PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 102(116) (2017), 247–262 DOI: https://doi.org/10.2298/PIM1716247Z

HOLOMORPHIC SERIES EXPANSION OF FUNCTIONS OF CARLEMAN TYPE ON THE INTERVAL $[-1, 1]$

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Abstract. We characterize the functions of some Carleman classes on the unit interval $[-1, 1]$ as sums of holomorphic functions in specific neighborhoods of $[-1, 1]$. As an application of our main theorem, we perform an alternative construction of the Dyn'kin's pseudoanalytic extension for these Carleman classes on $[-1, 1]$.

1. Introduction

In 1926 [**8**] Carleman raised the problem of the representation of the functions of a quasianalytic class in terms of their successive derivatives at a given point. He noticed that this problem should be solved by a decomposition method. This problem was also raised by Julia in 1925 [**13, 14, 15**], while he was looking for an algorithmic generalization of the classical Borel process which generates classes of quasi-analytic functions from sequences of complex numbers converging to 0. In 1962 [**2**] Badalyan gave, by his theory of quasi-powers (a generalization to quasianalytic Carleman classes of Taylor series expansion) the complete solution to Carleman's problem. In 1970 [**3**] Badalyan generalized his theory to some non-quasianalytic classes. In 1991 [**10**, pp. 249–253] Ecalle obtained for the functions of a regular Carleman class on a segment $[a, b]$, a series expansion into holomorphic functions on specific neighborhoods of $[a, b]$. In 2004, Belghiti obtained for certain Carleman classes on arbitrary bounded convex planar domains [**4**] a similar but more explicit holomorphic expansion series. Let us observe that the approach in [**4, 10**] relies mainly on the theorem of pseudoanalytic extension due to Dyn'kin [**9**].

Improving the methods of Ecalle and Belghiti, we obtained in [**5**] a characterization of the functions of a Gevrey class on $[-1, 1]$ as sums of series of holomorphic functions in suitable neighborhoods of $[-1, 1]$, and here we generalize this method to some Carleman classes on $[-1, 1]$. As an application of our main theorem, we

2010 *Mathematics Subject Classification:* Primary 30D60; Secondary 41A20.
Key words and phrases: Carleman classes.
Communicated by Stevan Pilipović.

derive an alternative construction of Dyn'kin's pseudoanalytic extension for these Carleman classes.

2. Preliminary notes

Let S be a nonempty subset of $\mathbb{C}$, $f\colon S \to \mathbb{C}$ a bounded function, and $\|f\|_{\infty,S} := \sup_{z\in S}|f(z)|$. For $z \in \mathbb{C}$ we set $\rho(z,S) := \inf_{u\in S}|z-u|$. For $r > 0$, $B(z,r)$ is the usual open ball in $\mathbb{C}$ with center z and radius r. We set also

$$S_r := S + B\,(0;r) := \{z + u : z \in S,\ u \in B(0,r)\}$$

Thus we have $S_r = \{z \in \mathbb{C} : \rho(z,S) < r\}$. $\mathcal{O}(S)$ denotes the set of holomorphic functions on some neighborhood of S.

Let $\alpha := (\alpha_1,\alpha_2)$, $\beta := (\beta_1,\beta_2) \in \mathbb{N}^2$. We write $\beta \leqslant \alpha$ if: $\beta_1 \leqslant \alpha_1$ and $\beta_2 \leqslant \alpha_2$. Given a property $\mathfrak{P}(x)$, with $x \in \mathbb{R}$, we say that $\mathfrak{P}(x)$ holds ultimately if there exists $a_0 \in \mathbb{R}$ such that $\mathfrak{P}(x)$ holds for all $x \geqslant a_0$. We define an equivalence relation on the set $\mathcal{F}$ of real valued functions which are defined on a real half-line by writing $f =_\infty g$ if we have $f(x) = g(x)$ ultimately. We denote by $[f]$ the class of equivalence of a function $f \in \mathcal{F}$ for the equivalence relation $=_\infty$. The quotient set $\mathcal{G} := \mathcal{F}/{=_\infty}$ is endowed with operations of addition and multiplication induced by those of $\mathcal{F}$ making $\mathcal{G}$ into a commutative ring. The classes of equivalence for this relations are called the germs of functions at $+\infty$. To simplify the writing we will identify the germ $[f]$ with its representant f. We consider the field $\mathbb{R}$ of real numbers as a subring of $\mathcal{G}$, by identifying a real number a with the germ of the function $x \mapsto a$.

