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A NEW PERSPECTIVE FOR MULTIVALUED WEAKLY PICARD OPERATORS

Gonca Durmaz and Ishak Altun

Abstract. This research contains some recent developments about multivalued weakly Picard operators on complete metric spaces. In addition, taking into account both multivalued θ-contraction and almost contraction on complete metric spaces, we present a new perspective for multivalued weakly Picard operators. Finally, we give a nontrivial example showing that the investigation of this paper is significant.

1. Introduction and Preliminaries

The concept of multivalued weakly Picard operator, which is introduced by Rus et al [**16**], is closely related to metric fixed point theory. Let (X, d) be a metric space and $T\colon X \to \mathcal{P}(X)$ be a mapping, where $\mathcal{P}(X)$ is the family of all nonempty subsets of X. Then T is said to be a multivalued weakly Picard (for short MWP) operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which is convergent and its limit is a fixed point of T. Berinde and Berinde [**2**] show that the type multivalued contractions on complete metric spaces considered by Nadler [**12**], Petruşel [**13**], Reich [**14**] and Rus [**15**] are MWP operators.

For the sake of completeness we recall some important concepts and results about multivalued mappings. In 1969, Nadler [**12**] initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and proved the following fundamental result:

Theorem 1.1 (Nadler [**12**]). *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ a multivalued mapping, where $\mathcal{CB}(X)$ is the family of all nonempty closed and bounded subsets of X. If T is a multivalued contraction, that is, there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leqslant Ld(x, y)$ for all $x, y \in X$, where H is the Pompeiu–Hausdorff metric on $\mathcal{CB}(X)$ defined by*

$$H(A, B) = \max \big\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A) \big\},$$

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and $d(x,B)=\inf\{d(x,y):y\in B\}$*, then there exists* $z\in X$ *such that* $z\in Tz$*.*

Inspired by his result, there has been vigorous and dense research activity for fixed point results concerning multivalued contraction, and by now, there are a number of results that generalize this result in many different directions and many researchers have given fantastic contributions to these areas (see [**3**–**5**, **9**–**11**]).

Recently, Berinde and Berinde [**2**] introduced the concepts of multivalued almost contraction (the original name was multivalued (δ,L)-weak contraction) and proved the following attracted result for MWP operators:

THEOREM 1.2 (Berinde and Berinde [**2**]). *Let* (X,d) *be a complete metric space and* $T\colon X\to\mathcal{CB}(X)$ *a given mapping. If* T *is a multivalued almost contraction, that is, there exist two constants* $\delta\in(0,1)$ *and* $L\geqslant 0$ *such that*

$$H(Tx,Ty)\leqslant \delta d(x,y)+Ld(y,Tx) \tag{1.1}$$

for all $x,y\in X$*, then* T *is an MWP operator.*

On the other hand, introducing a new type of contractive mapping, Jleli and Samet [**7**] presented an attracted generalization of the Banach contraction principle. Throughout this study we shall call the contraction defined in [**7**] the θ-contraction. Now, we recall basic definitions, relevant notions and some related results concerning the θ-contraction.

Let Θ be the set of all functions $\theta\colon(0,\infty)\to(1,\infty)$ satisfying the conditions:

- (θ_1) θ is nondecreasing;
- (θ_2) For each sequence $\{t_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\theta(t_n)=1$ and $\lim_{n\to\infty}t_n=0^+$ are equivalent;
- (θ_3) There exist $r\in(0,1)$ and $l\in(0,\infty]$ such that $\lim_{t\to 0^+}\frac{\theta(t)-1}{t^r}=l$.

Let (X,d) be a metric space and $\theta\in\Theta$. A mapping $T\colon X\to X$ is said to be a θ-contraction if there exists a constant $k\in[0,1)$ such that

$$\theta(d(Tx,Ty))\leqslant[\theta(d(x,y))]^k \tag{1.2}$$

for all $x,y\in X$ with $d(Tx,Ty)>0$.

Choosing some appropriate functions for θ, such as $\theta_1(t)=e^{\sqrt{t}}$ and $\theta_2(t)=e^{\sqrt{te^t}}$, we can obtain some different types of nonequivalent contractions from (1.2). Considering this new concept, Jleli and Samet proved that every θ-contraction on a complete metric space has a unique fixed point. In the literature some interesting papers concerning θ-contractions can be found (see [**1**, **8**]).

