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HARDY TYPE INEQUALITIES ON TIME SCALES

James A. Oguntuase

ABSTRACT. We obtain some new generalizations of Hardy type inequalities involving several functions on time scales. Furthermore, some new multidimensional Hardy–Knopp type inequalities on time scales are derived and discussed.

1. Introduction

Hardy [**4**] in a note published in 1920 announced (without proof) that if $p > 1$ and f is a nonnegative p-integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

Inequality (1.1), which is usually called the *classical Hardy inequality*, was proved in 1925 by Hardy in [**5**] (see also [**6, 7**]). Nowadays a well-known simple fact is that (1.1) can equivalently via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$, be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \leqslant \int_0^\infty h^p(x)\frac{dx}{x}, \tag{1.2}$$

and in this form it even holds with equality when $p = 1$. Observe that inequality (1.2) can easily be proved by using Jensen inequality and the Fubini theorem.

In a recent paper, Řehák [**12**] pioneered the time scale version of Hardy inequality by obtaining the following result:

$$\int_a^\infty \Big(\frac{F^\sigma(x)}{\sigma(x)-a}\Big)^p \Delta x \leqslant \Big(\frac{p}{p-1}\Big)^p \int_a^\infty f^p(x)\Delta x,$$

where $p > 1$, $F(x) := \int_0^x f(t)\Delta t$ and f is a nonnegative function.

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In a recent paper, Özkan and Yildirim [**10**] gave a time scale Hardy inequality involving several functions as follows:

THEOREM 1.1. *Let* $a \geqslant 0$ *and* $f_1, f_2, \dots, f_n$, $n \in \mathbb{Z}_+$, *be nonnegative integrable functions. Define* $F_k(x) = \frac{1}{\sigma(x)-a}\int_a^x f_k(t)\Delta t$, $k = 1, 2, \dots, n$. *Then*

$$\int_a^\infty \left(\prod_{k=1}^n F_k^\sigma(x)\right)^{p/n} \Delta x \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)^p \Delta x. \tag{1.3}$$

Furthermore, in the same paper [**10**] they also obtained the time scale Hardy–Knopp type inequality as follows:

THEOREM 1.2. *If* $u \in C_{rd}([a,b),\mathbb{R})$ *is a nonnegative function such that the delta integral* $\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x$ *exists as a finite number and the function* v *is defined by*

$$v(t) = (t-a)\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x, \quad t \in [a,b).$$

If $\phi : (c,d) \to \mathbb{R}$ *is continuous and convex, where* $c, d \in \mathbb{R}$, *then the inequality*

$$\int_a^b u(x)\phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \leqslant \int_a^b v(x)\phi(f(x))\frac{\Delta x}{x-a},$$

which holds for all delta integrable functions $f \in C_{rd}([a,b),\mathbb{R})$ *such that* $f(x) \in (c,d)$.

In 2009, Özkan and Yildirim [**11**] further obtained a generalization of Hardy–Knopp type inequality for several functions and also derived the Hardy–Knopp type inequality with a general kernel.

The aim of this paper is to obtain some new generalizations of Hardy type inequalities involving several functions and also some new multidimensional Hardy–Knopp type inequalities on time scales.

First we recall some basic concepts used in this paper and also refer interested reader to the books [**2, 3**] for a detailed theory of time scales. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

DEFINITION 1.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in T$.

The point t is said to be right-scattered if $\sigma(t) > t$, respectively left-scatted if $\rho(t) < t$. Points that are right-scattered and left-scattered at the same time are called isolated. The point t is called right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, respectively left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$. Finally, the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in T$.

A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if

(i) f is continuous at each right-dense point or maximal point of $\mathbb{T}$;
(ii) at each left-dense point $t \in \mathbb{T}$, $\lim_{s\to t^-} g(s) = g(t^-)$ exists.

The set of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is usually denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

2. Hardy integral inequality for several functions on time scales

In this section, we obtain generalization of Theorem 1.1. Before we give our results in this section, we make the following remark.

