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FIXED POINTS FOR ĆIRIĆ-G-CONTRACTIONS IN UNIFORM SPACES ENDOWED WITH A GRAPH

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Abstract. We investigate the notion of λ-generalized contractions introduced by Ćirić in uniform spaces endowed with a graph and discuss on the existence and uniqueness of fixed points for this type of contractions using the basic entourages.

1. Introduction and preliminaries

In [**6**], Ćirić introduced the notion of a λ-generalized contraction on a metric space X as follows:

$$d(Tx,Ty) \leqslant q(x,y)d(x,y) + r(x,y)d(x,Tx) + s(x,y)d(y,Ty) + t(x,y)\big(d(x,Ty)+d(y,Tx)\big) \qquad \big((x,y)\in X\big),$$

where q,r,s,t are nonnegative functions on $X \times X$ such that

$$\sup\big\{q(x,y)+r(x,y)+s(x,y)+2t(x,y) : x,y\in X\big\} = \lambda < 1.$$

Acharya [**1**] investigated some well-known types of contractions in uniform spaces and Rhoades [**10**] discussed λ-generalized type contractions in uniform spaces.

Recently, Jachymski [**8**] entered graphs in metric fixed point theory and generalized the Banach contraction principle in both metric and partially ordered metric spaces. For further works and results in metric and uniform spaces endowed with a graph, see, e.g., [**2, 3, 4, 5, 9**].

Here we investigate the notion of λ-generalized contractions in uniform spaces endowed with a graph and establish some results on the existence and uniqueness of fixed points via an entourage approach for this type of contractions. Despite the method given in [**5**] that the results therein may not be applied for (the partially ordered contractions induced by graph) their partially ordered counterparts, we will see that our contractions are both extensions of Ćirić–Reich–Rus operators

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in uniform spaces and may also be converted to the language of partially ordered metric or uniform spaces.

We start by reviewing a few basic notions in uniform spaces. For a widespread discussion on the uniform spaces, the reader can see, e.g., [**11**, pp. 238–277].

Suppose that X is a nonempty set and U and V are nonempty subsets of $X \times X$. We let

- $\Delta(X) = \{(x,x) : x \in X\}$ be the diagonal of X;
- $U^{-1} = \{(x,y) : (y,x) \in U\}$ be the inverse of U; and
- $U \circ V = \{(x,y) : \exists\, z \in X \text{ s.t. } (x,z) \in V,\ (z,y) \in U\}$.

Now assume that $\mathcal{U}$ is a nonempty family of subsets of $X \times X$ satisfying the following properties:

(1) Each member of $\mathcal{U}$ contains $\Delta(X)$;
(2) The intersection of each two members of $\mathcal{U}$ lies in $\mathcal{U}$;
(3) $\mathcal{U}$ contains the inverses of its members;
(4) For each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;
(5) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

Then $\mathcal{U}$ is called a uniformity on X and the pair $(X, \mathcal{U})$ (shortly denoted by X) is called a uniform space.

For instance, if (X,d) is a metric space, then the family of all the supersets of the sets $U_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$ where $\varepsilon > 0$, forms a uniformity on X called the uniformity induced by d.

It is well-known that a uniformity $\mathcal{U}$ on a set X is separating if the intersection of all members of $\mathcal{U}$ is exactly the diagonal $\Delta(X)$. If this is satisfied, then X is called a separated uniform space.

To remind the convergence and Cauchyness notions in uniform spaces, let $\{x_n\}$ be a sequence in a uniform space X. Then $\{x_n\}$ is said to be convergent to a point $x \in X$, denoted by $x_n \to x$, if for each $U \in \mathcal{U}$, there exists an $N > 0$ such that $(x_n, x) \in U$ for all $n \geqslant N$, and it is said to be Cauchy in X if for each $U \in \mathcal{U}$, there exists an $N > 0$ such that $(x_m, x_n) \in U$ for all $m, n \geqslant N$. The uniform space X is called sequentially complete if each Cauchy sequence in X is convergent to some point of X. It can be easily verified that if $x_n \to x$, then each subsequence of $\{x_n\}$ converges to x, and further in a separated uniform space, each sequence may converge to at most one point, i.e., the limits of convergent sequences is unique in separated uniform spaces.

Let $\mathcal{F}$ be a nonempty collection of (uniformly continuous) pseudometrics on X that generates the uniformity $\mathcal{U}$ (see, [**1**, Theorem 2.1]), and denote by $\mathcal{V}$, the family of all sets of the form $\bigcap_{i=1}^{m} \big\{(x,y) \in X \times X : \rho_i(x,y) < r_i\big\}$, where m is a positive integer, $\rho_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, m$. Then it has been shown that $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$, i.e., $\mathcal{V}$ satisfies (U1)–(U4) and each member of $\mathcal{U}$ contains a member of $\mathcal{V}$. Finally, if $V = \bigcap_{i=1}^{m} \big\{(x,y) \in X \times X : \rho_i(x,y) < r_i\big\} \in \mathcal{V}$ and $\alpha > 0$, then the set $\alpha V = \bigcap_{i=1}^{m} \big\{(x,y) \in X \times X : \rho_i(x,y) < \alpha r_i\big\}$ is still a member of $\mathcal{V}$.

The next lemma embodies some important properties about the above-mentioned sets. For other properties, the reader is referred to [**1**, Lemmas 2.1-2.6].