We denote by $\mathcal{G}_1$ the subring of $\mathcal{G}$ consisting of the germs of functions that are ultimately of class C^1. A subring $\aleph$ of $\mathcal{G}_1$ is called a Hardy field if $\aleph$ is a field which is stable by derivation. Functions belonging to a Hardy field $\aleph$ have the following properties: they are ultimately strictly monotone unless they are ultimately constant, they are ultimately of constant sign unless they are ultimately identically vanishing. It follows that for every $f \in \aleph$ the limit $\lim_{x\to+\infty} f(x)$ exists in $\mathbb{R} \cup \{+\infty, -\infty\}$, and that for every $f,\ g \in \aleph$ we have ultimately one of the following cases $f(x) < g(x)$, $g(x) < f(x)$, $f(x) = g(x)$.

We say that an element f of $\aleph$ is bounded if $\lim_{x\to+\infty} f(x) \in \mathbb{R}$, infinitesimal if $\lim_{x\to+\infty} f(x) = 0$, and infinite if $\lim_{x\to+\infty} |f(x)| = +\infty$.

If $f, g \in \aleph$ and g is infinite and ultimately positive, then $f \circ g \in \mathcal{G}_1$ is by definition the germ in $\mathcal{G}_1$ such that ultimately $(f \circ g\ )(x) = f(g(x))$.

In our work we will need the following results.

Theorem 2.1. [**1, 19**]. *Let f be an infinite and ultimately positive element of a Hardy field $\aleph$. Then there exists a Hardy field $\mathcal{H}$ and a germ $g \in \mathcal{H}$ such that g is an infinite and ultimately positive element of the Hardy field $\mathcal{H}$ and $(f \circ g)(x) = x$ ultimately. g is called the inverse of f and is denoted by $g^{\langle -1\rangle}$.*

Theorem 2.2. [**1, 18, 20**]. *Let $F(Y), G(Y) \in \aleph[Y]$ and $y \in \mathcal{G}_1$ be such that $G(y) \neq 0$ and $y'\, G(y) = F(y)$ (in $\mathcal{G}_1$). Then the ring of germs $\aleph[y]$ is an integral domain with fraction field $\aleph(y) \subset \mathcal{G}_1$, and $\aleph(y)$ is a Hardy field.*

As a consequence of this theorem, it follows that a Hardy field $\aleph$ can be enlarged to a Hardy field $\aleph_0$ containing the germ Id of the identity function and the germ ln of the logarithmic function. The following theorem provides a strong generalization of this remark.

THEOREM 2.3. [**7**] *Let $\aleph$ be a Hardy field. There exists a Hardy field $\aleph_1$ containing $\aleph$ such that the germs at $+\infty$ of the functions $\exp\circ f$, $\ln\circ|f|$ belong to $\aleph_1$ for every function $f\in\aleph_1$ which is not ultimately identically vanishing.*

A positive function measurable fuction f defined on some neighborhood of $+\infty$ is said to be regularly varying with index $\tau\in\mathbb{R}$ if $\lim_{x\to+\infty}\frac{f(Cx)}{f(x)}=C^{\tau}$, $C>0$. We set $\mathfrak{I}(f):=\tau$. If $\mathfrak{I}(f)=0$, then we will say that the function f is slowly varying.

If f is regularly varying with index τ, then there exists a slowly varying function L such that for $f(x)=x^{\tau}L(x)$, for sufficiently large values of x.

Let f be a function defined on an interval of the form $[a,+\infty[$ such that f is strictly positive and belongs as a germ at $+\infty$ to a Hardy field. Then according to [**12**], the function f is regularly varying if and only if $\lim_{x\to+\infty}\frac{xf'(x)}{f(x)}\in\mathbb{R}$. Then we have $\mathfrak{I}(f)=\lim_{x\to+\infty}\frac{xf'(x)}{f(x)}$.