Naturally, the concept of θ-contraction extended to multivalued mappings by Hançer et al [**6**] and they introduced the concept of multivalued θ-contraction: Let (X,d) be a metric space, $T\colon X\to\mathcal{CB}(X)$ be a mapping and $\theta\in\Theta$. Then T is said to be a multivalued θ-contraction if there exists a constant $k\in[0,1)$ such that

$$\theta(H(Tx,Ty))\leqslant[\theta(d(x,y))]^k \tag{1.3}$$

for all $x,y\in X$ with $H(Tx,Ty)>0$.

Consequently, they established some fixed point results for multivalued θ-contraction mappings on complete metric spaces as follows:

THEOREM 1.3. [**6**] *Let (X,d) be a complete metric space and $T\colon X \to \mathcal{K}(X)$ be given a multivalued mapping, where $\mathcal{K}(X)$ is the family of all nonempty compact subsets of X. If T is a multivalued θ-contraction, then T has a fixed point.*

Since the compactness of Tx for all $x \in X$ in Theorem 1.3 is a strong condition, it is intended to replace $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$. However, in the same paper they also gave an example [**6**, Example 2.4] showing that this is not impossible. Even so, this replacement is possible by adding the following weak condition on θ:

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Let Ω be the family of all functions θ satisfying (θ_1)-(θ_4). It is clear that $\Omega \subset \Theta$. If we define $\theta(t) = e^{\sqrt{t}}$ for $t < 1$ and $\theta(t) = 9$ for $t \geqslant 1$, then $\theta \in \Theta \backslash \Omega$. Note that, if θ is right continuous and satisfies (θ_1), then (θ_4) hold. Conversely, if (θ_4) hold, then θ is right continuous.

THEOREM 1.4. [**6**] *Let (X,d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be given a multivalued mapping. If T is a multivalued θ-contraction with $\theta \in \Omega$, then T has a fixed point.*

If we examine the proofs of Theorem 1.3 and Theorem 1.4, we can see that the mentioned multivalued mappings are MWP operators.

The aim of this paper is to give a new and general class of multivalued weakly Picard operators on complete metric space. For this, we will introduce a new type contraction for multivalued mappings taking into account both multivalued almost contraction and multivalued θ-contraction. Later, we give some fixed point results for mappings of this type on complete metric spaces.

2. Results

Our main results are based on the following new concept.

Let (X,d) be a metric space, $T\colon X \to \mathcal{CB}(X)$ be a given mapping and $\theta \in \Theta$. Then, we say that T is a multivalued almost θ-contraction if there exist two constants $k \in (0,1)$ and $\lambda \geqslant 0$ such that

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda d(y,Tx))]^k, \tag{2.1}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

Note that, taking into account the symmetry property of the metric, the multivalued almost θ-contractive condition includes the following dual one

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda d(x,Ty))]^k, \tag{2.2}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$. So, in order to check the multivalued almost θ-contractiveness of a multivalued mapping T, it is necessary to check both (2.1) and (2.2) or the following inequality:

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda \min\{d(y,Tx), d(x,Ty)\})]^k,$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

REMARK 2.1. Taking $\theta(t) = e^{\sqrt{t}}$ in inequality (2.1), then it turns to (1.1) with $\delta = k^2$ and $L = k^2\lambda$. Thus, every multivalued almost contraction is also multivalued almost θ-contraction. On the other hand, taking $\lambda = 0$ in inequality (2.1), then it turns to (1.3). Thus, every multivalued θ-contraction is also multivalued almost θ-contraction. Therefore, Theorems 1.1, 1.2 and 1.4 are special cases of the following of first result of ours.

In fact, our first result also presents a new class of multivalued weakly Picard operators on a complete metric space.

THEOREM 2.1. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be given a mapping. If T is an multivalued almost θ-contraction with $\theta \in \Omega$, then T is a MWP.*

PROOF. Define a set $X^* = \{x \in X : d(x, Tx) > 0\}$. Let $x_0 \in X^*$ be an arbitrary point and choose $x_1 \in Tx_0$. If $x_1 \notin X^*$, then x_1 is a fixed point of T. Suppose $x_1 \in X^*$, then $0 < d(x_1, Tx_1) \leqslant H(Tx_0, Tx_1)$ and so from (θ_1), we obtain $\theta(d(x_1, Tx_1)) \leqslant \theta(H(Tx_0, Tx_1))$. From (2.1), we can write

$$\begin{aligned} \theta(d(x_1, Tx_1)) &\leqslant \theta(H(Tx_0, Tx_1)) \\ &\leqslant [\theta(d(x_1, x_0) + \lambda d(x_1, Tx_0))]^k \leqslant [\theta(d(x_1, x_0))]^k. \end{aligned} \tag{2.3}$$

