Let $E(S)$ be the set of idempotents of the multiplicative reduct $(S, \cdot)$ of S, i.e., $E(S) = \{e \in S \mid e^2 = e\}$.

Thus we have directly the following result from Theorem 2.3 in [3].

THEOREM 3.1. *If $(S, \vee, \cdot)$ is a $\vee$-semilatticed inverse semigroup under the partial order $\leqslant$, then $(E(S), \vee, \cdot)$ is a semilatticed semilattice.*

When $E(S)$ is a distributive lattice for a $\vee$-semilatticed inverse semigroup S, we have

PROPOSITION 3.1. *If $(S, \vee, \cdot)$ is a $\vee$-semilatticed inverse semigroup under the partial order $\leqslant$, then the following conditions are equivalent:*

(i) *$\leqslant$ is an extension of $\preceq$ (i.e., $\preceq \subseteq \leqslant$);*
(ii) $(\forall e, f \in E(S))\, e \leqslant f \Leftrightarrow e \preceq f$;
(iii) $(\forall a, b \in S)\, a \vee ab^{-1}b = a$;
(iv) *$(E(S), \vee, \cdot)$ is a distributive lattice.*

PROOF. Suppose that $(S, \vee, \cdot)$ is a $\vee$-semilatticed inverse semigroup under the partial order $\leqslant$.

(i) $\Rightarrow$ (ii). Suppose that $e, f \in E(S)$. If $e \preceq f$, then it follows immediately from (i) that $e \leqslant f$. Conversely, if $e \leqslant f$, then pre-multiplying this by e, we have $e \leqslant ef$. On the other hand, since $ef \preceq e$, it follows from (i) that $ef \leqslant e$. Thus, $ef = e$ holds. That is to say, $e \preceq f$. This shows that $e \leqslant f \Leftrightarrow e \preceq f$, as required.

(ii) $\Rightarrow$ (i). Suppose that $a, b \in S$. If $a \preceq b$, then it follows from Proposition 5.2.1 in [**8**] that $b^{-1}a = a^{-1}a$, $aa^{-1} \preceq bb^{-1}$ and $a^{-1}a \preceq b^{-1}b$. Thus we have immediately from (ii) that $a^{-1}a \leqslant b^{-1}b$, $aa^{-1} \leqslant bb^{-1}$. Hence,

$$\begin{aligned} a &\leqslant bb^{-1}a \quad (\text{post-multiplying } aa^{-1} \leqslant bb^{-1} \text{ by } a) \\ &\leqslant ba^{-1}a \quad (b^{-1}a = a^{-1}a) \\ &\leqslant bb^{-1}b \quad (a^{-1}a \leqslant b^{-1}b) \\ &= b. \end{aligned}$$

That is to say, $a \leqslant b$. This shows that $\leqslant$ is an extension of $\preceq$.

(ii) $\Rightarrow$ (iii). It is clear that $a^{-1}ab^{-1}b \preceq a^{-1}a$ for any $a, b \in S$. Thus it follows directly from (*ii*) that $a^{-1}ab^{-1}b \leqslant a^{-1}a$. That is to say, $a^{-1}ab^{-1}b \vee a^{-1}a = a^{-1}a$. Premultiplying this by a, we have $a(a^{-1}ab^{-1}b \vee a^{-1}a) = ab^{-1}b \vee a = a(a^{-1}a) = a$. Hence, $ab^{-1}b \vee a = a$, as required.

(iii) $\Rightarrow$ (iv). It is clear from Theorem 3.1 that $(E(S), \vee, \cdot)$ is a semilatticed semilattice. Suppose that $e, f \in E(S)$. Then it follows directly from (*iii*) that $ef \vee e = e$ since $f^{-1} = f$. That is to say, $E(S)$ satisfies the absorption law and so it is a distributive lattice, as required.