THEOREM 2.4 (Potter's bounds, [**6**]). *Let f be a regularly varying function of index τ. For every $\varepsilon>0$, we have ultimately $(1-\varepsilon)x^{\tau-\varepsilon}\leqslant f(x)\leqslant(1+\varepsilon)x^{\tau+\varepsilon}$.*

Let $\mu\colon\mathbb{R}_+^*\to\mathbb{R}$ be a function of class C^2 on $\mathbb{R}_*^+$ which belongs, as a germ at $+\infty$, to a Hardy field $\aleph$ containing the germ at $+\infty$ of the function $x\mapsto\ln x$. Since the function μ belongs as a germ at $+\infty$ to the Hardy field $\aleph$, it follows that the limit $\sigma(\mu):=\lim_{t\to+\infty}\frac{\ln(t)}{\mu(t)}$ exists in $\mathbb{R}_+\cup\{+\infty\}$. $\sigma(\mu)$ is called the order of the function μ. We assume that $0<\sigma(\mu)<+\infty$. It follows then that we have

$$\lim_{t\to+\infty}\mu(t)=+\infty,\quad \mu(t)=\underset{t\to+\infty}{O}(t)$$

Furthermore, we have by virtue of L'Hopital's rule, $\lim_{t\to+\infty}t\mu'(t)=\frac{1}{\sigma(\mu)}$. Thence we have $\lim_{t\to+\infty}\frac{t\mu'(t)}{\mu(t)}=0$. Consequently the function μ is slowly varying.

Consider the function $\mathcal{M}_\mu$ defined on $]0,+\infty[$ by $\mathcal{M}_\mu(t):=t^te^{t\mu(t)}$, $t>0$. The functions Ω_μ and H_μ are defined on $\mathbb{R}_+^*$ by

$$\Omega_\mu(x):=\inf_{t>0}\left[\frac{\mathcal{M}_\mu(t)}{x^t}\right],\ \ x>0,\qquad H_{_\mu}(x)=\inf_{t>0}\left[\frac{\mathcal{M}_\mu(t)}{t^tx^t}\right],\ \ x>0$$

We consider also the sequence $M_\mu:=(M_n)_{n\in\mathbb{N}^*}$ defined by $M_n:=\mathcal{M}_\mu(n)$, $n\in\mathbb{N}^*$.

Let W be a nontrivial interval of $\mathbb{R}$. The Carleman class $C_{M_\mu}(W)$ is the set of functions f of class C^∞ on W such that $\sup_{x\in W}|f^{(n)}(x)|\leqslant C\rho^nM_n$, $n\in\mathbb{N}^*$ where $C,\rho>0$ are real constants.

We denote by Λ_{M_μ} the set of sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $|a_n|\leqslant C\rho^nM_n$, $n\in\mathbb{N}^*$ where $C,\rho>0$ are constants. We denote by ω_μ and h_μ the functions defined by $\omega_\mu(x):=-\ln[\Omega_\mu(x)]$ and $h_\mu(x):=-\ln[H_\mu(x)]$, respectively.

Let γ_μ denote the function ultimately defined by the system

$$x=t^2\mu'(t),\quad \gamma_\mu(x)=\mu(t)+t\mu'(t), \tag{2.1}$$

the parameter t being uniquely determined by x. We denote then t by $t_0(x)$.

Let φ_μ denote the function defined by $\varphi_\mu(x) := \omega_\mu(x) - x\omega'_\mu(x)$ for sufficiently large values of x.

The following propositions will play a crucial role in the proof of our main result.

PROPOSITION 2.1. 1. *The function ω_μ is ultimately well defined by the system*

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega_\mu(x) = t + t^2\mu'(t) \tag{2.2}$$

the parameter t being ultimately uniquely determined by x. We denote then t by $t_1(x)$.

2. *The function ω_μ is ultimately strictly concave.*

3. *The function φ_μ is ultimately well defined by the system*

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \varphi_\mu(x) = t^2\mu'(t) \tag{2.3}$$

the parameter t being ultimately uniquely determined by x.