LEMMA 1.1. [**1**] *Let X be a uniform space and $\mathcal{V}$ be as above. Then the following assertions hold.*

(1) *If $0 < \alpha \leqslant \beta$, then $\alpha V \subseteq \beta V$ for all $V \in \mathcal{V}$.*
(2) *If $\alpha, \beta > 0$, then $\alpha V \circ \beta V \subseteq (\alpha + \beta)V$ for all $V \in \mathcal{V}$.*
(3) *For each $x, y \in X$ and each $V \in \mathcal{V}$, there exists a positive number λ such that $(x, y) \in \lambda V$.*
(4) *For each $V \in \mathcal{V}$, there exists a pesudometric ρ on X such that $(x, y) \in V$ if and only if $\rho(x, y) < 1$.*

REMARK 1.1. The pseudometric ρ in Lemma 1.1 (iv) is called Minkowski's psudometric of V. Moreover, for any $\alpha > 0$, we have $(x, y) \in \alpha V$ if and only if $\rho(x, y) < \alpha$. In other words, $\frac{1}{\alpha}\rho$ is Minkowski's pseudometric of αV.

2. Main results

Throughout this section, the letter X is used to denote a nonempty set equipped with a uniformity $\mathcal{U}$ unless otherwise stated and $\mathcal{F}$ is a nonempty collection of (uniformly continuous) pseudometrics on X generating the uniformity $\mathcal{U}$. Furthermore, $\mathcal{V}$ is the collection of all sets of the form $\bigcap_{i=1}^{m} \{(x, y) \in X \times X : \rho_i(x, y) < r_i\}$, where m is a positive integer, $\rho_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, m$. The uniform space X is also endowed with a directed graph G without any parallel edges such that $V(G) = X$ and $E(G) \supseteq \Delta(X)$, i.e., $E(G)$ contains all loops, and by $\widetilde{G}$, it is meant the undirected graph obtained from G by ignoring the directions of the edges of G. The set of all fixed points of a self-mapping $T : X \to X$ is denoted by $\mathrm{Fix}(T)$ and we set $X_T = \{x \in X : (x, Tx) \in E(G)\}$.

The idea of following definition is taken from [**6**, 2.1. Definition] and [**8**, Definition 2.1].

DEFINITION 2.1. Let T be a mapping of X into itself. Then we call T a Ćirić-G-contraction if

(1) $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$, that is, T is edge-preserving;
(2) for all $x, y \in X$ and all $V_1, V_2, V_3, V_4, V_5 \in \mathcal{V}$,

$$(x, y) \in E(G) \cap V_1,\ (x, Tx) \in V_2,\ (y, Ty) \in V_3,\ (x, Ty) \in V_4,\ (y, Tx) \in V_5$$

imply

$$(Tx, Ty) \in a_1(x, y)V_1 \circ a_2(x, y)V_2 \circ a_3(x, y)V_3 \circ a_4(x, y)V_4 \circ a_4(x, y)V_5,$$

where a_1, a_2, a_3 and a_4 are positive-valued functions on $X \times X$ satisfying

$$\sup\big\{a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y) : x, y \in X\big\} = \alpha < 1. \tag{2.1}$$

Note that if (2.1) holds, then

$$\begin{aligned} a_1(x, y) + a_2(x, y) + a_4(x, y) + \alpha\big(a_3(x, y) + a_4(x, y)\big) \\ < a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y) \leqslant \alpha, \end{aligned}$$

for all $x, y \in X$. So

$$a_1(x, y) + a_2(x, y) + a_4(x, y) < \alpha\big(1 - a_3(x, y) - a_4(x, y)\big) \qquad (x, y \in X),$$

which yields

$$\frac{a_1(x,y)+a_2(x,y)+a_4(x,y)}{1-a_3(x,y)-a_4(x,y)}<\alpha \qquad (x,y\in X). \tag{2.2}$$

Example 2.1. (1) Since $E(G)$ and each member V of $\mathcal{V}$ contain $\Delta(X)$, it follows that each constant self-mapping $T : X \to X$ is a Ćirić-G-contraction with any positive-valued functions a_1, a_2, a_3 and a_4 satisfying (2.1).

(2) Let G_0 be the complete graph with $V(G_0) = X$, i.e., $E(G_0) = X \times X$. Then Ćirić-G_0-contractions (simply Ćirić-contractions) are precisely the counterparts of λ-generalized contractive mappings introduced by Ćirić in [**6**, 2.1. Definition] (the existence and uniqueness of fixed points for this type of contractions on sequentially complete separated uniform spaces were investigated by Rhoades [**10**, Theorem 1]).

(3) Let $\preceq$ be a partial order on X, and consider a graph G_1 by $V(G_1) = X$ and $E(G_1) = \{(x,y) \in X \times X : x \preceq y\}$. Then $E(G_1)$ contains all loops and Ćirić-G_1-contractions are precisely the nondecreasing order Ćirić contractions.

Example 2.2. Let (X,d) be a metric space and consider the set X with the uniformity induced by the metric d. Let $T : X \to X$ be a Ćirić G_0-contraction. For arbitrary $x, y \in X$ write

$$d(x,y)=r_1,\quad d(x,Tx)=r_2,\quad d(y,Ty)=r_3,$$
$$d(x,Ty)=r_4,\quad \text{and}\quad d(y,Tx)=r_5$$

and take $\varepsilon > 0$. Then it is clear that

$$(x,y)\in U_{r_1+\varepsilon},\quad (x,Tx)\in U_{r_2+\varepsilon},\quad (y,Ty)\in U_{r_3+\varepsilon},$$
$$(x,Ty)\in U_{r_4+\varepsilon},\quad \text{and}\quad (x,Ty)\in U_{r_5+\varepsilon}.$$

Hence it follows by (C2) that

$$\begin{aligned}(Tx,Ty)\in a_1(x,y)U_{r_1+\varepsilon}\circ a_2(x,y)U_{r_2+\varepsilon}\circ a_3(x,y)U_{r_3+\varepsilon}\\ \circ a_4(x,y)U_{r_4+\varepsilon}\circ a_4(x,y)U_{r_5+\varepsilon}.\end{aligned}$$

