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AN EXAMPLE OF BRUNS–GUBELADZE *K*-THEORY DEFINED BY THREE DIMENSIONAL POLYTOPE

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Abstract. For the Bruns–Gubeladze polytopal K-theory, we describe a new series of three dimensional balanced Col-divisible polytopes. Also we calculate the corresponding elementary groups and as a corollary obtain an expression of the polytopal K-groups in terms of the Quillen K-groups.

1. Introduction

In the series of papers [**1**–**4**] Bruns and Gubeladze have investigated polytopal algebras $k[P]$ where k is a field and P is a lattice polytope. The group of graded R-automorphisms $\mathrm{gr.aut}_R(k[P])$ of the algebra $k[P]$ is an analog of the group $\mathrm{GL}_n(k)$. The paper [**1**] introduces elementary automorphisms of $k[P]$ and establishes an important fact that every graded automorphism can be diagonalized by a sequence of elementary automorphism. In [**2**] it was shown that many graded retractions are conjugates of diagonal idempotents. So the natural question arises: is it possible to find polytopal analogs of the higher algebraic K-groups (for rings). The answer is positive and has been given by Bruns and Gubeladze in [**3**, **4**] for a wide class of balanced polytopes. For a commutative ring R and a balanced polytope P the group $\mathbb{E}_R(P)$ generated by elementary graded automorphisms is not perfect in general. Bruns and Gubeladze established a highly nontrivial stabilization procedure which on polytopal level works as "doubling along a facet". As an outcome of the stabilization procedure one obtains the stable elementary group $\mathbb{E}(R,P)$. It is important that the stable group $\mathbb{E}(R,P)$ in general is not a union of corresponding unstable groups, hence the "polytopal part" of the stabilization is essential. The group $\mathbb{E}(R,P)$ is perfect. In [**3**, **4**] the (stable) Steinberg group $\mathbb{S}\mathrm{t}(R,P)$ was defined and it was shown that for a balanced polytope P the natural homomorphism $\mathbb{S}\mathrm{t}(R,P) \to \mathbb{E}(R,P)$ is a universal central extension. Higher polytopal K-groups are defined by

$$K_i(R,P) = \pi_i(B\mathbb{E}(R,P)^+), \quad i \geq 2.$$

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If P coincides with the standard simplex Δ^k one obtains the Quillen K-theory. More detailed account on history of the question and on the motivation can be found in [**1**, **2**, **5**].

Some natural questions arise about the polytopal generalization of K-theory. First of all it is important to know equivalence relations on polytopes which lead to naturally equivalent polytopal K-theory. The projective equivalence of polytopes is obvious, but the E-equivalence (see [**3**, **4**] for definition) is more convenient. Also it is interesting to calculate $K_i(R, P)$ for various polytopes P. The case of 2-dimensional polytopes (i.e., polygones) was completely solved by Bruns and Gubeladze [**3**, **4**]. They proved that there are 6 classes of E-equivalence of the polygones and also they calculated the corresponding K-groups. The case of 3-dimensional polytopes was investigated in [**6**]. In that work a classification of the balanced 3-dimensional polytopes up to E-equivalence was proposed and the stable elementary groups of Col-divisible 3-dimensional polytopes were identified. In [**8**] the case of the pyramid over the unit square (it is balanced, but not Col-divisible) was investigated. This polytope appears in [**3**, **4**] several times as a polytope not satisfying some natural conditions (see also [**5**]). Some calculations for balanced, but not Col-divisible polytopes, can be found in [**9**].

Bruns and Gubeladze conjectured [**5**, Conjecture 8.3] that for a commutative ring R and a Col-divisible (balanced) polytope P of arbitrary dimension one has $K_i(R, P) = K_i(R) \oplus \cdots \oplus K_i(R)$ ($c(P)$ summands), where $c(P) < \dim P$ is a natural number explicitly defined by P (for some polytopes P a technical condition on the ring R is involved). In all known examples the conjecture holds even for balanced not Col-divisible polytopes.