(iv) $\Rightarrow$ (ii). Suppose that $E(S)$ is a distributive lattice and $e, f \in E(S)$. If $e \leqslant f$, then $e \vee f = f$. Pre-multiplying this by e, we can show $e \vee ef = ef$. Since the absorption law is satisfied in $E(S)$, it follows that $e = e \vee ef = ef$ and so $e \preceq f$. Conversely, if $e \preceq f$, then $ef = e$. This implies that $ef \vee f = e \vee f$. By using absorption law again, we have $f = ef \vee f = e \vee f$. That is to say, $e \leqslant f$. This shows that $e \leqslant f \Leftrightarrow e \preceq f$, as required. □

McAlister introduced amenable partial orders on inverse semigroups and studied amenable partially ordered inverse semigroups in [**10**], in which, amenable partial order is an extension of the natural partial order. McAlister gave the definition of amenable partial order on an inverse semigroup as follows.

Definition 3.1. Let $(S, \cdot, \leqslant)$ be a partially ordered inverse semigroup. The partial order $\leqslant$ is said to be a left(right) amenable partial order if it coincides with $\preceq$ on idempotents and for each $a, b \in S$, $a \leqslant b$ implies $a^{-1}a \preceq b^{-1}b\,(aa^{-1} \preceq bb^{-1})$. If $\leqslant$ is both a left amenable partial order and a right amenable partial order on S, then $\leqslant$ is called an amenable partial order and S is called an amenable partially ordered inverse semigroup.

Suppose that S is a Clifford semigroup. It is easy to see that both the left amenable partial order and the right amenable partial order coincide since Clifford semigroup satisfies $aa^{-1} = a^{-1}a$ for any $a \in S$. Thus we have

Lemma 3.1. *Suppose that $(S, \vee, \cdot)$ a $\vee$-semilatticed Clifford semigroup under the amenable partial order $\leqslant$. Then $(E(S), \vee, \cdot)$ is a distributive lattice and S satisfies*

$$(\forall a, b \in S)\,(a \vee b)^{-1}(a \vee b) = a^{-1}a \vee b^{-1}b.$$

Proof. Since $\leqslant$ is amenable, it follows from Definition that $\leqslant$ coincides with $\preceq$ on idempotents. By Propositon 3.1, we have that $E(S)$ is a distributive lattice.

Suppose that $a, b \in S$. It is clear that $a, b \leqslant a \vee b$. Since $\leqslant$ is left amenable, it follows that $a^{-1}a, b^{-1}b \preceq (a \vee b)^{-1}(a \vee b)$. Thus we have that $a^{-1}a, b^{-1}b \leqslant (a \vee b)^{-1}(a \vee b)$ and so $a^{-1}a \vee b^{-1}b \leqslant (a \vee b)^{-1}(a \vee b)$. On the other hand, it is obvious that $ab^{-1}b \preceq a, ba^{-1}a \preceq b$. It follows that $ab^{-1}b \leqslant a, ba^{-1}a \leqslant b$, since $\leqslant$ extends the natural partial order $\preceq$. Thus we have that $(a \vee b)(a^{-1}a \vee b^{-1}b) = a \vee ab^{-1}b \vee ba^{-1}a \vee b = a \vee b$. This implies that $(a \vee b)^{-1}(a \vee b)(a^{-1}a \vee b^{-1}b) = (a \vee b)^{-1}(a \vee b)$ and so $(a \vee b)^{-1}(a \vee b) \preceq a^{-1}a \vee b^{-1}b$. By Proposition 3.1, we have that $(a \vee b)^{-1}(a \vee b) \leqslant a^{-1}a \vee b^{-1}b$. This shows that $(a \vee b)^{-1}(a \vee b) = a^{-1}a \vee b^{-1}b$. □