So by Lemma 1.1 we get

$$\begin{aligned}d(Tx,Ty)&< a_1(x,y)r_1+a_2(x,y)r_2+a_3(x,y)r_3+a_4(x,y)\big(r_4+r_5\big)\\ &\quad+\big(a_1(x,y)+a_2(x,y)+a_3(x,y)+2a_4(x,y)\big)\varepsilon\\ &\leqslant a_1(x,y)d(x,y)+a_2(x,y)d(x,Tx)+a_3(x,y)d(y,Ty)\\ &\quad+a_4(x,y)\big(d(x,Ty)+d(y,Tx)\big)+\alpha\varepsilon,\end{aligned}$$

where $\alpha = \sup\{a_1(x,y)+a_2(x,y)+a_3(x,y)+2a_4(x,y) : x,y \in X\} < 1$. Since $\varepsilon > 0$ was arbitrary, we obtain

$$\begin{aligned}d(Tx,Ty)\leqslant a_1(x,y)d(x,y)+a_2(x,y)d(x,Tx)+a_3(x,y)d(y,Ty)\\ +\ a_4(x,y)\big(d(x,Ty)+d(y,Tx)\big).\end{aligned}$$

Consequently, T is an α-generalized contraction in the sense of Ćirić [**6**].

Example 2.3. Let (X,d) be a metric space with the following condition:

- For all $x, y \in X$ and $r_1, r_2 > 0$ satisfying $d(x, y) < r_1 + r_2$, there exists a $z \in X$ such that $d(x, z) < r_1$ and $d(y, z) < r_2$.

Consider the set X with the uniformity induced by the metric d and let $T : X \to X$ be a λ-generalized contraction. Assume that $x, y \in X$ and $r_1, r_2, r_3, r_4, r_5 > 0$ are such that

$$(x, y) \in U_{r_1},\ (x, Tx) \in U_{r_2},\ (y, Ty) \in U_{r_3},\ (x, Ty) \in U_{r_4}, \quad \text{and} \quad (y, Tx) \in U_{r_5}.$$

Then

$$\begin{aligned} d(Tx, Ty) &\leqslant q(x, y)d(x, y) + r(x, y)d(x, Tx) + s(x, y)d(y, Ty) \\ &\qquad + t(x, y)\big(d(x, Ty) + d(y, Tx)\big) \\ &\qquad < q(x, y)r_1 + r(x, y)r_2 + s(x, y)r_3 + t(x, y)r_4 + t(x, y)r_5. \end{aligned}$$

Using (†) four times, we see that there exist $z_1, z_2, z_3, z_4 \in X$ such that

$$\begin{gathered} d(Tx, z_1) < t(x, y)r_5, \quad d(z_1, z_2) < t(x, y)r_4, \quad d(z_2, z_3) < s(x, y)r_3, \\ d(z_3, z_4) < r(x, y)r_2, \quad \text{and} \quad d(z_4, Ty) < q(x, y)r_1, \end{gathered}$$

that is,

$$\begin{gathered} (Tx, z) \in t(x, y)U_{r_5}, \quad (z_1, z_2) \in t(x, y)U_{r_4}, \quad (z_2, z_3) \in s(x, y)U_{r_3}, \\ (z_3, z_4) \in r(x, y)U_{r_2}, \quad \text{and} \quad (z_4, Ty) \in q(x, y)U_{r_1}. \end{gathered}$$

Therefore,

$$(Tx, Ty) \in q(x, y)U_{r_1} \circ r(x, y)U_{r_2} \circ s(x, y)U_{r_3} \circ t(x, y)U_{r_4} \circ t(x, y)U_{r_5}.$$

Hence T is a Ćirić G_0-contraction.

According to Examples 2.2 and 2.3, all Ćirić G_0-contractions are λ-generalized contraction and the converse holds in metric spaces satisfying (†).

In the next example, we see that the self-mapping T given in [**6**, Example 1] is a Ćirić G-contraction in the uniformity induced by the usual metric on $[0, 2]$ for some graphs G.

Example 2.4. Consider the set $X = [0, 2]$ with the usual metric and define a self-mapping $T : X \to X$ by the rule $Tx = \frac{x}{9}$ if $0 \leqslant x \leqslant 1$, and $Tx = \frac{x}{10}$ if $1 < x \leqslant 2$ for all $x \in X$. Then T is not a contraction on X since

$$\Big|T\frac{1001}{1000} - T\frac{999}{1000}\Big| = \frac{109}{10000} > \frac{1}{500} = \Big|\frac{1001}{1000} - \frac{999}{1000}\Big|.$$

On the other hand, putting

$$a_1(x, y) = \frac{1}{10},\ a_2(x, y) = a_3(x, y) = \frac{1}{4}, \quad \text{and} \quad a_4(x, y) = \frac{1}{6} \qquad (x, y \in X),$$

we have

$$\sup\big\{a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y) : x, y \in X\big\} = \frac{14}{15} < 1$$

and it is not hard to see that T is a $\frac{14}{15}$-generalized contraction. Furthermore, because X satisfies (†), it follows by Example 2.3 that T is a Ćirić G_0-contraction.

More generally, T is a Ćirić G-contraction for all graphs G whose edges are preserved by T.

To investigate the existence and uniqueness of fixed points for Ćirić-G-contractions, we need the following lemmas.

LEMMA 2.1. *Let $T : X \to X$ be a Ćirić-G-contraction and $V \in \mathcal{V}$. If $x \in X_T$ is such that $(x, Tx) \in V$, then $(T^n x, T^{n+1}x) \in \alpha^n V$ $n = 0, 1, \ldots,$ where α is as in* (2.1).

