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THE SEMIRING VARIETY GENERATED BY ANY FINITE NUMBER OF FINITE FIELDS AND DISTRIBUTIVE LATTICES

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Abstract. We study the semiring variety $\mathbf{V}$ generated by any finite number of finite fields $F_1, \ldots, F_k$ and two-element distributive lattice B_2, i.e., $\mathbf{V} = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$. It is proved that $\mathbf{V}$ is hereditarily finitely based, and that, up to isomorphism, B_2 and all subfields of $F_1, \ldots, F_k$ are the only subdirectly irreducible semirings in $\mathbf{V}$.

1. Introduction and preliminaries

Semirings are the natural generalization of rings and distributive lattices. Besides the two well-known examples of semirings: the set of nonnegative integers $\mathbb{N}$ with the usual addition and multiplication as the most trivial one, and the first nontrivial example given by Dedekind [**2**] in connection with algebra of ideals of commutative ring, history of semirings date back, at least, to Vandiver [**22**]. The intensive study of semirings was initiated during the late 1960's when their significant applications were found. Thus, nowadays, semirings have both a developed algebraic theory as well as important practical applications. More about applications of semiring theory within analysis, fuzzy set theory, the theory of discrete-event dynamical systems, automata and formal language theory can be found in the trilogy [**4**]–[**6**] and in [**15**]. Recently, new examples of applications of semiring constructions have been investigated in [**11**]–[**14**].

All semirings $(S, +, \cdot)$ occurring in the literature satisfy at least the following axioms: $(S, +)$, the additive reduct, and $(S, \cdot)$, the *multiplicative reduct* of a semiring S are semigroups, and the multiplication distributes over addition from both sides, i.e.,

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(SR1) $x + (y + z) \approx (x + y) + z$;
(SR2) $x(yz) \approx (xy)z$;
(SR3) $x(y + z) \approx xy + xz, \quad (x + y)z \approx xz + yz$.

It is, as well, often assumed that $(S, +)$ is commutative, i.e.,

(SR4) $x + y \approx y + x$.

Note that the variety considered in the present paper satisfy this identity too.

Let S be a semiring. We can distinguish, in general, the following three subsets of idempotents (if there are any) of S: $E(S)_\bullet$ the set of all multiplicative idempotents of $(S, \cdot)$; $E(S)_+$ the set of all additive idempotents of $(S, +)$, and $E(S) = E(S)_\bullet \cap E(S)_+$. A semiring S is *idempotent* if $S = E(S)$, i.e., if it satisfies

$$x + x \approx x \approx x^2.$$

An idempotent semiring S is called a *bisemilattice* if both the additive and multiplicative reducts $(S, +)$ and $(S, \cdot)$ of S are semilattices. A *distributive lattice* is a bisemilattice which satisfies the absorption law

$$x + xy \approx x.$$

The variety of all distributive lattices is denoted by **D**. The smallest nontrivial distributive lattice, the two-element boolean algebra B_2, given by

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

is the only subdirectly irreducible (moreover, B_2 is congruence simple too) member of **D** and we have $\mathbf{D} = \mathrm{HSP}\{B_2\}$.

Kelarev in [**9**] described the ring variety generated by a finite number of finite fields with *pairwise distinct characteristics* and proved that such varieties are finitely based. Some of their properties, including the one that such a ring variety is arithmetical, are given in [**18, 25**]. Specially, in [**10**], it is proved that the ring variety generated by a finite ring is finitely based. Thus, in [**23**] the ring variety of square root rings is considered, and it is proved that it is generated by the finite field F_{2^k}. In [**1**] it is proved that the ring variety generated by a finite number of finite fields with *pairwise distinct characteristics* is finitely based and used in term rewriting. Shao and Ren in [**20**] proved that the semiring variety generated by distributive lattices and any finite number of prime fields are finitely based. In [**21**], it is proved that the semiring variety generated by a finite number of finite fields with *pairwise distinct characteristics* and distributive lattices are finitely based.

As we know, the "simplest" semiring variety generated by finite fields and distributive lattices is the the variety of Boolean semirings generated by B_2 and the smallest nontrivial finite field Z_2, the field of integers modulo 2 or 2-element Boolean ring, given by

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

.