Suppose that $(S, \vee, \cdot)$ a $\vee$-semilatticed Clifford semigroup under the amenable partial order $\leqslant$. Since the multiplicative reduct of S is a Clifford semigroup, it follows that $aa^{-1} = a^{-1}a$ for any $a \in S$. By Theorem II.1.4 in [**15**], we have that $\mathcal{H}$ is the least semilattice congruence of the Clifford semigroup $(S, \cdot)$ and every $\mathcal{H}$-class is a maximal subgroup of $(S, \cdot)$. For any $a \in S$, H_a denotes the $\mathcal{H}$-class containing a, and a^0 denotes the identity of subgroup H_a. It can be easily seen that $a\,\mathcal{H}\,b$ if and only if $a^0 = b^0$ for any $a, b \in S$. Thus we have that $E(S) = \{a^0 \mid a \in S\}$. By Lemma 3.1 we have

Corollary 3.1. *Suppose that $(S, \vee, \cdot)$ a $\vee$-semilatticed Clifford semigroup under the partial order $\leqslant$. If $\leqslant$ is amenable then $(\forall a, b \in S)\,(a \vee b)^0 = a^0 \vee b^0$.*

By [**4**], we have

Lemma 3.2. *Suppose that $(G,\cdot,\leqslant)$ is a partially ordered group and $a,b\in G$. Then the following statements are equivalent:*

(i) *there exists the least upper bound $a\vee b$ of a and b;*
(ii) *there exists the greatest lower bound $a\wedge b$ of a and b;*
(iii) *there exists the least upper bound of a^{-1} and b^{-1};*
(iv) *there exists the greatest lower bound of a^{-1} and b^{-1}.*

In particular, if there exists $a\vee b$, then for any $c,d\in G$ we have

$$ca\vee cb=c(a\vee b),\quad ad\vee bd=(a\vee b)d$$
$$a\wedge b=(a^{-1}\vee b^{-1})^{-1},\quad a\wedge b=a(a\vee b)^{-1}b.$$

Thus, $(G,\vee,\wedge,\cdot)$ is a lattice ordered group.

Suppose that $(S,\vee,\wedge,\cdot)$ is a lattice ordered Clifford semigroup under the partial order $\leqslant$. If $\leqslant$ is amenable then $(S,\vee,\wedge,\cdot)$ is called an amenably lattice ordered Clifford semigroup. Thus we have

Theorem 3.2. *Suppose that $(S,\vee,\cdot)$ a $\vee$-semilatticed Clifford semigroup under the amenable partial order $\leqslant$. Then $(S,\vee,\wedge,\cdot)$ is an amenably lattice ordered Clifford semigroup.*

Proof. Suppose that $a,b\in S$. It is easy to see that $(ab^0,ba^0)\in\mathcal{H}$. That is to say $ab^0,ba^0\in H_{a^0b^0}$. By Corollary 3.1, we have that $H_{a^0b^0}$ is a $\vee$-semilatticed group, it follows from Lemma 3.2 that $H_{a^0b^0}$ is lattice ordered group. For any $x,y\in H_{a^0b^0}$ we denote by $x\wedge y$ the great lower bound of x and y. Thus, there exists an element $c\in H_{a^0b^0}$ is the great lower bound of ab^0 and ba^0, i.e., $c=b^0\wedge ba^0$. Hence we have from $\leqslant$ is amenable that $c\leqslant ab^0\leqslant a$ and $c\leqslant b$.

Suppose that $d\in S$ such that $d\leqslant a,b$. Then $d^0\preceq a^0,d^0\preceq b^0$. Thus we have that $d=db^0\leqslant ab^0,d=da^0\leqslant ba^0$ and so $d\leqslant ab^0\wedge ba^0$, that is to say $d\leqslant c$. This shows that $ab^0\wedge ba^0$ is the greatest lower bound of a and b. We denote by $a\wedge b$ the greatest lower bound of a and b. Thus we have that $a\wedge b=ab^0\wedge ba^0$. Furthermore, we have that

$$a\wedge b=ab^0\wedge ba^0=((ab^0)^{-1}\vee(ba^0)^{-1})^{-1}=(a^{-1}\vee b^{-1})^{-1}a^0b^0.$$

In the following we will prove that $(S,\vee,\wedge,\cdot)$ is a lattice ordered Clifford semigroup.