PROOF. If $n = 0$, then there is nothing to prove. Let $n \geqslant 1$ and denote by ρ, Minkowski's pseudometric of V. Write

$$\rho(T^{n-1}x, T^n x) = r_1, \quad \rho(T^n x, T^{n+1}x) = r_2, \quad \text{and} \quad \rho(T^{n-1}x, T^{n+1}x) = r_3$$

and let $\varepsilon > 0$. Then it is clear that

$$\begin{aligned} &(T^{n-1}x, T^n x) \in (r_1 + \varepsilon)V, \quad (T^n x, T^{n+1}x) \in (r_2 + \varepsilon)V, \\ &(T^{n-1}x, T^{n+1}x) \in (r_3 + \varepsilon)V, \quad \text{and} \quad (T^n x, T^n x) \in \varepsilon V. \end{aligned}$$

Note that by (C1), we have $(T^{n-1}x, T^n x) \in E(G)$. Hence it follows by (C2) and Lemma 1.1 that

$$\begin{aligned} (T^n x, T^{n+1}x) \in\ & a_1(T^{n-1}x, T^n x)(r_1 + \varepsilon)V \circ a_2(T^{n-1}x, T^n x)(r_1 + \varepsilon)V \\ &\circ a_3(T^{n-1}x, T^n x)(r_2 + \varepsilon)V \circ a_4(T^{n-1}x, T^n x)(r_3 + \varepsilon)V \\ &\circ a_4(T^{n-1}x, T^n x)\varepsilon V \\ \subseteq\ & \Big(\big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x)\big)r_1 + a_3(T^{n-1}x, T^n x)r_2 \\ &+ a_4(T^{n-1}x, T^n x)r_3 + \big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x) \\ &+ a_3(T^{n-1}x, T^n x) + 2a_4(T^{n-1}x, T^n x)\big)\varepsilon\Big)V \\ \subseteq\ & \Big(\big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x)\big)r_1 \\ &\quad + a_3(T^{n-1}x, T^n x)r_2 + a_4(T^{n-1}x, T^n x)r_3 + \alpha\varepsilon\Big)V, \end{aligned}$$

where α is as in (2.1). Because ρ is Minkowski's pseudometric of V, it follows by Remark 1.1 that

$$\begin{aligned} \rho(T^n x, T^{n+1}x) <\ & \big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x)\big)r_1 \\ &+ a_3(T^{n-1}x, T^n x)r_2 + a_4(T^{n-1}x, T^n x)r_3 + \alpha\varepsilon \\ =\ & \big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x)\big)\rho(T^{n-1}x, T^n x) \\ &+ a_3(T^{n-1}x, T^n x)\rho(T^n x, T^{n+1}x) \\ &+ a_4(T^{n-1}x, T^n x)\rho(T^{n-1}x, T^{n+1}x) + \alpha\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\begin{aligned} \rho(T^n x, T^{n+1}x) \leqslant\ & \big(a_1(T^{n-1}x, T^n x) + a_2(T^{n-1}x, T^n x)\big)\rho(T^{n-1}x, T^n x) \\ &+ a_3(T^{n-1}x, T^n x)\rho(T^n x, T^{n+1}x) \end{aligned}$$

$$
\begin{aligned}
&+ a_4(T^{n-1}x,T^nx)\rho(T^{n-1}x,T^{n+1}x)\\
&\leqslant \big(a_1(T^{n-1}x,T^nx)+a_2(T^{n-1}x,T^nx)\big)\rho(T^{n-1}x,T^nx)\\
&+ a_3(T^{n-1}x,T^nx)\rho(T^nx,T^{n+1}x)\\
&+ a_4(T^{n-1}x,T^nx)\big(\rho(T^{n-1}x,T^nx)+\rho(T^nx,T^{n+1}x)\big).
\end{aligned}
$$

Therefore, by (2.2),

$$
\begin{aligned}
(2.3)\quad \rho(T^nx,T^{n+1}x) &\leqslant \frac{a_1(T^{n-1}x,T^nx)+a_2(T^{n-1}x,T^nx)+a_4(T^{n-1}x,T^nx)}{1-a_3(T^{n-1}x,T^nx)-a_4(T^{n-1}x,T^nx)}\\
&\quad\times \rho(T^{n-1}x,T^nx) < \alpha\rho(T^{n-1}x,T^nx) < \cdots < \alpha^n\rho(x,Tx).
\end{aligned}
$$

Because $(x,Tx)\in V$, it follows that $\rho(x,Tx)<1$, and hence using (2.3), one has $\rho(T^nx,T^{n+1}x)<\alpha^n$, that is, $(T^nx,T^{n+1}x)\in\alpha^nV$. □

LEMMA 2.2. *Let $T : X\to X$ be a Ćirić-G-contraction. Then the sequence $\{T^nx\}$ is Cauchy in X for all $x\in X_T$.*

PROOF. Let $x\in X_T$ and $V\in\mathcal{V}$ be given. Then Lemma 1.1 ensures the existence of a positive number λ such that $(x,Tx)\in\lambda V$, and so, by Lemma 2.1 we have $(T^nx,T^{n+1}x)\in(\alpha^n\lambda)V$, $n=0,1,\dots$, where α is as in (2.1). Now, if ρ is Minkowski's pseudometric of V, then by Remark 1.1, $\rho(T^nx,T^{n+1}x)<\alpha^n\lambda$ for all $n\geqslant 0$, and since $\alpha<1$, it follows that

$$
\sum_{n=0}^{\infty}\rho(T^nx,T^{n+1}x)\leqslant\sum_{n=0}^{\infty}\alpha^n\lambda=\frac{\lambda}{1-\alpha}<\infty.
$$

An easy argument shows that $\rho(T^mx,T^nx)\to 0$ as $m,n\to\infty$. Hence there exists an $N>0$ such that $\rho(T^mx,T^nx)<1$ for all $m,n\geqslant N$. Therefore, $(T^mx,T^nx)\in V$ for all $m,n\geqslant N$, and because $V\in\mathcal{V}$ was arbitrary, it is concluded that the sequence $\{T^nx\}$ is Cauchy in X. □

We are now ready to prove our main theorem.