This note appeared as a result of an attempt to find a counterexample to the Bruns–Gubeladze conjecture. In fact, we did not succeed, but instead we found a series of balanced Col-divisible polytopes which had not been known before and was omitted in the Faramarzi's classification theorem [**6**, Theorem 3.2]. We calculate the corresponding elementary groups and as a corollary obtain an expression of the polytopal K-groups in terms of the Quillen K-groups. Despite [**6**], a classification of balanced (and balanced Col-divisible) 3-dimensional polytopes remains open.

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2. Basic definitions

The details of the Bruns–Gubeladze construction can be found in their original works [**3**, **4**]. Here we present only an outline of necessary definition and constructions.

Let P be a convex polytope in $\mathbb{R}^n$ with vertices in the integral lattice $\mathbb{Z}^n$. We always suppose that n is minimal, that is the minimal affine subspace containing P coincides with $\mathbb{R}^n$. Such a polytope is called *a lattice polytope*. For any facet F there is a unique surjective homomorphism $\langle F, -\rangle \colon \mathbb{Z}^n \to \mathbb{Z}$, with the kernel consisting of vectors parallel to the facet F and such that $\langle F, p - q\rangle \geq 0$ for any $p \in P$ and $q \in F$.

A vector $u \in \mathbb{Z}^n$ is called *a column vector* for a lattice polytope P if there exists a facet $P_u \subset P$ such that $\langle P_u, u\rangle = -1$ and for any other facet $F \subset P$ one has $\langle F, u\rangle \geq 0$. In this case the facet P_u is called *the base facet* for the column vector u. For a given column vector the base facet is defined uniquely, but two different column vectors can have the same base facet. A collection of all column vectors with the base facet F is denoted by $\mathrm{Col}\,(F)$. A collection of all column vectors of the lattice polytope P is denoted $\mathrm{Col}\,(P)$.

Assume $u \in \mathrm{Col}\,(F)$. Then for any point $p \in P \cap \mathbb{Z}^n$ there exists a unique nonnegative integer k such that $p + ku \in F$. This number is called a *height* of the point p over the base facet F and is denoted by $\mathrm{ht}_F(p)$. One has the equality $\mathrm{ht}_F(p) = \langle F, p - q\rangle$ where q is an arbitrary point from F.

One can define a natural partial multiplication on the set $\mathrm{Col}\,(P)$. Suppose $u, v \in \mathrm{Col}\,(P)$, $u + v \in \mathrm{Col}\,(P)$ and $P_{u+v} = P_u$. Then the product uv is defined to be $u + v$. The product uv is defined not for any pair of column vectors u, v. Obviously if uv exists, then vu is not defined.

As a basic example consider the simplex Δ^n in $\mathbb{R}^n$, with one vertex $(0, \dots, 0)$ in the origin and other n vertices of form $(0, \dots, 1, \dots, 0)$. The description of column vectors and their partial product is simple. For any two vertices $p_i, p_j \in \Delta^n$, there are two column vectors $\delta_i^j = p_j - p_i$ and $\delta_j^i = -\delta_i^j$. The base facet of the column vector δ_j^i is $\Delta^n \cap \{x_j = 0\}$ for $j \neq 0$ and $\Delta^n \cap \{\sum_k x_k = 1\}$ for $j = 0$. The partial product is described by the relation $\delta_i^j \delta_j^k = \delta_i^k$. There are no other column vectors, and no other product is defined.

Balanced polytopes. A lattice polytope P is called *balanced* if for any $u, v \in \mathrm{Col}\,(P)$ one has $\langle P_u, v\rangle \leq 1$. The simplex Δ^n is balanced while the triangle $\mathrm{conv}\,\{(0,0), (1,0), (0,2)\}$ is not.

Note that for a balanced polytope and its base facet F one has inequality $|\langle F, u\rangle| \leq 1$ for any column vector v. Also $\langle F, u\rangle = -1$ iff $v \in \mathrm{Col}\,(F)$. Obviously $\langle F, u\rangle = 0$ iff u is parallel to F.

Doubling along a facet. Like in the classical Quillen K-theory we need some kind of a stabilization procedure. Let P be a lattice polytope. Choose its facet F. Without loss of generality one can assume that the origin belongs to the facet F and that F is contained in the hyperplane $x_n = 0$. Consider the standard embedding of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ onto the hyperplane $x_{n+1} = 0$. Let us turn the polytope P by $\pi/2$ around the plane $x_n = x_{n+1} = 0$ in $\mathbb{R}^{n+1}$. The image of P under the rotation is denoted by $P^{|}$.