For any $c\in S$ we have that

$$\begin{aligned}ac\wedge bc&=[(ac)^{-1}\vee(bc)^{-1}]^{-1}(ac)^0(bc)^0=[c^{-1}(a^{-1}\vee b^{-1})]^{-1}a^0b^0c^0\\&=(a^{-1}\vee b^{-1})^{-1}ca^0b^0c^0=(a^{-1}\vee b^{-1})^{-1}a^0b^0c=(a\wedge b)c,\end{aligned}$$

Dually, we have that

$$\begin{aligned}ca\wedge cb&=[(ca)^{-1}\vee(cb)^{-1}]^{-1}(ca)^0(cb)^0=[(a^{-1}\vee b^{-1})c^{-1}]^{-1}a^0b^0c^0\\&=c(a^{-1}\vee b^{-1})^{-1}a^0b^0c^0=cc^0(a^{-1}\vee b^{-1})^{-1}a^0b^0=c(a\wedge b).\end{aligned}$$

This shows that $(S,\wedge,\cdot)$ is a $\wedge$-semiltticed semigroup.

Since $(S,\vee,\cdot)$ a $\vee$-semilatticed Clifford semigroup, it follows that $(S,\vee,\wedge,\cdot)$ is a lattice ordered Clifford semigroup. □

Suppose that $(S, \vee, \wedge, \cdot)$ is an amenably lattice ordered Clifford semigroup. It follows from Theorem 3.2 that $(a \wedge b)^0 = a^0 b^0 = a^0 \wedge b^0$ since $a \wedge b = ab^0 \wedge ba^0$ and $H_{a^0 b^0}$ is a lattice ordered group. Assume that $a, b \in S$ such that $(a, b) \in \mathcal{H}$. For any $c \in S$, we have that $(a \vee c)^0 = a^0 \vee c^0 = b^0 \vee c^0 = (b \vee c)^0$. This implies that $(a \vee c, b \vee c) \in \mathcal{H}$ and so $\mathcal{H}$ is a congruence on $(S, \vee)$. Similarly, we can obtain that $\mathcal{H}$ is a congruence on $(S, \wedge)$. This shows that $\mathcal{H}$ is a congruence on S. It follows that $S/\mathcal{H}$ is a distributive lattice. Also, we can define a binary relation σ on S as follows:

$$(\forall a, b \in S)\, (a, b) \in \sigma \Leftrightarrow (\exists e \in E(S))\, ae = be.$$

It follows from Proposition 5.3.2 in [**8**] that σ is the least group congruence on the multiplicative reduct of S. Assume that $a, b \in S$ and that $(a, b) \in \sigma$. Then there exists $e \in E(S)$ such that $ae = be$. For any $c \in S$, we have that $ae \vee ce = be \vee ce$. That is to say, $(a \vee c)e = (b \vee c)e$. This implies that $(a \vee c, b \vee c) \in \sigma$. Since $(S, \vee)$ is a semilattice, we also have that $(c \vee a, c \vee b) \in \sigma$. This shows that σ is a congruence on $(S, \vee)$. Similarly, we can obtain that σ is a congruence on $(S, \wedge)$. This shows that σ is a congruence on S. Thus we have that σ is a congruence on S. It follows that S/σ is a lattice ordered group.

Suppose that $(S, \vee, \wedge, \cdot)$ is an amenably lattice ordered Clifford semigroup. If the multiplicative reduct of S is E-unitary, then it follows from Corollary 4.3.6 in [**15**] that $\mathcal{H} \cap \sigma = \Delta$. By Theorem 2.2, Corollary 4.3.6 in [**15**] and Theorem 3.5 in [**21**] we have

THEOREM 3.3. *Let $(S, \vee, \wedge, \cdot)$ be an amenably lattice ordered Clifford semigroup. Then the following statements are equivalent:*