REMARK 2.1. Observe that inequality (1.3) follows directly by using the time scale Hardy inequality (see [**12**])

$$\int_a^\infty \Big(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\Big)^p \Delta x \leqslant \Big(\frac{p}{p-1}\Big)^p \int_a^\infty f^p(x)\Delta x \tag{2.1}$$

and the Arithmetic-Geometric Mean inequality

$$\Big(\prod_{k=1}^n F_k^\sigma(x)\Big)^{1/n} \leqslant \frac{1}{n}\sum_{k=1}^n F_k^\sigma(x) = \frac{1}{n}\int_a^{\sigma(x)}\Big(\sum_{k=1}^n f_k(t)\Big)\Delta t.$$

REMARK 2.2. If $a=0$, $\mathbb{T}=\mathbb{R}$ and $\sigma(t)=t$, $t\in\mathbb{T}$ (i.e., t is right dense), we obtain the classical Hardy inequality (1.1).

Our first result reads:

THEOREM 2.1. *Let $a\geqslant 0$, $p\neq 1$ and $n\in\mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^\infty$ be a positive sequence such that $\sum_{k=1}^\infty \alpha_k = 1$ and $\{f_k\}_{k=1}^\infty$ be a sequence of nonnegative delta integrable functions and let*

$$F_k(x) = \int_a^{\sigma(x)} f_k(t)\Delta t,\ \ k=1,2,\dots.$$

Then the inequality

$$\int_a^\infty \Big(\prod_{k=1}^\infty \Big[\frac{1}{(\sigma(x)-a)}F_k^\sigma(x)\Big]^{\alpha_k}\Big)^p \Delta x \leqslant \Big(\frac{p}{|p-1|}\Big)^p \int_a^\infty \Big(\sum_{k=1}^\infty \alpha_k f_k(x)\Big)^p \Delta x \tag{2.2}$$

holds if and only if $p>1$. If, in addition, $\frac{\mu(t)}{t}\to 0$ as $t\to\infty$, then the constant is sharp.

REMARK 2.3. By letting $p>1$ and

$$\alpha_k = \begin{cases} 1/k & k=1,2,\dots,n \\ 0 & k\geqslant n+1,\end{cases}$$

we obtain Theorem 4.1 of Özkan and Yildirim [**10**] (i.e., inequality (1.3)).

PROOF. First, assume that $p>1$. Then by using a more general arithmetic–geometric mean inequality (see [**9**])

$$\prod_{k=1}^\infty g_k^{\alpha_k}(x) \leqslant \sum_{k=1}^\infty \alpha_k g_k(x),$$

we easily obtain that

$$\Big(\prod_{k=1}^\infty [F_k^\sigma]^{\alpha_k}(x)\Big)^p \leqslant \Big(\sum_{k=1}^\infty \alpha_k F_k^\sigma(x)\Big)^p = \Big(\int_0^{\sigma(x)}\Big(\sum_{k=1}^\infty \alpha_k f_k(t)\Big)\Delta t\Big)^p. \tag{2.3}$$

By using the time scale Hardy inequality (2.1) with the functions $\sum_{k=1}^{\infty} \alpha_k f_k(t)$ and inequality (2.3), then inequality (2.2) is proved. The constant in the inequality is sharp since by applying it with $f_k(t) = f(t)$, $k = 1, 2, \ldots$, and the fact that $\frac{\mu(t)}{t} \to 0$ as $t \to \infty$ yields inequality (2.1). It is known that the constant in this inequality is sharp if $\frac{\mu(t)}{t} \to 0$ as $t \to \infty$ (see [**12**]).

Now, let $0 < p < 1$. Then (2.3) still holds. But then (2.2) cannot hold in general since by applying it with $f_k(x) = f(x)$, $k = 1, 2, \ldots$, it reduces to the inequality

$$\int_a^{\infty} \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f(t) \Delta t \right)^p \Delta x \leqslant \left(\frac{p}{1-p} \right)^p \int_a^{\infty} f^p(x) \Delta x$$

but it is well known that this is not true. In fact, it just holds in the reversed direction. The proof is complete. $\square$

Remark 2.4. For $p < 0$ it is known that (2.1) still holds but now (2.3) holds in the reversed direction so our proof above does not work so we leave it as an open question whether (2.2) holds in this case or not.