The polytope $P^{\lrcorner F}$ is defined as the convex hull of P and $P^{|}$ and is called *doubling of P along the facet F*. If $v \in \mathrm{Col}\,(F)$ then one can write $P^{\lrcorner v}$ instead of $P^{\lrcorner F}$.

After doubling along F the number of facets increases by 1. For any facet G different from F denote by $G^{\lrcorner F}$ the facet of $P^{\lrcorner F}$ which is the convex hull of G and its image $G^{|}$ after rotation of the polytope P by $\pi/2$. Let $F^{\lrcorner F} = P^{|}$ for the facet F.

Let us describe the structure of $\mathrm{Col}\,(P^{\lrcorner F})$. Choose a column vector $v \in \mathrm{Col}\,(P)$ with base facet G. Then v is also a column vector for $P^{\lrcorner F}$ with the base facet $G^{\lrcorner F}$ (think of the inclusion $P \subset P^{\lrcorner F}$).

Describe new column vectors which arise after doubling. First of all there are two column vectors δ^+ and $\delta^- = -\delta^+$ with the base facets $P^{|}$ and $P^- = P$

correspondingly. Secondly, for a column vector $v \in \mathrm{Col}\,(P)$ the vector $v^{|}$ (the image of v after rotation of P by $\pi/2$) is a column vector for $P^{\lrcorner F}$. If the vector v is parallel to the facet F, then $v^{|}$ coincides with v. If v is not parallel to F then $v \neq v^{|}$ and we have two possibilities. Namely, if $v \in \mathrm{Col}\,(F)$, then one has the relations

$$\delta^- v = v^{|}, \quad \delta^+ v^{|} = v, \tag{1}$$

and if $v \notin \mathrm{Col}\,(F)$ (therefore $\langle F, v\rangle > 0$), then one has another relations

$$v\delta^+ = v^{|}, \quad v^{|}\delta^- = v. \tag{2}$$

One can show that for a balanced polytope P its doubling $P^{\lrcorner F}$ is also balanced and that the set $\mathrm{Col}\,(P^{\lrcorner F})$ is a union (not necessarily disjoint)

$$\mathrm{Col}\,(P^{\lrcorner F}) = \mathrm{Col}\,(P) \cup \mathrm{Col}\,(P)^{|} \cup \{\delta^+, \delta^-\}.$$

A sequence of polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset P_2 \subset \ldots)$ is called *a doubling spectrum* if (1) P_{k+1} is a doubling of P_k along a base facet and (2) for any $i \in \mathbb{Z}_+$, $v \in \mathrm{Col}\,(P_i)$ there exists $j \geq i$, such that $P_{j+1} = P_j^{\lrcorner v}$.

For any doubling spectrum there is a natural inclusion $\mathrm{Col}\,(P_i) \subset \mathrm{Col}\,(P_{i+1})$, therefore the direct limit $\mathrm{Col}\,(\mathfrak{P}) = \lim \mathrm{Col}\,(P_i)$ is defined.

Elementary automorphisms and Steinberg group. Consider a lattice polytope P. Let $S(P)$ be the additive semigroup generated by pairs $(p, 1) \in \mathbb{R}^{n+1}$ where $p \in P \cap \mathbb{Z}^n$. For a given associative commutative ring R with unit consider the semigroup ring $R[P] = R[S(P)]$. It has natural grading defined on the generators of the ring by formula $\deg(p, d) = d$.

Denote by $\mathrm{gr.aut}_R\,(R[P])$ the group of graded R-automorphisms of $R[P]$. An element $\phi \in \mathrm{gr.aut}_R\,(R[P])$ is called *an elementary automorphism* if there exist a column vector $v \in \mathrm{Col}\,(P)$ and an element $\lambda \in R$ such that for every $x \in S(P)$, one has

$$\phi(x) = (1 + \lambda v)^{\mathrm{ht}_{P_v}(x)} x.$$

Denote this automorphism by e_v^λ. The subgroup of $\mathrm{gr.aut}_R\,(R[P])$ generated by elementary automorphisms is denoted by $\mathbb{E}_R(P)$.