In [**8**] it is proved that this variety is finitely based, and it is equivalent to the category of partially Stone spaces. This motivates us to give a little progress in that direction.

The main subject here is the semiring variety $\mathbf{V} = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$ generated by B_2 and any finite number of finite fields $F_1, \ldots, F_k$. In our consideration, we *do not need that finite fields $F_1, \ldots, F_k$ have pairwise distinct characteristics.* We prove that $\mathbf{V} = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$ is hereditarily finitely based and characterize all subdirectly irreducible semirings in $\mathbf{V}$. We refer to [**4**]–[**6**] as sources of references on semirings. For notions and terminology not given here, we refer to [**16**] as the background on finite fields, [**17**] on universal algebras, and [**7, 19**] for semigroup theory.

2. On the semiring variety $\mathbf{V} = \mathrm{HSP}\,\{B_2, F_1, \ldots, F_k\}$

Let $p_1, \ldots, p_k$ be primes and $q_1 = p_1{}^{n_1}, \ldots, q_k = p_k{}^{n_k}$ for some positive integers $n_1, \ldots, n_k$. Assume that d is the least common multiple of $p_1, \ldots, p_k$ and that m is a positive integer such that $m-1$ is the product of $q_1 - 1, \ldots, q_k - 1$. Let $\mathbf{W}$ denote the variety of semirings defined by identities (SR1–4) and the following ones

(W1) $(d+1)\cdot x \approx x$;
(W2) $x^m \approx x$;
(W3) $d \cdot x^2 \approx d \cdot x$;
(W4) $x + d \cdot xy \approx x$;
(W5) $xy \approx yx$.

Let S be a semiring in $\mathbf{W}$. We denote by $E(S)_+$ the set of all idempotents of the additive reduct $(S, +)$ of S. By Theorems 1.1, 1.2, 2.1 and Lemma 2.1 in [**21**], we have

THEOREM 2.1. *Let S be a semiring in $\mathbf{W}$. Then the following statements are true:*

(i) *$E(S)_+ = \{d \cdot a | a \in S\}$, and $(E(S)_+, +, \cdot)$ is a distributive lattice;*
(ii) *$(S, +)$ is an E-unitary Clifford semigroup;*
(iii) *If R is a subdirectly irreducible semiring in $\mathbf{W}$, then R is two-element distributive lattice or R is a finite field.*

In this section we assume that $F_1, \ldots, F_k$ is any given finite number of finite fields with characteristics $p_1, \ldots, p_k$ and sizes $q_1, \ldots, q_k$. In what follows the semiring variety $\mathbf{V} = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$ will be considered.

It suffices to consider the following cases:

- $\mathbf{V}_1 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$, in which there exist at least two finite fields in $\{F_1, \ldots, F_k\}$ such that their characteristics are distinct.
- $\mathbf{V}_2 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$, in which $F_1, \ldots, F_k$ have the same characteristics.

We firstly consider the variety $\mathbf{V}_1$. Clearly, $\mathbf{V}_1$ satisfies (W1–5) so it is a subvariety of $\mathbf{W}$. We also have that B_2 and finite fields $F_1, \ldots, F_k$ satisfy the following identities

$$\text{(W6)} \qquad \frac{d}{p_i} \cdot x^{q_i} \approx \frac{d}{p_i} \cdot x \quad (1 \leqslant i \leqslant k),$$

which implies that $\mathbf{V}_1 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$ satisfies (W1–6). In fact, we have

THEOREM 2.2. *Let* $\mathbf{V}_1 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$. *Then*

(i) $\mathbf{V}_1$ *is finitely based;*

(ii) *if* S *is a subdirectly irreducible semiring in* $\mathbf{V}_1$*, then* S *is isomorphic to* B_2*, or there exists a field* F *in* $\{F_1, \ldots, F_k\}$ *such that* S *is isomorphic to a subfield of* F.