THEOREM 2.1. *Suppose that the uniform space X is sequentially complete and separated, and has the following property:*

($*$) *If a sequence $\{x_n\}$ converges to some point $x\in X$ and it satisfies $(x_n,x_{n+1})\in E(G)$ for all $n\geqslant 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in E(G)$ for all $k\geqslant 1$.*

Then a Ćirić-G-contraction $T : X\to X$ has a fixed point if and only if $X_T\neq\emptyset$. Furthermore, this fixed point is unique if

(1) *the functions a_2 and a_3 in* (C2) *coincide on $X\times X$; and*
(2) *for all $x,y\in X$, there exists a $z\in X$ such that $(x,z),(y,z)\in E(\widetilde{G})$.*

PROOF. It is clear that each fixed point of T is an element of X_T. For the converse, let $x\in X_T$. Then by Lemma 2.2, the sequence $\{T^nx\}$ is Cauchy in X. By sequential completeness of X, there exists an $x^*\in X$ such that $T^nx\to x^*$. On the other hand, since $x\in X_T$ and T is edge-preserving, it follows that $(T^nx,T^{n+1}x)\in$

$E(G)$ for all $n \geqslant 0$. Therefore, by Property $(*)$, there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $(T^{n_k}x, x^*) \in E(G)$ for all $k \geqslant 1$. We shall show that $T^{n_k+1}x \to Tx^*$. To this end, let V be an arbitrary member of $\mathcal{V}$ and denote by ρ, Minkowski's pseudometric of V. Let $k \geqslant 1$; write

$$\rho(T^{n_k}x, x^*) = r_1, \quad \rho(T^{n_k}x, T^{n_k+1}x) = r_2, \quad \rho(x^*, Tx^*) = r_3,$$
$$\rho(T^{n_k}x, Tx^*) = r_4, \quad \text{and} \quad \rho(x^*, T^{n_k+1}x) = r_5,$$

and take $\varepsilon > 0$. Then it is clear that

$$(T^{n_k}x, x^*) \in (r_1+\varepsilon)V, \quad (T^{n_k}x, T^{n_k+1}x) \in (r_2+\varepsilon)V, \quad (x^*, Tx^*) \in (r_3+\varepsilon)V,$$
$$(T^{n_k}x, Tx^*) \in (r_4+\varepsilon)V, \quad \text{and} \quad (x^*, T^{n_k+1}x) \in (r_5+\varepsilon)V.$$

Therefore, by (C2) and Lemma 1.1, we have

$$\begin{aligned}
(T^{n_k+1}x, Tx^*) &\in a_1(T^{n_k}x, x^*)(r_1+\varepsilon)V \circ a_2(T^{n_k}x, x^*)(r_2+\varepsilon)V \\
&\quad \circ a_3(T^{n_k}x, x^*)(r_3+\varepsilon)V \circ a_4(T^{n_k}x, x^*)(r_4+\varepsilon)V \\
&\quad \circ a_4(T^{n_k}x, x^*)(r_5+\varepsilon)V \\
&\subseteq \Big(a_1(T^{n_k}x, x^*)r_1 + a_2(T^{n_k}x, x^*)r_2 + a_3(T^{n_k}x, x^*)r_3 \\
&\qquad + a_4(T^{n_k}x, x^*)\big(r_4+r_5\big) + \big(a_1(T^{n_k}x, x^*) \\
&\qquad + a_2(T^{n_k}x, x^*) + a_3(T^{n_k}x, x^*) + 2a_4(T^{n_k}x, x^*)\big)\varepsilon\Big)V \\
&\subseteq \Big(a_1(T^{n_k}x, x^*)r_1 + a_2(T^{n_k}x, x^*)r_2 + a_3(T^{n_k}x, x^*)r_3 \\
&\qquad\qquad + a_4(T^{n_k}x, x^*)\big(r_4+r_5\big) + \alpha\varepsilon\Big)V,
\end{aligned}$$

where α is as in (2.1). Now by Remark 1.1, we get

$$\begin{aligned}
\rho(T^{n_k+1}x, Tx^*) &< a_1(T^{n_k}x, x^*)r_1 + a_2(T^{n_k}x, x^*)r_2 + a_3(T^{n_k}x, x^*)r_3 \\
&\quad + a_4(T^{n_k}x, x^*)\big(r_4+r_5\big) + \alpha\varepsilon \\
&= a_1(T^{n_k}x, x^*)\rho(T^{n_k}x, x^*) + a_2(T^{n_k}x, x^*)\rho(T^{n_k}x, T^{n_k+1}x) \\
&\quad + a_3(T^{n_k}x, x^*)\rho(x^*, Tx^*) \\
&\quad + a_4(T^{n_k}x, x^*)\big(\rho(T^{n_k}x, Tx^*) + \rho(x^*, T^{n_k+1}x)\big) + \alpha\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\begin{aligned}
\rho(T^{n_k+1}x, Tx^*) &\leqslant a_1(T^{n_k}x, x^*)\rho(T^{n_k}x, x^*) + a_2(T^{n_k}x, x^*)\rho(T^{n_k}x, T^{n_k+1}x) \\
&\quad + a_3(T^{n_k}x, x^*)\rho(x^*, Tx^*) \\
&\quad + a_4(T^{n_k}x, x^*)\big(\rho(T^{n_k}x, Tx^*) + \rho(x^*, T^{n_k+1}x)\big) \\
&\leqslant a_1(T^{n_k}x, x^*)\rho(T^{n_k}x, x^*) + a_2(T^{n_k}x, x^*)\rho(T^{n_k}x, T^{n_k+1}x) \\
&\quad + a_3(T^{n_k}x, x^*)\big(\rho(x^*, T^{n_k+1}x) + \rho(T^{n_k+1}x, Tx^*)\big) \\
&\quad + a_4(T^{n_k}x, x^*)\big(\rho(T^{n_k}x, T^{n_k+1}x) \\
&\qquad\qquad + \rho(T^{n_k+1}x, Tx^*) + \rho(x^*, T^{n_k+1}x)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(T^{n_k+1}x, Tx^*) &\leqslant \frac{1}{1 - a_3(T^{n_k}x, x^*) - a_4(T^{n_k}x, x^*)}\Big(a_1(T^{n_k}x, x^*)\rho(T^{n_k}x, x^*)\\ &\quad + \big(a_2(T^{n_k}x, x^*) + a_4(T^{n_k}x, x^*)\big)\rho(T^{n_k}x, T^{n_k+1}x)\\ &\quad + \big(a_3(T^{n_k}x, x^*) + a_4(T^{n_k}x, x^*)\big)\rho(T^{n_k+1}x, x^*)\Big)\\ &\leqslant \frac{1}{1-\alpha}\Big(\alpha\rho(T^{n_k}x, x^*) + \alpha\rho(T^{n_k}x, T^{n_k+1}x) + \alpha\rho(T^{n_k+1}x, x^*)\Big)\\ &= \frac{\alpha}{1-\alpha}\Big(\rho(T^{n_k}x, x^*) + \rho(T^{n_k}x, T^{n_k+1}x) + \rho(T^{n_k+1}x, x^*)\Big).\end{aligned}$$