For any $v \in \mathrm{Col}\,(\mathfrak{P})$ there exists $i \in \mathbb{N}$ such that $v \in \mathrm{Col}\,(P_j)$ for all $j \geq i$. Hence elementary automorphisms $e_v^\lambda \in \mathbb{E}_R(P_j)$, $j \geq i$, form a compatible system. Therefore they define a graded automorphism of $R[\mathfrak{P}]$, which we call “elementary” and denote by e_v^λ. The group $\mathbb{E}(R, \mathfrak{P})$ is the subgroup of $\mathrm{gr.aut}_R\,(R[\mathfrak{P}])$ generated by elementary automorphisms.

In [**3**, **4**] it was shown that the group $\mathbb{E}(R, \mathfrak{P})$ does not depend on a choice of a doubling spectrum of the polytope P, hence one uses notation $\mathbb{E}(R, P)$ instead of $\mathbb{E}(R, \mathfrak{P})$. The group $\mathbb{E}(R, P)$ is perfect. For a balanced lattice polytope P elementary automorphisms satisfy relations which are similar to the relations between elementary matrices:

(i) $e_v^\lambda e_v^\mu = e_v^{\lambda+\mu}$ for all $v \in \mathrm{Col}\,(\mathfrak{P})$ and $\lambda, \mu \in R$;

(ii) for all $u, v \in \mathrm{Col}\,(\mathfrak{P})$ and $\lambda, \mu \in R$

$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu}, & \text{if } uv \text{ is defined,} \\ 1, & \text{if } u + v \notin \mathrm{Col}\,(P) \cup \{0\}. \end{cases}$$

Fix a doubling spectrum $\mathfrak{P}$ of P. Define the Steinberg group $\mathbb{S}\mathrm{t}(R,P)$ to be the group generated by symbols x_v^λ, $v \in \mathrm{Col}\,(\mathfrak{P})$, $\lambda \in R$, and relations

$$x_v^\lambda x_v^\mu = x_v^{\lambda+\mu} \quad \text{for all} \quad v \in \mathrm{Col}\,(\mathfrak{P}), \quad \lambda, \mu \in R; \tag{3}$$

$$[x_u^\lambda, x_v^\mu] = \begin{cases} x_{uv}^{-\lambda\mu}, & \text{if product } uv \text{ is defined,} \\ 1, & \text{if } u+v \notin \mathrm{Col}\,(\mathfrak{P}) \cup \{0\} \end{cases} \tag{4}$$

for all $u, v \in \mathrm{Col}\,(\mathfrak{P})$ and $\lambda, \mu \in R$.

The group $\mathbb{S}\mathrm{t}(R,P)$ does not depend on a choice of a doubling spectrum $\mathfrak{P}$ of P. For a balanced polytope P the natural epimorphism $\mathbb{S}\mathrm{t}(R,P) \to \mathbb{E}(R,P)$ is a universal central extension.

Example: For $P = \Delta^n$ the groups $\mathbb{E}(R,P)$ and $\mathbb{S}\mathrm{t}(R,P)$ are isomorphic to usual $E(R)$ and $St(R)$.

3. Polytopes $F(t,s)$

For integers $t \geq s \geq 1$ define the polytope $F(t,s)$ to be the convex hull of the points $(0,0,0), (2t,0,0), (1,2,0), (t+1,1,0), (0,0,1), (1,0,2), (2t-s+1,0,1) \in \mathbb{R}^3$.

The Col-structure of $F(t,s)$ can be described as follows. There are two base facets: $F_w = F(t,s) \cap \{y = 0\}$ with the column vectors $v_\tau = (\tau-1,-1,0)$, $\tau = 1,\ldots,t$, and $w = (0,-1,1)$ and $F_u = F(t,s) \cap \{z = 0\}$ with the column vectors$v_\sigma = (\sigma-1,0,-1)$, $\sigma = 1,\ldots,s$. The only relations between the column vectors are $wu_\sigma = v_\sigma$, $\sigma = 0,\ldots,s$. If $t > s$, then the column vectors $v_{s+1},\ldots,v_t$ cannot be decomposed as a product of other column vectors.