4. *The function φ_μ is an increasing diffeomorphism between neighborhoods of $+\infty$. The inverse function $\mathcal{N}_\mu := \varphi_\mu^{\langle -1\rangle}$ is ultimately defined by the system*

$$x = t^2\mu'(t), \quad \mathcal{N}_\mu(x) = et\exp[\mu(t) + t\mu'(t)] \tag{2.4}$$

the parameter t being ultimately uniquely determined by x

5. *The function h_μ is ultimately well defined by the system*

$$x = \exp[\mu(t) + t\mu'(t)], \quad h_\mu(x) = t^2\mu'(t) \tag{2.5}$$

the parameter t being utimatey uniquely determined by x. Furthermore h_μ is ultimately positive and infinite so it has an inverse $h_\mu^{\langle -1\rangle}$ which belongs to a Hardy field.

6. *Each of the function ω_μ, φ_μ, h_μ, $\mathcal{N}_\mu$, γ_μ belongs to a Hardy field.*

7. *The function γ_μ is slowly varying and the function ω_μ is regularly varying of index*

$$\mathfrak{I}(\omega_\mu) = \frac{\sigma(\mu)}{1+\sigma(\mu)}. \tag{2.6}$$

8. *The function γ_μ is ultimately positive and infinite and we have*

$$\gamma_\mu(x) - \mu(x) = \underset{x\to+\infty}{O}(1) \tag{2.7}$$

9. *We have ultimately*

$$\omega'_\mu(\mathcal{N}_\mu(x)) = \frac{e^{-\gamma_\mu(x)}}{e}, \tag{2.8}$$

$$\gamma_\mu(x)) = \ln(h_\mu^{\langle -1\rangle}(x)). \tag{2.9}$$

10. *The following relations hold for every $\alpha \in \mathbb{R}_+^*$*

$$\mu(\alpha x) - \mu(x) = \underset{x\to+\infty}{O}(1), \tag{2.10}$$

$$\lim_{x\to+\infty} \frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)} = 0. \tag{2.11}$$

PROOF. 1. Thanks to [**4**], the function h_μ is ultimately well defined by the system

$$x = \exp[\mu(t) + t\mu'(t)], \quad h_\mu(x) = t^2\mu'(t)$$

Consider then the function $\bar{\mu}\colon x \mapsto \mu(x) + \ln(x)$. It belongs to the Hardy field $\aleph$ and we have $\sigma(\bar{\mu}) = \frac{\sigma(\mu)}{\sigma(\mu)+1} \in]0, +\infty[$. It follows that the function $h_{\bar{\mu}}$ is ultimately well defined by the system

$$x = \exp[\bar{\mu}(t) + t\bar{\mu}'(t)], \quad h_{\bar{\mu}}(x) = t^2\bar{\mu}'(t)$$

that is by the system

$$x = et\exp[\mu(t) + t\mu'(t)], \quad h_{\bar{\mu}}(x) = t + t^2\mu'(t)$$

But we know that $h_{\bar{\mu}} = \omega_\mu$, thence the function ω_μ is ultimately well defined by the system

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega_\mu(x) = t + t^2\mu'(t)$$

the parameter t being ultimately uniquely determined by x.

On the other hand, since

$$t^2\mu'(t) \underset{t\to+\infty}{\sim} \frac{1}{\sigma(\mu)}t, \quad \exp[\mu(t) + t\mu'(t)] \underset{t\to+\infty}{\sim} e^{\frac{1}{\sigma(\mu)}}e^{\mu(t)}$$

it follows that $\lim_{x\to+\infty} h_\mu(x) = +\infty$. Consequently h_μ is ultimately positive and infinite. Thence according to Theorem 2.1 above, the function h_μ has an inverse $h_\mu^{\langle -1\rangle}$ which belongs to a Hardy field.

2. It follows from the definition of the function ω_μ that it is ultimately of class C^1. Direct computations from the system (2.2) prove then that the function ω'_μ has ultimately the following parametrical representation

$$x = et\exp[\mu(t) + t\mu'(t)], \quad \omega'_\mu(x) = \frac{1}{e\exp[\mu(t) + t\mu'(t)]} \tag{2.12}$$

It follows that the function ω'_μ is ultimately strictly decreasing. Then that the function ω_μ is ultimately strictly concave.