Consequently, from $T^n x \to x^*$, there exists a $k_0 > 0$ such that

$$(T^{n_k}x, x^*) \in \frac{1-\alpha}{3\alpha}\cdot V,\ (T^{n_k}x, T^{n_k+1}x) \in \frac{1-\alpha}{3\alpha}\cdot V,\ \text{and}\ (T^{n_k+1}x, x^*) \in \frac{1-\alpha}{3\alpha}\cdot V,$$

for all $k \geqslant k_0$. Therefore,

$$\rho(T^{n_k+1}x, Tx^*) < \frac{\alpha}{1-\alpha}\Big(\frac{1-\alpha}{3\alpha} + \frac{1-\alpha}{3\alpha} + \frac{1-\alpha}{3\alpha}\Big) = 1 \qquad (k \geqslant k_0),$$

that is, $(T^{n_k+1}x, Tx^*) \in V$ for all $k \geqslant k_0$. Since $V \in \mathcal{V}$ was arbitrary, it is seen that $T^{n_k+1}x \to Tx^*$. On the other hand, since $T^{n_k+1}x \to x^*$ and X is separated, we must have $x^* = Tx^*$, and therefore x^* is a fixed point for T.

To see that x^* is the unique fixed point for T whenever (i) and (ii) are satisfied, let $y^* \in X$ be a fixed point for T. If $V \in \mathcal{V}$, then we consider the following two cases to show that $(x^*, y^*) \in V$:

Case 1: (x^*, y^*) is an edge of G. Let ρ be Minkowski's pseudometric of V. Take any arbitrary $\varepsilon > 0$ and write $\rho(x^*, y^*) = r$. Then $(x^*, y^*) \in (r+\varepsilon)V$ and so by (C2) and Lemma 1.1, we have

$$\begin{aligned}(x^*, y^*) = (Tx^*, Ty^*) &\in a_1(x^*, y^*)(r+\varepsilon)V \circ a_2(x^*, y^*)(r+\varepsilon)V\\ &\quad \circ a_2(x^*, y^*)(r+\varepsilon)V \circ a_4(x^*, y^*)(r+\varepsilon)V\\ &\quad \circ a_4(x^*, y^*)(r+\varepsilon)V\\ &\subseteq \Big(\big(a_1(x^*, y^*) + 2a_2(x^*, y^*) + 2a_4(x^*, y^*)\big)r\\ &\qquad + \big(a_1(x^*, y^*) + 2a_2(x^*, y^*) + 2a_4(x^*, y^*)\big)\varepsilon\Big)V\\ &\subseteq (\alpha r + \alpha\varepsilon)V.\end{aligned}$$

Therefore, $\rho(x^*, y^*) < \alpha r + \alpha\varepsilon = \alpha\rho(x^*, y^*) + \alpha\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\rho(x^*, y^*) \leqslant \alpha\rho(x^*, y^*)$, and since $\alpha < 1$, it follows that $\rho(x^*, y^*) = 0$, that is, $(x^*, y^*) \in V$.

Case 2: (x^*, y^*) is not an edge of G. In this case, by (ii), there exists a $z \in X$ such that $(x^*, z), (y^*, z) \in E(\widetilde{G})$. First assume that $(x^*, z), (y^*, z) \in E(G)$. Pick a $W \in \mathcal{V}$ such that $W \circ W \subseteq V$ and denote by ρ', Minkowski's pseudometric of W and let $n \geqslant 1$. Since T preserves the edges of G, it follows that $(x^*, T^n z) =$

$(T^n x^*, T^n z) \in E(\widetilde{G})$. Now write

$$\rho'(x^*, T^{n-1}z) = r_1, \quad \rho'(T^{n-1}z, T^n z) = r_2, \quad \text{and} \quad \rho'(x^*, T^n z) = r_3.$$

Then, clearly,

$$(x^*, T^{n-1}z) \in (r_1+\varepsilon)W, \quad (x^*, x^*) \in \varepsilon W, \quad (T^{n-1}z, T^n z) \in (r_2+\varepsilon)W,$$
$$(x^*, T^n z) \in (r_3+\varepsilon)W, \quad \text{and} \quad (T^{n-1}z, x^*) \in (r_1+\varepsilon)W.$$

Therefore, from (C2) and Lemma 1.1, we have

$$\begin{aligned}(x^*, T^n z) = (T^n x^*, T^n z) &\in a_1(x^*, T^{n-1}z)(r_1+\varepsilon)W \circ a_2(x^*, T^{n-1}z)\varepsilon W \\ &\quad \circ a_2(x^*, T^{n-1}z)(r_2+\varepsilon)W \circ a_4(x^*, T^{n-1}z)(r_3+\varepsilon)W \\ &\quad \circ a_4(x^*, T^{n-1}z)(r_1+\varepsilon)W \\ &\subseteq \Big(\big(a_1(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)r_1 \\ &\qquad + a_2(x^*, T^{n-1}z)r_2 + a_4(x^*, T^{n-1}z)r_3 + \big(a_1(x^*, T^{n-1}z) \\ &\qquad\qquad + 2a_2(x^*, T^{n-1}z) + 2a_4(x^*, T^{n-1}z)\big)\varepsilon\Big)W \\ &\subseteq \Big(\big(a_1(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)r_1 \\ &\qquad + a_2(x^*, T^{n-1}z)r_2 + a_4(x^*, T^{n-1}z)r_3 + \alpha\varepsilon\Big)W.\end{aligned}$$