From this description it follows that the polytopes $F(t,s)$ are balanced and Col-divisibe. Note that

(1) for $t = s > 1$ the polytope $F(t,t)$ is E-equivalent to the polytope $P_{e_4}(t)$,
(2) for $t > 1$ the polytope $F(t,1)$ is E-equivalent to the polytope $P_e(t,1)$ (for definition of $P_{e_4}(t)$ and $P_e(t,1)$ see [**6**]),
(3) the polytope $F(1,1)$ (though it is three dimensional) is E-equivalent to the polygon of type (c) (for definition see [**4**]).

It appears that the condition $t \geq s$ is essential.

Proposition 1. *Suppose P is a polytope such that there are at least two base facets F_u and F_w. Assume that for the base facet F_u there are s column vectors u_σ, $1 \leq \sigma \leq s$, which are parallel to F_w and that for the base facet F_w there is a column vector w which is not parallel to the F_u. Then the vectors $v_\sigma = w + u_\sigma$ are column vectors for the base facet F_w.*

In other words, in assumptions of the proposition there are at least s column vectors for the base facet F_w such that $wu_\sigma = v_\sigma$.

Proof. Fix $\sigma = 1,\ldots,s$. As w and u_σ are column vectors, one has $\langle G, w+u_\sigma\rangle = \langle G, w\rangle + \langle G, u_\sigma\rangle \geq 0$ for any facet G of the polytope P different from F_u and F_w. For the facet F_u one has $\langle F_u, w+u_\sigma\rangle = \langle F_u, w\rangle - 1 \geq 0$ and for the facet F_w: $\langle F_w, w+u_\sigma\rangle = -1+0 = -1$. Hence $w+u_\sigma$ is a column vector for the base facet F_w. $\square$

Fix integers $t \geq s \geq 1$ and consider a doubling spectrum $P_0, P_1, P_2, \dots$ of the polytope $P_0 = F(t, s)$. Let us describe the column structure of the polytopes P_j (by induction on j). From this we obtain the description of the elementary group and the Steinberg group for the polytope P_0 and identify the corresponding K-theory.

First of all we describe the structure of the base facets of the polytopes P_j. For every polytope P_j we divide its base facets into two families $\mathbf{A}^1(0)$, $\mathbf{A}^2(0)$ and enumerate the facets in the families. For the polytope P_0 define $\mathbf{A}^1(0) = \{F_w\}$, $\mathbf{A}^2(0) = \{F_u\}$.

Suppose P_{j+1} is a doubling of P_j along a facet F, which belongs to $\mathbf{A}^1(j)$. Assume $\mathbf{A}^1(j) = \{A_1, A_2, \dots, A_n\}$. Define $\mathbf{A}^1(j+1)$ to be $\{A'_1, A'_2, \dots, A'_n, A'_{n+1}\}$, where $A'_{n+1} = P_j$ (recall that P_j is a facet of P_{j+1}) and $A'_j = A_j^{\lrcorner F}$ for $j \leq n$. Assume $\mathbf{A}^2(j) = \{B_1, B_2, \dots, B_m\}$. Define $\mathbf{A}^2(j+1)$ to be $\{B'_1, B'_2, \dots, B'_m\}$, $B'_j = B_j^{\lrcorner F}$. Definition of $\mathbf{A}^r(j+1)$ for the case $F \in \mathbf{A}^2(j)$ is analogous.

Now describe the structure of Col-vectors. We shall do it in two steps.

First of all we describe what happens to the vectors u_σ, v_τ, w under consecutive doublings. Let $a_r(j) = \#\mathbf{A}^r(j)$

Lemma 1. *For the polytope P_j there are the column vectors w_i^k, $u_{\sigma,k}$, $v_{\tau,i}$, where $1 \leq i \leq a_1(j)$, $1 \leq k \leq a_2(j)$ such that:*

(1) $w_i^k u_{\sigma,k} = v_{\sigma,i}$, *for all k and $1 \leq \sigma \leq s$,*

(2) $v_{\tau,i}$ *is a column vector for the i-th base facet from $\mathbf{A}^1(j)$, and it is parallel to all other base facets,*

(3) $u_{\sigma,k}$ *is a column vector for the k-th base facet from $\mathbf{A}^2(j)$, and it is parallel to all other base facets,*

(4) w_i^k *is a column vector for the i-th base facet from $\mathbf{A}^1(j)$, it has height 1 over the k-th base facet from $\mathbf{A}^2(j)$ and it is parallel to all other base facets.*

Proof. The case of P_0 is obvious.