Next, we give the following multidimensional weighted version of Theorem 2.1. In what follows we use bold letters to denote the n-tuples of real numbers, e.g., $\mathbf{x} = (x_1, \ldots, x_n)$ or $\mathbf{t} = (t_1, \ldots, t_n)$, $\Delta \mathbf{t} = (\boldsymbol{\Delta} t_1 \ldots \boldsymbol{\Delta} t_1)$. In particular, we set $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$.

Theorem 2.2. *Let $p > 0$, $p \neq 1$, $m \neq 1$ and $n \in \mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a positive sequence such that $\sum_{k=1}^{\infty} \alpha_k = 1$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence of delta integrable functions on $[\mathbf{a}, \mathbf{b}]$, $0 \leqslant \mathbf{b} \leqslant \infty$, and let*

$$F_k(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} f_k(\mathbf{t}) \Delta t_1 \ldots \Delta t_n, \ k = 1, 2, \ldots.$$

Then the inequality

$$\begin{aligned} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-m} \left(\prod_{k=1}^{\infty} [F_k^{\sigma}(\mathbf{x})]^{\alpha_k} \right)^p \Delta x_1 \ldots \Delta x_n \\ \leqslant \left(\frac{p}{|m-1|} \right)^{np} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{i=1}^{n} \left[1 - \left(\frac{x_i - a_i}{b_i - a_i} \right) \right]^{\frac{m-1}{p}} \left(\sum_{k=1}^{\infty} \alpha_k f_k(\mathbf{x}) \right)^p \\ \times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{p-m} \Delta x_1 \ldots \Delta x_n \end{aligned}$$

holds if and only if $p > 1$ and the constant $\left(\frac{p}{m-1}\right)^{np}$ is sharp.

Proof. We just use Theorem 3.1 in [**8**] instead of the time scale Hardy's inequality (2.1) and the proof is similar to the proof of Theorem 2.1. We omit the details. $\square$

Remark 2.5. For the case $n = 1$, if we set $m = p > 1$, then the result in Theorem 2.2 coincides with that in Theorem 2.1.

REMARK 2.6. In Theorem 2.2, if we let $a_i = 0$, $i = 1, 2, \dots, n$ and let the point t be right-dense (i.e., $\sigma(t) = t$), then we obtain Theorem 2.2 in [**9**].

If under the same assumptions of Theorem 2.2, if we set $b_i = \infty$, $i = 1, 2, \dots, n$, then we obtain the following result.

COROLLARY 2.1. *Let $p > 0$, $p \neq 1$, $m \neq 1$ and $n \in \mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a positive sequence such that $\sum_{k=1}^{\infty} \alpha_k = 1$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence of delta integrable functions on $[\mathbf{a}, \mathbf{b}]$, $0 \leqslant \mathbf{b} \leqslant \infty$, and let*

$$F_k(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} f_k(\mathbf{t}) \Delta t_1 \dots \Delta t_n, \quad k = 1, 2, \dots.$$

Then the inequality

$$\int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \prod_{i=1}^{n} \big(\sigma(x_i) - a_i\big)^{-m} \bigg(\prod_{k=1}^{\infty} [F_k^{\sigma}(\mathbf{x})]^{\alpha_k} \bigg)^p \Delta x_1 \dots \Delta x_n \\ \leqslant \Big(\frac{p}{|m-1|} \Big)^{np} \int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} \bigg(\sum_{k=1}^{\infty} \alpha_k f_k(\mathbf{x}) \bigg)^p \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{p-m} \Delta x_1 \dots \Delta x_n. \tag{2.4}$$

holds if and only if $p > 1$ and the constant $\big(\frac{p}{m-1}\big)^{np}$ is sharp.