PROOF. (i) Let $\mathbf{V}^*$ be the variety of semirings defined by (SR1-4) and (W1-6). It is easy to see that $\mathbf{V}^*$ is a subvariety of $\mathbf{W}$ and that $\mathbf{V}_1$ is a subvariety of $\mathbf{V}^*$. In what follows we will prove that $\mathbf{V}_1 = \mathbf{V}^*$.

Suppose that S is a subdirectly irreducible semiring in $\mathbf{V}^*$. It follows from Theorem 2.1 that S, up to isomorphism, is B_2 or a finite field. If S is a finite field, then S satisfies the identity (W1). Thus, the characteristic of S is equal to some p_i $(1 \leqslant i \leqslant k)$ since d is the least common multiple of $p_1, \ldots, p_k$. Next, S satisfies $\frac{d}{p_i} \cdot x^{q_i} = \frac{d}{p_i} \cdot x$, which implies that S satisfies $x^{q_i} = x$, so the size of S divides q_i. Thus, up to isomorphism, S is a subfield of F_i. Since every subfield of F_i is in the variety $\mathbf{V}_1 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$, we have that S belongs to $\mathbf{V}_1$. This shows that every subdirectly irreducible semiring of $\mathbf{V}^*$ is in $\mathbf{V}_1$ and so $\mathbf{V}^* \subseteq \mathbf{V}_1$ and so $\mathbf{V}^* = \mathbf{V}_1$. This shows that $\mathbf{V}_1$ is finitely based.

(ii) If S is a subdirectly irreducible semiring in $\mathbf{V}_1$, then it follows directly from the proof of (i) that S is isomorphic to B_2, or there exists a field F in $\{F_1, \ldots, F_k\}$ such that S is isomorphic to a subfield of F. □

In general, $\mathbf{V}_1$ can be a proper subvariety of $\mathbf{W}$. This can be shown by the following example.

EXAMPLE 2.1. Let us consider the variety $\mathrm{HSP}\{B_2, F_3, F_{2^2}, F_{2^3}, F_{7^2}\}$ and the semiring variety $\mathbf{W}(2, 3, 7, 2017)$ defined by the additional identities

(1) $x + 42 \cdot x \approx x$; (3) $42 \cdot x^2 \approx 42 \cdot x$; (5) $xy \approx yx$.
(2) $x^{2017} \approx x$; (4) $x + 42 \cdot xy \approx x$;

It is easy to see that $\mathrm{HSP}\{B_2, F_3, F_{2^2}, F_{2^3}, F_{7^2}\}$ satisfies identities (1)–(5). It is a routine matter to verify that F_{3^2} is in $\mathbf{W}(2, 3, 7, 2017)$. By Theorem 2.2 we have that F_{3^2} does not belong to $\mathrm{HSP}\{B_2, F_3, F_{2^2}, F_{2^3}, F_{7^2}\}$. This implies that $\mathrm{HSP}\{B_2, F_3, F_{2^2}, F_{2^3}, F_{7^2}\}$ is a proper subvariety of $\mathbf{W}(2, 3, 7, 2017)$. This means that, for $\mathbf{V}_1$, the identity (W6) is indispensable.

In the following we will discuss the variety $\mathbf{V}_2 = \mathrm{HSP}\{B_2, F_1, \ldots, F_k\}$ generated by B_2 and a finite number of finite fields with the same characteristic. Without loss of generality, we assume that there exists a prime p such that the characteristics of $F_1, \ldots, F_k$ are equal to p. Thus, there exist positive integers $n_1, \ldots, n_k$ such that $|F_i| = p^{n_i}$ $(1 \leqslant i \leqslant k)$.

Let n be a positive integer such that $n-1$ is the product of $p^{n_1} - 1, \ldots, p^{n_k} - 1$. It is easy to verify that $\mathbf{V}_2$ satisfy

(FSR1) $(p + 1) \cdot x \approx x$;
(FSR2) $x^n \approx x$;

(FSR3) $p \cdot x^2 \approx p \cdot x$;
(FSR4) $x + p \cdot xy \approx x$;
(FSR5) $x + (x^{p^{n_1}} + (p-1) \cdot x) \dots (x^{p^{n_k}} + (p-1) \cdot x) \approx x$;
(W5) $xy \approx yx$.