From (θ_4), we know that $\theta(d(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y))$, and so, from (2.3), we have

$$\inf_{y \in Tx_1} \theta(d(x_1, y)) \leqslant [\theta(d(x_0, x_1))]^k < [\theta(d(x_0, x_1))]^s, \tag{2.4}$$

where $s \in (k, 1)$. Then, from (2.4) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leqslant [\theta(d(x_0, x_1))]^s.$$

If $x_2 \notin X^*$, then x_2 is a fixed point of T. Otherwise, by the same way, we can find $x_3 \in Tx_2$ such that $\theta(d(x_2, x_3)) \leqslant [\theta(d(x_1, x_2))]^s$. Therefore, continuing recursively, we can obtain a sequence $\{x_n\}$ in X^* such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leqslant [\theta(d(x_{n-1}, x_n))]^s, \tag{2.5}$$

for all $n \in \mathbb{N}$ (Otherwise T has a fixed point). Denote $c_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N}$. Then $c_n > 0$ for all $n \in \mathbb{N}$ and, using (2.5), we have

$$\theta(c_n) \leqslant [\theta(c_{n-1})]^s \leqslant [\theta(c_{n-2})]^{s^2} \leqslant \cdots \leqslant [\theta(c_1)]^{s^{n-1}}.$$

Thus, we obtain

$$1 < \theta(c_n) \leqslant [\theta(c_1)]^{s^{n-1}} \tag{2.6}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.6), we obtain $\lim_{n\to\infty} \theta(c_n) = 1$. From (θ_2), $\lim_{n\to\infty} c_n = 0^+$ and so, from (θ_3), there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(c_n) - 1}{(c_n)^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\Big|\frac{\theta(c_n) - 1}{(c_n)^r} - l\Big| \leqslant B.$$

This implies that, for all $n \geqslant n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geqslant l - B = B.$$

Then, for all $n \geqslant n_0$, we have $n(c_n)^r \leqslant An[\theta(c_n) - 1]$, where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geqslant B.$$

This implies that, for all $n \geqslant n_0$, we have $n(c_n)^r \leqslant An[\theta(c_n) - 1]$, where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that $n(c_n)^r \leqslant An[\theta(c_n) - 1]$, for all $n \geqslant n_0$. Using (2.6), we obtain $n(c_n)^r \leqslant An\big[[\theta(c_1)]^{s^{n-1}} - 1\big]$, for all $n \geqslant n_0$. Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} n(c_n)^r = 0$. Thus, there exits $n_1 \in \mathbb{N}$ such that $n(c_n)^r \leqslant 1$ for all $n \geqslant n_1$. So, we have for all $n \geqslant n_1$

$$c_n \leqslant \frac{1}{n^{1/r}}. \tag{2.7}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. Using the triangular inequality for the metric and from (2.7), we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= c_n + c_{n+1} + \cdots + c_{m-1} = \sum_{i=n}^{m-1} c_i \leqslant \sum_{i=n}^{\infty} c_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

Now, from (θ_1) and (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y) + \lambda d(y, Tx)$$

and so, for all $x, y \in X$, we get $H(Tx, Ty) \leqslant d(x, y) + \lambda d(y, Tx)$. Therefore,

$$d(x_{n+1}, Tz) \leqslant H(Tx_n, Tz) \leqslant d(x_n, z) + \lambda d(z, Tx_n) \leqslant d(x_n, z) + \lambda d(z, x_{n-1}).$$

Letting to limit $n \to \infty$ in the above inequality, we obtain $d(z, Tz) = 0$. Thus, we get $z \in Tz$. Therefore, it can be seen that, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which is convergent and its limit is a fixed point of T. That is, T is a weakly Picard operator. Therefore T is a MWP. □

Now, we give a nontrivial example showing that T is a MWP because of it is multivalued almost θ-contraction on a complete metric space. Nevertheless, taking into account Theorem 1.2 (or Theorem 1.4), we can not guarantee that T is a MWP since it is not both multivalued almost contraction and multivalued θ-contraction.

EXAMPLE 2.1. Let $X = [0,1] \cup \{2,3,\cdots\}$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |x-y|, & \text{if } x,y \in [0,1] \\ x+y, & \text{if one of } x,y \notin [0,1] \end{cases}$$

Then (X,d) is a complete metric space. Define a mapping $T\colon X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \{x\}, & x \in [0,1] \\ \{1, x-1\}, & x \in \{2,3,\dots\} \end{cases}.$$

First, suppose that T is a multivalued almost contraction. Then there exists two constants $\delta \in (0,1)$ and $L \geqslant 0$ satisfying $H(Tx,Ty) \leqslant \delta d(x,y) + Ld(y,Tx)$ for all $x,y \in X$. Now, for $y = 1$ and $x > 2$, since $d(y,Tx) = 0$, we get

$$x = H(Tx,Ty) \leqslant \delta d(x,y) + Ld(y,Tx) = \delta(x+1)$$

and so $\frac{x}{x+1} \leqslant \delta$ for all $x \in X$, which is impossible.