3. Direct computations from system (2.2) lead to the system representing ultimately the function φ_μ.

4. It is clear that the function $F_1\colon t \to e^{\mu(t)+t\mu'(t)}$ which belongs as a germ at $+\infty$ to the Hardy field $\aleph$, is ultimately strictly increasing and satisfies $\lim\limits_{x\to+\infty} F_1(t) = +\infty$. Thence, according to Theorem 2.1, the function F_1 has an inverse g belonging to a Hardy field $\aleph_1$ which contains the identity.

The function h_μ is ultimately of the class C^1, and, according to (2.5), we have ultimately

$$h'_\mu(x) = \frac{\frac{d(t^e\mu'(t))}{dt}(g(x))}{\frac{d(e^{\mu(t)+t\mu'(t)})}{dt}(g(x))} = \frac{g(x)}{e^{\mu(g(x))+g(x)\mu'(t(x))}} = \frac{g(x)}{x}$$

Hence we have ultimately $xh'_\mu(x) = g(x)$. It follows, according to Theorem 2 above, that $\aleph_1[h_\mu]$ is an integral domain whose fraction field $\aleph_1(h_\mu)$ is a Hardy field which contains the function h_μ as a germ at $+\infty$. By a similar proof we obtain that $h_{\bar{\mu}}$

belongs to a Hardy field. Since $\omega_\mu = h_{\bar{\mu}}$ it follows then that the function ω_μ belongs as a germ at $+\infty$ to a Hardy field.

It is obvious that the function φ_μ belongs to the same Hardy field $\aleph$ as ω_μ. Furthermore direct computations, based on the system representing φ_μ on some neighborhood of $+\infty$, show that φ_μ is infinite and ultimately positive. It follows then that φ_μ is ultimately strictly increasing. Thence φ_μ is a diffeomorphism between neighborhoods of $+\infty$ whose inverse $\mathcal{N}_\mu$ belongs to a Hary field. It is clear that the function $\mathcal{N}_\mu$ is ultimately well defined by the system

$$x = t^2\mu'(t), \quad \mathcal{N}_\mu(x) = et\exp[\mu(t) + t\mu'(t)]$$

the parameter t being ultimately uniquely determined by x.

Direct computations from systems (2.1), (2.2), (2.4) prove that we have ultimately $\gamma_\mu(h_\mu(x)) = \ln(x)$, that is $\gamma_\mu(x) = \ln(h_\mu^{\langle -1\rangle}(x))$

It follows, according to Theorem 3 above, that there exists a Hardy field containing the function γ_μ. It follows also from the relation (2.9) that γ_μ is ultimately positive and infinite.

Direct computations from systems (2.1), (2.2), (2.4) prove also that relation (2.8) holds ultimately.

5. The function ω_μ belongs to a Hardy field and we have

$$\lim_{x\to+\infty} \frac{x\omega_\mu'(x)}{\omega_\mu(x)} = \lim_{x\to+\infty} \frac{t_1(x)}{t_1(x) + t_1(x)^2\mu'(t_1(x))}$$
$$= \lim_{x\to+\infty} \frac{1}{1 + t_1(x)\mu'(t_1(x))} = \frac{\sigma(\mu)}{1+\sigma(\mu)}$$

Thence the function ω_μ is regularly varying with index $\mathfrak{I}(\omega_\mu) = \frac{\sigma(\mu)}{1+\sigma(\mu)}$.

6. The function γ_μ belongs as a germ at $+\infty$ to a Hardy field and we have

$$\lim_{x\to+\infty} \frac{x\gamma_\mu'(x)}{\gamma_\mu(x)} = \lim_{t\to+\infty} \frac{t\mu'(t)}{\mu(t) + t\mu'(t)} = \lim_{t\to+\infty} \frac{\frac{t\mu'(t)}{\mu(t)}}{1 + \frac{t\mu'(t)}{\mu(t)}} = 0$$

Thence γ_μ is slowly varying.

7. Since γ_μ is slowly varying it follows, according to Theorem 2.4 above, that we have ultimately $0 \leqslant \gamma_\mu(x) \leqslant \sqrt{x}$. It follows that $\gamma_\mu(x) = o_{x\to+\infty}(x)$.