PROOF. The proof follows directly from the proof of Theorem 2.2 and so the details are omitted. □

REMARK 2.7. By setting $a_i = 0$, $i = 1, \dots, n$, then inequality (2.4) yields

$$\int_{0}^{\infty} \dots \int_{0}^{\infty} \prod_{i=1}^{n} (\sigma(x_i))^{-m} \bigg(\prod_{k=1}^{\infty} [F_k^{\sigma}(\mathbf{x})]^{\alpha_k} \bigg)^p \Delta x_1 \dots \Delta x_n \\ \leqslant \Big(\frac{p}{|m-1|} \Big)^{np} \int_{0}^{\infty} \dots \int_{0}^{\infty} \bigg(\sum_{k=1}^{\infty} \alpha_k f_k(\mathbf{x}) \bigg)^p \prod_{i=1}^{n} (\sigma(x_i))^{p-m} \Delta x_1 \dots \Delta x_n.$$

3. Multidimensional Hardy–Knopp type inequality on time scale

Throughout this section, we assume that $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ are two time scale measures. Let $U \subset \mathbb{R}^m$ be a closed convex set, $\phi \in C(U, \mathbb{R})$ is convex such that $f(\Lambda) \subset U$. In particular, we take

$$\Omega = \Lambda = [a_1, b_1)_{\mathbb{T}} \times [a_1, b_1)_{\mathbb{T}} \times \dots \times [a_1, b_1)_{\mathbb{T}}, 0 \leqslant a_i < b_i \leqslant \infty$$

for all $i = 1, 2, \dots, n$, where $\mathbb{T}$ is a time scale.

Before we we state our main results in this section, we recall Jensen's inequality and Fubini's theorem on time scales which will be used in the proofs of our main results:

LEMMA 3.1 (Jensen's Inequality). [**2**, Theorem 6.17] *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g : [a, b] \to (c, d)$ is rd-continuous and $\phi : (c, d) \to \mathbb{R}$ is continuous and convex, then*

$$\phi\bigg(\frac{1}{b-a} \int_a^b g(t) \Delta(t) \bigg) \leqslant \frac{1}{b-a} \int_a^b \phi(g(t)) \Delta(t). \tag{3.1}$$

Lemma 3.2 (Fubini Theorem). [**1**, Theorem 1.1] *If $f:\Omega\times\Lambda\to\mathbb{R}$ is a $\mu_\Delta\times\lambda_\Delta$-integrable functions and if we define the function $\varphi=\int_\Omega f(x,y)\Delta x$ for a.e. $y\in\Lambda$ and $\psi(x)=\int_\Lambda f(x,y)\Delta y$ for a.e. $y\in\Omega$, then φ is λ_Δ-integrable on Λ, ψ is μ_Δ-integrable on Ω and*

$$\int_\Omega \Delta x\int_\Lambda f(x,y)\Delta y=\int_\Lambda \Delta y\int_\Omega f(x,y)\Delta x. \tag{3.2}$$

Our first result in this section reads:

Theorem 3.1. *Let $u:\Omega\to\mathbb{R}_+$ be a nonnegative function such that the delta integral*

$$\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\frac{u(\mathbf{x})}{\prod_{i=1}^n(x_i-a_i)(\sigma(x_i)-a_i)}\Delta x_1\dots\Delta x_n,$$

exists as a finite number and the function v be given by

$$v(\mathbf{t})=\prod_{i=1}^n(t_i-a_i)\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\frac{u(\mathbf{x})}{\prod_{i=1}^n(x_i-a_i)(\sigma(x_i)-a_i)}\Delta x_1\dots\Delta x_n,\quad t_i\in[a_i,b_i). \tag{3.3}$$

If $U\subset\mathbb{R}^m$ is a closed convex set such that the function $\phi:U\to\mathbb{R}$ is convex and continuous, then the inequality

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}u(\mathbf{x})\,\phi\bigg(\frac{1}{\prod_{i=1}^n(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}f(\mathbf{t})\Delta t_1\dots\Delta t_n\bigg) \\ \times\frac{\Delta x_1\dots\Delta x_n}{\prod_{i=1}^n(x_i-a_i)}\leqslant\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}v(\mathbf{x})\,\phi(f(\mathbf{x}))\frac{\Delta x_1\dots\Delta x_n}{(x_1-a_1)\dots(x_n-a_n)}. \tag{3.4}$$

holds for all delta integrable functions $f:\Lambda\to\mathbb{R}^m$ such that $f(\Lambda)\subset U$.