Hence by Remark 1.1,

$$\begin{aligned}\rho'(x^*, T^n z) &< \big(a_1(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)r_1 + a_2(x^*, T^{n-1}z)r_2 \\ &\quad + a_4(x^*, T^{n-1}z)r_3 + \alpha\varepsilon \\ &= \big(a_1(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)\rho'(x^*, T^{n-1}z) \\ &\quad + a_2(x^*, T^{n-1}z)\rho'(T^{n-1}z, T^n z) + a_4(x^*, T^{n-1}z)\rho'(x^*, T^n z) + \alpha\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\begin{aligned}\rho'(x^*, T^n z) &\leqslant \big(a_1(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)\rho'(x^*, T^{n-1}z) \\ &\quad + a_2(x^*, T^{n-1}z)\rho'(T^{n-1}z, T^n z) + a_4(x^*, T^{n-1}z)\rho'(x^*, T^n z) \\ &\leqslant \big(a_1(x^*, T^{n-1}z) + a_2(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)\rho'(x^*, T^{n-1}z) \\ &\quad + \big(a_2(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)\big)\rho'(x^*, T^n z),\end{aligned}$$

which accompanied with (2.2) yields

$$\begin{aligned}\rho'(x^*, T^n z) &\leqslant \frac{a_1(x^*, T^{n-1}z) + a_2(x^*, T^{n-1}z) + a_4(x^*, T^{n-1}z)}{1 - a_2(x^*, T^{n-1}z) - a_4(x^*, T^{n-1}z)} \cdot \rho'(x^*, T^{n-1}z) \\ &< \alpha\rho'(x^*, T^{n-1}z) = \alpha\rho'(T^{n-1}x^*, T^{n-1}z) < \cdots < \alpha^n \rho'(x^*, z).\end{aligned}$$

Similarly, one can show that $\rho'(y^*, T^n z) \leqslant \alpha^n \rho'(y^*, z)$. Now, for sufficiently large n, we have $\alpha^n \rho'(x^*, z) < 1$ and $\alpha^n \rho'(y^*, z) < 1$, that is, $(x^*, T^n z), (y^*, T^n z) \in W$. Since W is symmetric, that is $W = W^{-1}$, we get $(x^*, y^*) \in W \circ W \subseteq V$.

Finally, since every member of $\mathcal{V}$ is symmetric, with a similar argument to that of above one can show that in the other three cases, namely, $(x^*, z), (z, y^*) \in E(G)$, $(z, x^*), (y^*, z) \in E(G)$, and $(z, x^*), (z, y^*) \in E(G)$, we again get $(x^*, y^*) \in W \circ W \subseteq V$.

Consequently, in both Cases 1 and 2, we have $(x^*, y^*) \in V$. Since $V \in \mathcal{V}$ was arbitrary and X is separated, it follows that $y^* = x^*$. □

Setting $G = G_0$ and $G = G_1$ in Theorem 2.1, we get the next results in uniform spaces and partially ordered uniform spaces, respectively. Note that Corollary 2.1 is a generalization of [**6**, 2.5 Theorem].

Corollary 2.1. *Let the uniform space X be sequentially complete and separated and $T : X \to X$ be a Ćirić-contraction. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T. Moreover, if a_2 and a_3 in* (C2) *coincide on $X \times X$, then this fixed point is unique, i.e., there exists a unique $x^* \in \mathrm{Fix}(T)$ such that $\{T^n x\}$ converges to x^* for all $x \in X$.*

Corollary 2.2. *Let $\preceq$ be a partial order on the sequentially complete and separated uniform space X satisfying the following property:*

> *If a nondecreasing sequence $\{x_n\}$ converges to some point $x \in X$, then it contains a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \preceq x$ for all $k \geqslant 1$.*

Then a nondecreasing order Ćirić-contraction $T : X \to X$ has a fixed point if and only if there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$. Moreover, this fixed point is unique if

(1) *the functions a_2 and a_3 in* (C2) *coincide on $X \times X$; and*
(2) *every two elements of X has either a lower or an upper bound.*

Our next result is a generalization of the fixed point theorem for Hardy and Rogers-type contraction [**7**] from metric spaces to uniform spaces endowed with a graph. It also generalizes Banach, Kannan and Chatterjea contractions provided that $0V = \Delta(X)$.

Corollary 2.3. *Suppose that the uniform space X is sequentially complete and separated, and satisfies the following properties:*

- *If a sequence $\{x_n\}$ converges to some point $x \in X$ and it satisfies $(x_n, x_{n+1}) \in E(G)$ for all $n \geqslant 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geqslant 1$;*
- *For all $x, y \in X$, there exists a $z \in X$ such that $(x, z), (y, z) \in E(\widetilde{G})$.*

Let $T : X \to X$ be an edge-preserving self-mapping satisfying the following contractive condition:

For all $x, y \in X$ and all $V_1, V_2, V_3, V_4, V_5 \in \mathcal{V}$,

$$(x, y) \in E(G) \cap V_1,\ (x, Tx) \in V_2,\ (y, Ty) \in V_3,\ (x, Ty) \in V_4, \quad \text{and} \quad (y, Tx) \in V_5$$

imply

$$(Tx, Ty) \in aV_1 \circ bV_2 \circ bV_3 \circ cV_4 \circ cV_5,$$

where a, b and c are positive real numbers such that $a + 2b + 2c < 1$.

Then T has a unique fixed point if and only if $X_T \neq \emptyset$.