Thus we have

Theorem 2.3. *Let* $\mathbf{V}_2 = \mathrm{HSP}\{B_2, F_1, \dots, F_k\}$ *be the variety generated by* B_2 *and a finite number of finite fields with the same characteristic* p*. Then*

(i) $\mathbf{V}_2$ *is finitely based;*
(ii) *if* S *is a subdirectly irreducible semiring in* $\mathbf{V}_2$*, then* S *is isomorphic to* B_2*, or there exists a field* F *in* $\{F_1, \dots, F_k\}$ *such that* S *is isomorphic to a subfield of* F*.*

Proof. (i) We denote by $\mathbf{V}'$ the variety of semirings defined by (SR1–4), (FSR1–5) and (W5). It is easy to see that $\mathbf{V}_2$ is a subvariety of $\mathbf{V}'$. In what follows, we will prove that $\mathbf{V}_2 = \mathbf{V}'$.

Suppose that S is a subdirectly irreducible semiring in $\mathbf{V}'$. It follows from Theorem 2.1 that S, up to isomorphism, is B_2 or a finite field. If S is a finite field, then S satisfies the identity (FSR1). This implies that the characteristic of S is equal to p. Since $(S, +, \cdot)$ is a finite field, we denote by 0 and 1 the zero element and the identity of S, respectively. Thus we have that $(S \setminus \{0\}, \cdot)$ is a cyclic group of a finite order. Without loss of generality, we suppose that $(S \setminus \{0\}, \cdot)$ can be generated by the element a and the order of $(S \setminus \{0\}, \cdot)$ is equal to q, i.e., $|S \setminus \{0\}| = q$. From (FSR5) we have that $a + (a^{{p_1}^{n_1}} + (p-1) \cdot a) \dots (a^{{p_k}^{n_k}} + (p-1) \cdot a) = a$. It follows that $(a^{p^{n_1}} + (p-1) \cdot a) \dots (a^{p^{n_k}} + (p-1) \cdot a) = 0$ since $(S, +)$ is a group. Furthermore, there exists $1 \leqslant j \leqslant k$ such that $a^{p^{n_j}} + (p-1) \cdot a = 0$ and so $a^{p^{n_j}} + (p-1) \cdot a + a = a$. Since the characteristic of S is equal to p, $a = a^{p^{n_j}} + (p-1) \cdot a + a = a^{p^{n_j}} + p \cdot a = a^{p^{n_j}}$ and so $a^{p^{n_j}-1} = 1$. This shows the size q of $(S \setminus \{0\}, \cdot)$ divides $p^{n_j} - 1$ and so the size of S divides p^{n_j}. Thus, S is isomorphisic to the subfield of F_j. Since every subfield of F_i is in the variety $\mathbf{V}_2 = \mathrm{HSP}\{B_2, F_1, \dots, F_k\}$, we have that S belongs to $\mathbf{V}_2$. This shows that every subdirectly irreducible semiring of $\mathbf{V}'$ is in $\mathbf{V}_2$ and so $\mathbf{V}' = \mathbf{V}_2$. This means that $\mathbf{V}_2$ is finitely based.

(ii) If S is a subdirectly irreducible semiring in $\mathbf{V}_2$, then it follows directly from the proof of (i) that S is isomorphic to B_2, or there exists a field F in $\{F_1, \dots, F_k\}$ such that S is isomorphic to a subfield of F. □

In general, $\mathbf{V}_2$ can be a proper subvariety of $\mathbf{W}$. For example, let us consider the variety $\mathrm{HSP}\{B_2, F_{3^3}, F_{3^5}, F_{3^7}\}$ and the semiring variety $\mathbf{W}(3, 13754313)$ defined by the additional identities

(1) $4 \cdot x \approx x$;	(3) $3 \cdot x^2 \approx 3 \cdot x$;	(5) $xy \approx yx$.
(2) $x^{13754313} \approx x$;	(4) $x + 3 \cdot xy \approx x$;	