Second, T is not also multivalued θ-contraction, since $H(T0,T1) = 1 = d(0,1)$, then for all $\theta \in \Omega$ and any $k \in (0,1)$, we have

$$\theta(H(Tx,Ty)) = \theta(1) > [\theta(1)]^k = [\theta(d(x,y))]^k.$$

Finally, we claim that T is multivalued almost θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$, $k = \frac{1}{\sqrt{2}}$ and $\lambda = 1$. To see this, we have to show that

$$\frac{H(Tx,Ty)e^{H(Tx,Ty)-d(x,y)-\min\{d(y,Tx),d(x,Ty)\}}}{d(x,y)+\min\{d(y,Tx),d(x,Ty)\}} \leqslant \frac{1}{2}, \tag{2.8}$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$. Note that, $H(Tx,Ty) > 0$ if and only if $(x,y) \notin \Delta \cup \{(1,2),(2,1)\}$, where $\Delta = \{(x,x) : x \in X\}$. Now, for shortness we will assign the left side of (2.8) as $A(x,y)$. Without loss of generality, we may assume $x > y$ in the following three cases:

Case 1. For $x,y \in [0,1]$, since

$$H(Tx,Ty) = d(x,y) = \min\{d(y,Tx),d(x,Ty)\} = x-y,$$

we have

$$A(x,y) = \frac{x-y}{2(x-y)}e^{-(x-y)} < \frac{x-y}{2(x-y)} = \frac{1}{2}.$$

Case 2. For $y \in [0,1]$ and $x \in \{2,3,\dots\}$, since

$$H(Tx,Ty) = x+y-1, \quad d(x,y) = x+y, \quad \min\{d(y,Tx),d(x,Ty)\} = 1-y,$$

we have

$$A(x,y) = \frac{x+y-1}{x+1}e^{y-2} < e^{-1} < \frac{1}{2}.$$

Case 3. For $x, y \in \{2, 3, \dots\}$, since

$$H(Tx, Ty) = x + y - 2, \quad d(x, y) = x + y, \quad \min\{d(y, Tx), d(x, Ty)\} = 1 + y,$$

we have

$$A(x, y) = \frac{x + y - 2}{x + 2y + 1} e^{-3-y} < e^{-1} < \frac{1}{2}.$$

This shows that T is multivalued almost θ-contraction. Thus, all conditions of Theorem 2.1 are satisfied and so T is a MWP.

By taking $\theta(t) = e^{\sqrt{t^2+t}}$ in Theorem 2.1, we obtain the following corollary:

COROLLARY 2.1. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be a mapping. Suppose that, there exists two constants $l \in (0, 1)$ and $\lambda \geqslant 0$ such that*

$$\frac{H(Tx, Ty)[H(Tx, Ty) + 1]}{[d(x, y) + \lambda d(y, Tx)][d(x, y) + \lambda d(y, Tx) + 1]} \leqslant l,$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Then, T has a fixed point.

The following result is interested in the mapping $T\colon X \to \mathcal{K}(X)$. Here, we can remove the condition (θ_4) on the function θ.

THEOREM 2.2. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{K}(X)$ be given a mapping. If T is a multivalued almost θ-contraction, then T is a MWP.*

PROOF. As in proof of Theorem 2.1, we get

$$\theta(d(x_1, Tx_1)) \leqslant \theta(H(Tx_0, Tx_1)) \leqslant [\theta(d(x_1, x_0))]^k. \tag{2.9}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. From (2.9),

$$\theta(d(x_1, x_2)) \leqslant \theta(H(Tx_0, Tx_1)) \leqslant [\theta(d(x_1, x_0))]^k.$$

By induction, we obtain a sequence $\{x_n\}$ in X^* with the property that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leqslant [\theta(d(x_n, x_{n-1}))]^k$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 2.1. ☐

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Department of Mathematics
Faculty of Science
Çankırı Karatekin University
Çankırı
Turkey
gncmatematik@hotmail.com

King Saud University
College of Science
Riyadh
Saudi Arabia

Department of Mathematics
Faculty of Science and Arts
Kirikkale University
Kirikkale
Turkey
ishakaltun@yahoo.com