Remark 2.1. In [**5**], Bojor established some results on the existence and uniqueness of fixed points for edge-preserving self-mappings T, called G-Ćirić–Reich–Rus operators, on a metric space (X,d) endowed with a T-connected graph G satisfying

$$d(Tx,Ty) \leqslant ad(x,y) + bd(x,Tx) + cd(y,Ty) \qquad \big(x,y \in X,\ (x,y) \in E(G)\big),$$

where $a,b,c \geqslant 0$ and $a+b+c<1$. Let us review the notion of T-connectedness introduced by Bojor: Let (X,d) be a metric space endowed with a graph G (see, [**8**, Section 2]) and T be a self-mapping on X. The graph G is said to be T-connected if for all $x,y \in X$ with $(x,y) \notin E(G)$, there exists a finite sequence $(x_i)_{i=0}^N$ of vertices of G such that $x_0 = x$, $x_N = y$, $(x_{i-1},x_i) \in E(G)$ for $i=1,\dots,N$, and $x_i \in X_T$ for $i=1,\dots,N-1$.

Note that if $\preceq$ is a partial order on X and the graph G_1 is T-connected, then for all $x,y \in X$ with $x \npreceq y$, there exists a finite sequence $(x_i)_{i=0}^N$ of vertices of G_1 such that $x_0 = x$, $x_N = y$, $x_{i-1} \preceq x_i$ for $i=1,\dots,N$, and $x_i \preceq Tx_i$ for $i=1,\dots,N-1$. Hence by the transitivity of $\preceq$, we get $x \preceq y$, which is impossible. Therefore, the graph G_1 is T-connected if and only if X is a singleton. More generally, a transitive graph G (that is, $(x,y),(y,z) \in E(G)$ implies $(x,z) \in E(G)$ for all $x,y,z \in V(G)$) is T-connected if and only if the set of its vertices is a singleton. Hence, in [**5**, Theorem 6], to get a nontrivial result, the graph G should not be replaced with the graph G_1 induced by a partial order $\preceq$.

In Theorem 2.1 of the present work, we have proved the existence of a fixed point for a Ćirić-G-contraction T with $X_T \neq \emptyset$ in a sequentially complete and separated uniform space X endowed with a graph G using $(*)$ rather than the T-connectedness of G. Moreover, by (i) and (ii) we have obtained the uniqueness of the fixed point. Therefore, the fact that every partial order $\preceq$ on X induces the graph G_1 implies that Theorem 2.1 may be restated in a partially ordered form (see, Corollary 2.2).

If a metric space (X,d) endowed with a graph G satisfies (†) (given in Example 2.3), then by Examples 2.2 and 2.3, Ćirić G-contractions are precisely the edge-preserving self-mappings $T : X \to X$ such that

$$\begin{aligned} d(Tx,Ty) \leqslant{}& a_1(x,y)d(x,y) + a_2(x,y)d(x,Tx) + a_3(x,y)d(y,Ty) \\ &+ a_4(x,y)\big(d(x,Ty)+d(y,Tx)\big) \quad \big(x,y \in X,\ (x,y) \in E(G)\big), \end{aligned}$$

where $a_1,a_2,a_3,a_4 : X \times X \to (0,1)$ satisfy (2.1). Moreover, a G-Ćirić-Reich-Rus operator in metric spaces endowed with a graph given in [**5**, Definition 7] may have a counterpart in uniform spaces endowed with a graph G as follows: Let X be a uniform space endowed with a graph G. An edge-preserving self-mapping $T : X \to X$ is called a G-Ćirić–Reich–Rus operator if for all $x,y \in X$ and all $V_1,V_2,V_3 \in \mathcal{V}$,

$$(x,y) \in V_1 \cap E(G),\ (x,Tx) \in V_2,\ (y,Ty) \in V_3$$

imply $(Tx,Ty) \in aV_1 \circ bV_2 \circ cV_3$, where $a,b,c \geqslant 0$ and $a+b+c<1$.

It is easily seen that in this case, every G-Ćirić-Reich-Rus operator is a Ćirić G-contraction. Therefore, a uniformity version of [**5**, Theorem 6] may be obtained from Theorem 2.1 as follows:

Suppose that X is a sequentially complete and separated uniform space endowed with a graph G satisfying $(*)$ and $T : X \to X$ is a G-Ćirić–Reich–Rus operator. Then T has a fixed point if and only if $X_T \neq \emptyset$. Furthermore, if $b = c$ and for all $x, y \in X$, there exists $z \in X$ such that $(x, z), (y, z) \in E(\widetilde{G})$, then the fixed point is unique.

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References

1. S. P. Acharya, *Some results on fixed points in uniform spaces*, Yokohama Math. J. **22** (1974), 105–116.
2. A. Aghanians, K. Fallahi, K. Nourouzi, *An entourage approach to the contraction principle in uniform spaces endowed with a graph*, Panamer. Math. J. **23** (2013), 87–102.
3. ______, *Fixed points for G-contractions on uniform spaces endowed with a graph*, Fixed Point Theory Appl. **2012**:182 (2012), 12 pages.
4. F. Bojor, *Fixed point of φ-contraction in metric spaces endowed with a graph*, An. Univ. Craiova Ser. Mat. Inform. **37** (2010), 85–92.
5. ______, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. **75** (2012), 3895–3901.
6. Lj. B. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) (N.S.) **12**(26) (1971), 19–26.
7. G. E. Hardy, T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.
8. J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2008), 1359–1373.
9. A. Nicolae, D. O'Regan, A. Petruşel, *Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph*, Georgian Math. J. **18** (2011), 307–327.
10. B. E. Rhoades, *Fixed point theorems in a uniform space*, Publ. Inst. Math., Nouv. Sér. **25(39)** (1979), 153–156.
11. S. Willard, *General Topology*, Addison-Wesley, Reading, Mass.-London-Don Mills, Ont., 1970.

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