By induction assume that doubling of P_j was made along the facet $A_i \in \mathbf{A}^1(j)$. Then besides the vectors $\delta^\pm$ we obtain new vectors $(w_i^k)^|$ and $v_{\tau,i}^|$ for all k and τ. Denote them by $w_{a_1(j)+1}^k$ and $v_{\tau,a_1(j)+1}$ correspondingly. All these vectors are column vectors for the new base facet $A_{a_1(j)+1}$.

Assume doubling of P_j was made along the facet $B_k \in \mathbf{A}^2(j)$. Then besides the vectors $\delta^\pm$ we obtain new vectors $(w_i^k)^|$ and $u_{\sigma,k}^|$ for all i and τ. Denote them by $w_i^{a_2(j)+1}$ and $v_{\tau,a_2(j)+1}$ correspondingly.

Statement (1) is a straightforward consequence of the relation $wu_\sigma = v_\sigma$. Also note that there is no such relation for the vectors $v_{\sigma+1,i}, \dots, v_{\tau,i}$. $\square$

Secondly, we describe column vectors which appear as the $\delta^\pm$-vectors or vectors they produce under doublings.

Lemma 2. *For any P_j and for any two different facets $A_k, A_l \in \mathbf{A}^r(j)$ there is a column vector $\delta_k^l(r)$ for the base facet A_k, which has height 1 over the facet A_l and is parallel to all other base facets. The vectors $\delta_k^l(r)$ satisfy the relations $\delta_k^l(r)\delta_l^p(r) = \delta_k^p(r)$ for all r, k, l, p.*

Proof. For $j = 0$ the statement is trivial.

Assume by induction that doubling of P_j was made along the facet $A_i \in \mathbf{A}^1(j)$. Then we have new column vectors $\delta^{\pm}$ and $\delta_i^k(1)^|$, $\delta_k^i(1)^|$ for all $k \neq i$. Denote $\delta_i^k(1)^|$ by $\delta^k_{a_1(j)+1}(1)$, $\delta_k^i(1)^|$ by $\delta_k^{a_1(j)+1}(1)$, δ^+ by $\delta_i^{a_1(j)+1}$, and δ^- by $\delta^i_{a_1(j)+1}$. If one of the indices k, l, p coincides with $a_1(j)+1$, then the relation $\delta_k^l(r)\delta_l^p(r) = \delta_k^p(r)$ follows from one of relations (1), (2). For example relation (2) $(\delta_k^i)^|\delta^- = \delta_k^i$ is the same as $\delta_k^{a_1+1}\delta^i_{a_1+1} = \delta_k^i$ and relation (1) $\delta^+(\delta_i^k)^| = \delta_i^k$ is the same as $\delta_i^{a_1+1}\delta^k_{a_1+1} = \delta_i^k$.

The collection of the vectors $\delta_k^l(2)$ is unchanged as all of them are parallel to the facet A_i.

The case of doubling along the facet from $\mathbf{A}^2(j)$ is analogous. □

Lemma 3. *The vectors $\delta_i^k(r), u_{\sigma,p}, v_{\tau,i}, w_i^p$ satisfy the relations*

$$\delta_k^i(1)v_{\tau,i} = v_{\tau,k} \tag{5}$$

$$\delta_q^p(2)u_{\sigma,p} = u_{\sigma,q} \tag{6}$$

$$\delta_k^i(1)w_i^p = w_k^p \tag{7}$$

$$w_i^p\delta_p^q(2) = w_i^q \tag{8}$$

for all i, k, p, q, r.

The relations (5)-(7) follow from (1), and the relation (8) follows from (1).

4. Representation of $\mathbb{S}\mathrm{t}(R, F(t,s))$

The purpose of this section is to construct a kind of a matrix representation of $\mathbb{S}\mathrm{t}(R, F(t,s))$ and to deduce from it the description of $\mathbb{E}(R, F(t,s))$. From now on fix $t \geq s \geq 1$ and choose a doubling spectrum $\mathfrak{P}$ of $P_0 = F(t,s)$.