On the other hand, according to (2.1), we have ultimately

$$\gamma_\mu(x) - \mu(x) = \mu(t_0(x)) + t_0(x)\mu'(t_0(x)) - \mu(x)$$
$$= \frac{(t_0(x) - x)}{v} v\mu'(v) + t_0(x)\mu'(t_0(x))$$

where v lies between x and $t_0(x)$. Since

$$x = t_0(x)^2\mu'(t_0(x)) \underset{x\to+\infty}{\sim} \frac{1}{\sigma(\mu)} t_0(x)$$

it follows that $\frac{t_0(x)-x}{v} = O_{x\to+\infty}(1)$. Consequently we have

$$\gamma_\mu(x) - \mu(x) = \underset{x\to+\infty}{O}(1)$$

8. We have

$$\lim_{x\to+\infty}\frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)}=\lim_{t\to+\infty}\frac{e[1+2t\mu'(t)+t^2\mu''(t)]\exp[\mu(t)+t\mu'(t)-\alpha t^2\mu'(t)]}{2t\mu'(t)+t^2\mu''(t)}$$

Thence we have by virtue of L'Hopital's rule that

$$\lim_{t\to+\infty}-t^2\mu''(t)=\lim_{t\to+\infty}t\mu'(t)=\lim_{t\to+\infty}\frac{\mu(t)}{\ln t}=\frac{1}{\sigma(\mu)}$$

Consequently the following estimate holds

$$\frac{e[1+2t\mu'(t)+t^2\mu''(t)]\exp[\mu(t)+t\mu'(t)-\alpha t^2\mu'(t)]}{2t\mu'(t)+t^2\mu''(t)} \underset{t\to+\infty}{\sim} e(1+\sigma(\mu))e^{\frac{1}{\sigma(\mu)}}\exp[\mu(t)-\alpha t^2\mu'(t)].$$

But $\mathfrak{I}(\mu)=0$ and $\mathfrak{I}(t\mapsto\alpha t^2\mu'(t))=1$, hence we have, according to Theorem 2.4 above, that $\mu(t)=O_{t\to+\infty}(\alpha t^2\mu'(t))$. It follows that

$$\lim_{t\to+\infty}e(1+\sigma(\mu))e^{\frac{1}{\sigma(\mu)}}\exp[\mu(t)-\alpha t^2\mu'(t)]=0.$$

Consequently we have

$$\lim_{x\to+\infty}\frac{e^{-\alpha\varphi_\mu(x)}}{\varphi'_\mu(x)}=0.$$

On the other hand, according to the mean value theorem, we have for all $x>0$

$$|\mu(\alpha x)-\mu(x)|=|\alpha-1||x\mu'(u)|=|\alpha-1|\frac{x}{u}|u\mu'(u)|$$

where u lies between αx and x. It follows that

$$|\mu(\alpha x)-\mu(x)|\leqslant|\alpha-1|\max\Big(\alpha,1/\alpha\Big)|u\mu'(u)|$$

Since $\lim_{s\to+\infty}t\mu'(t)=\frac{1}{\sigma(\mu)}<+\infty$, it follows then that

$$\mu(\alpha x)-\mu(x)=\underset{x\to+\infty}{O}(1) \qquad \square$$

Proposition 2.2. *Let I be a nontrivial compact interval of $\mathbb{R}$ and $a\in I$. The so-called Borel mapping $\mathcal{T}\colon C_{M_\mu}(I)\to\Lambda_{M_\mu}$, $f\mapsto(f^{(n)}(a))_{n\in\mathbb{N}}$ is surjective.*

Proof. Following Petzsche [**17**, p. 300], we set

$$m_p^*:=\frac{M_p}{pM_{p-1}},\quad p\in\mathbb{N}^*.$$

We have then for every $p\in\mathbb{N}^*$

$$\begin{aligned}\frac{m_{2p}^*}{m_p^*}=\frac12\frac{M_{2p}/M_{2p-1}}{M_p/M_{p-1}}&=\frac{2^{2p}p^{2p}}{2(2p-1)^{2p-1}}\frac{(p-1)^{p-1}}{p^p}\\&\quad\times\exp[2p\mu(2p)-(2p-1)\mu(2p-1)-p\mu(p)-(p-1)\mu(p-1)]\\&=\frac{(1-\frac1p)^{p-1}}{(1-\frac{1}{2p})^{2p-1}}\exp[2p\mu(2p)-(2p-1)\mu(2p-1)-p\mu(p)-(p-1)\mu(p-1)].\end{aligned}$$