(i) *S is a sturdy distributive lattice of lattice ordered groups;*
(ii) *S is a subdirect product of a distributive lattice and a lattice ordered group;*
(iii) *the multiplicative reduct of S is E-unitary;*
(iv) *the multiplicative reduct of S is a sturdy semilattice of groups;*
(v) *the multiplicative reduct of S is a subdirect product of a semilattice and a group.*

REMARK 3.1. It is clear that both a lattice ordered Clifford semigroup and a lattice ordered group are algebras of type $(2, 2, 2)$. Also, a distributive lattice can be considered as an algebra of type $(2, 2, 2)$ whose multiplication and meet coincide. Thus Theorem 3.3 characterizes the amenably lattice ordered Clifford semigroup which can be expressed as a sturdy distributive lattice (as an algebra of type $(2, 2, 2)$) of lattice ordered groups.

References

1. H. J. Bandelt, M. Petrich, *Subdirect products of rings and distributive lattices*, Proc. Edinburgh Math. Soc. **25** (1982), 155–171.
2. S. Burris, H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, New York, 1981.
3. S. Ghosh, *A characterization of semirings which are subdirect products of a distributive lattice and a ring*, Semigroup Forum **59** (1999), 106–120.
4. M. S. Gomes, E. G. Donald, D. B. McAlister, *On a class of lattice ordered inverse semigroups*, J. Algebra **230**(1) (2000), 496–517.

5. Y. Q. Guo, K. P. Shum, M. K. Sen, *The semigroup structure of left Clifford semirings*, Acta Math. Sci. (English Ed.) **19**(4) (2003), 783–792.
6. G. Grätzer, *Universal Algebra*, Springer-Verlag, New York, 1979.
7. U. Hebisch, H. J. Weinert, *Semirings-Algebraic Theory and Applications in Computer Science*, World Scientific, Singapore, 1998.
8. J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press/Oxford University Press, New York, 1995.
9. M. Kuřil, L. Polák, *On varieties of semilattice-ordered semigroups*, Semigroup Forum **71** (2005), 27–48.
10. D. B. McAlister, *Amenable ordered inverse semigroup*, J. Algebra **65**(1) (1980), 118–146.
11. H. Mitsh, *Subdirect products of E-inversive semigroups*, J. Aust. Math. Soc. (Series A) **48** (1990), 66–78.
12. F. Pastijn, *Idempotent distributive semirings II*, Semigroup Forum **26** (1983), 151–166.
13. F. Pastijn, A. Romanowska, *Idempotent distributive semirings I*, Acta Sci. Math. (Szeged) **44**(3–4) (1983), 239–253.
14. M. Petrich, *Regular semigroups which are subdirect products of a band and a semilattice of groups*, Glasgow. Math. J. **14**(1) (1973), 27–49.
15. ———, *Lectures in Semigroups*, Akad. Verlag, Berlin, 1977.
16. M. Petrich, N. R. Reilly, *Completely Regular Semigroups*, Wiley, New York, 1999.
17. J. Płonka, *On a method of contruction of abstract algebras*, Fund. Math. **61** (1967), 183–189.
18. A. Romanowska, *Idempotent distributive semirings with a semilattice reduct*, Math. Japon. **27**(4) (1982), 483–493.
19. M. K. Sen, Y. Q. Guo, K. P. Shum, *On a class of idempotent semirings*, Semigroup Forum **60** (2000), 351–367.
20. M. K. Sen, S. K. Maity, K. P. Shum, *Clifford semirings and generalized Clifford semirings*, Taiwanese J. Math. **9**(3) (2005), 433–444.
21. Y. Shao, X. Z. Zhao, *Semirings which are distributive lattice of M-rectangular divieded-semirings*, Algebra Colloq. **20**(2) (2013), 243–250.
22. X. Z. Zhao, Y. Q. Guo, K. P. Shum, *Sturdy frames of type* $(2, 2)$ *algebras and their applications to semirings*, Fund. Math. **179**(1) (2003), 69–81.

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