Denote by $\mathrm{St}(n)$, $n \geq 0$, a group generated by the symbols x_v^λ and the Steinberg relations (3), (4) where $v \in \mathrm{Col}\, P_n$ and $\lambda \in R$. There is a canonical homomorphism $\phi_n : \mathrm{St}(n) \to \mathrm{St}(n+1)$. It can be shown that $\mathbb{S}\mathrm{t}(R, P_0) = \lim \mathrm{St}(n)$.

Let $a_r(n) = \#\mathbf{A}^r(n)$ (for simplicity we write a_r). Denote by M_{kl} (or $M_{kl}(R)$) a set of all matrices (with k rows and l columns) with entries in R. Let $M(n)$ be a set of block matrices of the form

$$\begin{pmatrix} M_{a_1a_1} & M_{a_1a_2} & \bigoplus_{1\leq\tau\leq t} M_{a_11} \\ 0 & M_{a_2a_2} & \bigoplus_{1\leq\sigma\leq s} M_{a_21} \\ 0 & 0 & 1 \end{pmatrix}$$

(here $M_{a_1a_1}$ acts on $\bigoplus_\tau M_{a_11}$ diagonally, etc.).

We define the map $\psi_n : \mathrm{St}(n) \to M(n)$ as follows. Consider the “scheme”

$$\begin{pmatrix} \delta(1) & w & v_\tau \\ 0 & \delta(2) & u_\sigma \\ 0 & 0 & 1 \end{pmatrix}.$$

Let v be one of the column vectors $\delta_i^j(r)$, w_i^j, $v_{\tau,i}$ or $u_{\sigma,i}$ $(r = 1,2)$ of the polytope P_n. Define ψ_n on the generator x_v^λ to be the matrix from $M(n)$ with zero entries except 1 on the diagonal and λ placed in the block with the same “name” as the vector v at the intersection of i-th row and j-th column if v is $\delta_i^j(r)$ or w_i^j, or just in i-th row if $v = u_{\sigma,i}$ or $v = v_{\tau,i}$.

Proposition 2. *The map ψ_n is an epimorphism of* $\mathrm{St}(n)$ *onto the group*

$$E(n) = \begin{pmatrix} E_{a_1}(R) & M_{a_1a_2}(R) & \bigoplus_\tau M_{a_11}(R) \\ 0 & E_{a_2}(R) & \bigoplus_\sigma M_{a_21}(R) \\ 0 & 0 & 1 \end{pmatrix}.$$

The proof is straightforward.

While passing from P_n to P_{n+1} one of the numbers a_1, a_2 increases by 1. So we have obvious stabilization maps $\eta_n : M(n) \to M(n+1)$ and their restrictions $\eta_n : E(n) \to E(n+1)$. The diagram

$$\begin{array}{ccc} \mathrm{St}(n) & \xrightarrow{\phi_n} & \mathrm{St}(n+1) \\ \Big\downarrow{\scriptstyle \psi_n} & & \Big\downarrow{\scriptstyle \psi_{n+1}} \\ E(n) & \xrightarrow{\eta_n} & E(n+1) \end{array}$$

commutes, hence we have a homomorphism of the stable groups $\psi : \mathbb{S}\mathrm{t}(R, P_0) \to E(\infty)$, where $E(\infty)$ is the group

$$\begin{pmatrix} E(R) & M(R) & \bigoplus_\tau V(R) \\ 0 & E(R) & \bigoplus_\sigma V(R) \\ 0 & 0 & 1 \end{pmatrix},$$

$E(R) = \lim_n E_n(R)$, $M(R) = \lim_{m,n} M_{mn}(R)$, $V(R) = \lim_n M_{n1}(R)$.

5. Elementary group $\mathbb{E}(R, F(t,s))$ and K-theory

The representation ψ of $\mathbb{S}\mathrm{t}(R, F(t,s))$ is not exact.

Proposition 3. $\ker\psi = Z(\mathrm{St}(R, P_0))$.

Proof. The inclusion $\ker\psi \subset Z(\mathrm{St}(R, P_0))$ is obvious since $Z(E(\infty)) = 0$.

For the inverse inclusion we need the following statement, which generalizes Milnor's arguments from [**7**, Theorem 5.1].