It is easy to see that $\mathrm{HSP}\{B_2, F_{3^3}, F_{3^5}, F_{3^7}\}$ satisfies identities (1)–(5). It is routine to verify that F_{3^2} is in $\mathbf{W}(3, 13754313)$. By Theorem 2.3 it follows that F_{3^2} does not belong to the variety $\mathrm{HSP}\{B_2, F_{3^3}, F_{3^5}, F_{3^7}\}$. This implies that $\mathrm{HSP}\{B_2, F_{3^3}, F_{3^5}, F_{3^7}\}$ is a proper subvariety of $\mathbf{W}(3, 13754313)$. This means that, for $\mathbf{V}_2$, the identity (FSR5) is indispensable.

By Theorems 2.2 and 2.3, we can establish the following result:

THEOREM 2.4. *Let* $\mathbf{V}$ *be the variety generated by* B_2 *and a finite number of finite fields* $\{F_1, \ldots, F_k\}$. *Then*

(i) $\mathbf{V}$ *is finitely based;*
(ii) *if* S *is a subdirectly irreducible semiring in* $\mathbf{V}$, *then* S *is isomorphic to* B_2, *or there exists a field* F *in* $\{F_1, \ldots, F_k\}$ *such that* S *is isomorphic to a subfield of* F.

Theorem 2.4 extends and enriches the main results of [**8**]–[**10**], [**20**] and [**21**].

A variety is said to be hereditarily finitely based if every variety contained in it is finitely based. In the rest of this section, we will show that the variety $\mathbf{V}$ considered in Theorem 2.4 is hereditarily finitely based. By Theorem 2.4 (ii) we immediately have that, up to isomorphism, there are finitely many subdirectly irreducible members in $\mathbf{V}$. Let $\mathcal{T}$ denote the set of all subdirectly irreducible members in $\mathbf{V}$. Since every subvariety of $\mathbf{V}$ is generated by a subset of $\mathcal{T}$, it follows that the lattice of all subvarieties of $\mathbf{V}$ is finite. Let $\mathcal{A} \subseteq \mathcal{T}$. To show that HSP($\mathcal{A}$) is finitely based, we need only to consider the following cases:

- $\mathcal{A} = \emptyset$. It is clear that HSP($\mathcal{A}$) is the trivial variety.
- $\mathcal{A} = \{B_2\}$. HSP($\mathcal{A}$) = $\mathbf{D}$ is finitely based.
- $\mathcal{A}$ consists of B_2 and a finite number of finite fields. Then, by Theorem 2.4(i) it follows that HSP($\mathcal{A}$) is finitely based.
- $\mathcal{A}$ consists of a finite number of finite fields. Without loss of generality, we assume that $\mathcal{A} = \{F_{s_1}, \ldots, F_{s_t}\}$, in which every finite field F_{s_j} is a subfield of some F_i. Let b the least common multiple of characteristics of $F_{s_1}, \ldots, F_{s_t}$. It is easy to see that every finite field in $\{F_{s_1}, \ldots, F_{s_t}\}$ satisfies the identity $b \cdot x \approx b \cdot y$, but B_2 does not satisfy $b \cdot x \approx b \cdot y$. Thus, HSP($\mathcal{A}$) is a subvariety of HSP($\mathcal{A} \cup \{B_2\}$) determined by additional identity $b \cdot x \approx b \cdot y$. Suppose that K is a subdirectly irreducible semiring in the subvariety HSP($\mathcal{A} \cup \{B_2\}$) determined by additional identity $b \cdot x \approx b \cdot y$. It follows by Theorem 2.4 (ii) that K is a subfield of some finite field in $\mathcal{A}$ and so K belongs to HSP($\mathcal{A}$). This shows that HSP($\mathcal{A}$) is the subvariety of HSP($\mathcal{A} \cup \{B_2\}$) determined by additional identity $b \cdot x \approx b \cdot y$. Hence, HSP($\mathcal{A}$) is finitely based.

From above it follows that every subvariety of $\mathbf{V}$ is finitely based. We now have

THEOREM 2.5. *The semiring variety generated by a two-element distributive lattice and any finite number of finite fields is hereditarily finitely based.*

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