Remark 3.1. If ϕ is concave, then (3.4) holds in the reverse direction.

Proof. By application of Jensen's inequality (3.1) and Fubini theorem (3.2) on time scales, we find that

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}u(\mathbf{x})\phi\bigg(\frac{1}{\prod_{i=1}^n(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}f(\mathbf{t})\Delta t_1\dots\Delta t_n\bigg)\frac{\Delta x_1\dots\Delta x_n}{\prod_{i=1}^n(x_i-a_i)}\\
&\leqslant\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\bigg(\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}\phi(f(\mathbf{t}))\Delta t_1\dots\Delta t_n\bigg)\frac{u(\mathbf{x})\Delta x_1\dots\Delta x_n}{\prod_{i=1}^n(x_i-a_i)(\sigma(x_i)-a_i)}\\
&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\phi(f(\mathbf{t}))\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\bigg(\frac{u(\mathbf{x})\Delta x_1\dots\Delta x_n}{\prod_{i=1}^n(\sigma(x_i)-a_i)(x_i-a_i)}\bigg)\Delta t_1\dots\Delta t_n\\
&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\phi(f(\mathbf{t}))\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}\bigg(\frac{u(\mathbf{x})\Delta x_1\dots\Delta x_n}{\prod_{i=1}^n(\sigma(x_i)-a_i)(x_i-a_i)}\bigg)\Delta t_1\dots\Delta t_n\\
&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\phi(f(\mathbf{t}))v(t)\frac{\Delta t_1\dots\Delta t_n}{(t_1-a_1)\dots(t_n-a_n)}.
\end{aligned}$$

The proof is complete. □

EXAMPLE 3.1. If we set the weight function $u(\mathbf{x}) = 1$ in Theorem 3.1, then the weight function (3.3) yields

$$v(\mathbf{t}) = \begin{cases} \prod_{i=1}^{n}\left(1 - \frac{t_i - a_i}{b_i - a_i}\right) & \text{if } b_i < \infty \\ 1 & \text{if } b_i = \infty. \end{cases}$$

Hence, inequality (3.4) in this setting for the case $b_i < \infty$ reads

$$\begin{aligned}(3.5)\qquad \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \phi\Big(\frac{1}{\prod_{i=1}^{n}(\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} f(\mathbf{t})\Delta t_1 \dots \Delta t_n\Big) \\ \times \frac{\Delta x_1 \dots \Delta x_n}{\prod_{i=1}^{n}(x_i - a_i)} \\ \leqslant \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^{n}\Big(1 - \frac{t_i - a_i}{b_i - a_i}\Big)\phi(f(\mathbf{x})) \times \frac{\Delta x_1 \dots \Delta x_n}{(x_1 - a_1) \dots (x_n - a_n)},\end{aligned}$$

while the case $b_i = \infty$ yields

$$\begin{aligned}\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \phi\Big(\frac{1}{\prod_{i=1}^{n}(\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} f(\mathbf{t})\Delta t_1 \dots \Delta t_n\Big) \\ \times \frac{\Delta x_1 \dots \Delta x_n}{\prod_{i=1}^{n}(x_i - a_i)} \\ \leqslant \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{i=1}^{n} \phi(f(\mathbf{x})) \frac{\Delta x_1 \dots \Delta x_n}{(x_1 - a_1) \dots (x_n - a_n)}.\end{aligned}$$

REMARK 3.2. If we set $n = 1$, then Example 3.1 coincides with Corollary 2.1 in [**10**]. Also, in the special case $n = 2$, inequality (3.5) reduces to Theorem 3.2 in [**10**].

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Department of Mathematics
Federal University of Agriculture
P.M.B. 2240
Abeokuta
Nigeria
oguntuase@yahoo.com

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