Proposition 4. [**4**, proof of Proposition 8.2] *Assume that $\mathfrak{Q} = (Q = Q_0 \subset Q_1 \subset Q_2 \subset \ldots)$ is a doubling spectrum of a polytope Q. For every $i \in \mathbb{N}\cup\{0\}$ define two sets of column vectors $U^{i+1} = \{u \in \mathrm{Col}\,(Q_{i+1}) \mid \langle Q_i, u\rangle = 1\}$ and $V^{i+1} = \{v \in \mathrm{Col}\,(Q_{i+1}) \mid \langle Q_i, v\rangle = -1\}$. Consider subgroups $\mathfrak{U}^{i+1}, \mathfrak{V}^{i+1} \subset \mathrm{St}_{i+1}(R, Q)$ generated by all x_u^λ and x_v^μ correspondingly (here $u \in U^{i+1}$ and $v \in V^{i+1}$, $\lambda, \mu \in R$). Suppose for a group G, there is given an epimorphism $\pi\colon \mathrm{St}(R, Q) \to G$ which is injective on $\mathfrak{U}^{i+1}$ and $\mathfrak{V}^{i+1}$. Then $\ker\pi \subset Z(\mathrm{St}(R, Q))$.*

Apply this proposition to the doubling spectrum $\mathfrak{P}$ of P_0, $G = E(\infty)$ and $\pi = \psi$. From Lemmas 1 and 2 we can identify the sets U^n and V^n.

First of all suppose that doubling $P_{n-1} \subset P_n$ was done along a facet $A_i \in \mathbf{A}^1$. Then the set U^n consists of the vectors $\delta_j^i(1)$, $j = 1, \ldots, a_1(n)$, $j \neq i$. The set V^n

consists of the vectors $v_{\tau,i}$, $\delta_i^j(\mathbf{A}^1)$, $j = 1, \dots, a_1(n)$, $j \neq i$, and the vectors w_i^k, $k = 1, \dots, a_2(n)$.

Image of $\mathfrak{V}^n$ consists of the matrices in $E(n) \subset E(\infty)$ with 1 on the diagonal and other nonzero entries in the i-th rows of the blocks $\delta(1)$, w and $\oplus v_\tau$. Image of $\mathfrak{U}^n$ consists of matrices in $E(n) \subset E(\infty)$ with 1 on the diagonal and other nonzero entries in the i-th column of the block $\delta(\mathbf{A}^1)$.

To prove the injectivity one should note that from the Steinberg relations, it follows that $\mathfrak{U}^n$ are $\mathfrak{V}^n$ are abelian groups. Moreover, using arguments from Lemma 5.2 of [**7**] or from Proposition 8.2 of [**4**] one can show that these abelian groups are isomorphic to R^N for suitable numbers $N \in \mathbb{N}$. Then simple counting of dimensions shows that ψ is injective on $\mathfrak{U}^n$ and $\mathfrak{V}^n$.

The case of doubling $P_{n-1} \subset P_n$ along a facet from $\mathbf{A}^2$ is analogous. □

Theorem 1. *The groups $E(\infty)$ and $\mathbb{E}(R, P)$ are naturally isomorphic.*

Proof. The kernel of the natural homomorphism $\mathbb{St}(R, P_0) \to \mathbb{E}(R, P_0)$ coincides with $Z(\mathbb{St}(R, P_0))$ as $\mathbb{St}(R, P_0)$ is the universal central extension of $\mathbb{E}(R, P_0)$. It Proposition 3 it was shown that $\ker(\psi : \mathbb{St}(R, P_0) \to \mathbb{E}(\infty))$ also coincides with $Z(\mathbb{St}(R, P_0))$. Therefore there exists a natural isomorphism of $E(\infty)$ and $\mathbb{E}(R, P)$. □

Recall that a ring R is called an $S(n)$-ring if there exist elements $x_1, \dots, x_n \in R^*$ such that sum of any subset of them is a unit. The ring R *has many units* if R is an $S(n)$-ring for any $n \in \mathbb{N}$.

Corollary 1. *There is a natural isomorphism*

$$K_i(R, F(t, s)) = K_i(R) \oplus K_i(R), \quad i \geq 2,$$

provided R has many units.

The proof can be done in the same way as the proof of Theorem 9.2 from [**3**] with suitable minor changes.

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