But we have

$$2p\mu(2p) - (2p-1)\mu(2p-1) = z_{2p}\mu'(z_{2p}) + \mu(z_{2p}),$$
$$p\mu(p) - (p-1)\mu(p-1) = z_p\mu'(z_p) + \mu(z_p)$$

where $z_{2p} \in [2p-1, 2p]$ and $z_p \in [p-1, p]$. It follows that there exists $w_p \in [p-1, 2p]$ such that

$$\begin{aligned} 2p\mu(2p) - (2p-1)\mu(2p-1) - [p\mu(p) - (p-1)\mu(p-1)] \\ &= z_{2p}\mu'(z_{2p}) + \mu(z_{2p}) - (z_p\mu'(z_p) + \mu(z_p)) \\ &= (z_{2p} - z_p)(w_p\mu''(w_p) + 2\mu'(w_p)) \\ &= (z_{2p} - z_p)\mu'(w_p)\Big[\frac{w_p\mu''(w_p) + \mu'(w_p)}{\mu'(w_p)} + 1\Big] \end{aligned}$$

On the other hand, the limit $\lim_{x\to+\infty} \frac{x\mu''(x))+\mu'(x)}{\mu'(x)}$ exists and we have

$$\lim_{x\to+\infty} \frac{x\mu'(x)}{\mu(x)} = 0.$$

From L'Hopital's rule, it follows

$$\lim_{x\to+\infty} \frac{x\mu''(x)) + \mu'(x)}{\mu'(x)} = 0 \tag{2.13}$$

Furthermore we have $z_{2p} - z_p \geqslant \frac{1}{2}w_p - 1$. Thence we have for large values of p

$$(z_{2p} - z_p)\mu'(w_p) \geqslant \frac{1}{2}\Big[\frac{w_p - 2}{w_p}\Big]w_p\mu'(w_p) \tag{2.14}$$

We conclude from (2.13) and (2.14) that

$$\liminf_{p\to+\infty} [2p\mu(2p) - (2p-1)\mu(2p-1)] - [p\mu(p) - (p-1)\mu(p-1)] \geqslant \frac{1}{2\sigma(\mu)}$$

It follows that

$$\liminf_{p\to+\infty} \frac{m^*_{2p}}{m^*_p} \geqslant e^{\frac{1}{2\sigma(\mu)}} > 1.$$

Thence a slight refinement of a theorem in [**17**, pp. 300 and 311] yields that the Borel mapping $\mathcal{T}$ is surjective. ☐

Direct computations show that μ and γ_μ can be extended to $\mathbb{R}^*_+$ in a way to be functions of class C^1 on $\mathbb{R}^*_+$ such that $-\varepsilon \leqslant \mu(x) - \gamma_\mu(x) \leqslant \varepsilon, x \in \mathbb{R}^*_+$ where ε is a positive constant. From now on we will do so and we will set for every $A > 0$, $n \in \mathbb{N}$ and for every nonempty subset S of $\mathbb{C}$

$$S_{\mu,A,n} := S_{Ae^{-\mu(n)}}, \quad S_{\gamma_\mu,A,n} := S_{Ae^{-\gamma_\mu(n)}}.$$

Thence the following inclusions hold for all $n \in \mathbb{N}$.

$$S_{\gamma_\mu,Ae^{-\varepsilon},n} \subset S_{\mu,A,n} \subset S_{\gamma_\mu,Ae^{\varepsilon},n}.$$

3. Statement of the main result

The main result of this paper is the following.

THEOREM 3.1. 1. *Let* $f \in C_{M_\mu}([-1,1])$*; then there exists constants* $C > 0$, $A > 0$, $0 < \rho < 1$ *and a sequence* $(P_n)_{n\geqslant 1}$ *of rational functions defined on* $\mathbb{C}\setminus\{i,-i\}$ *such that* $\sum P_n$ *is uniformly convergent on* $[-1,1]$ *to* f *and*

$$\|P_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n, \quad n \in \mathbb{N}, \qquad f(x) = \sum_{n=1}^{\infty} P_n(x), \quad x \in [-1,1]$$

2. *Conversely, let us assume that there exist some constants* $C > 0$, $A > 0$, $0 < \rho < 1$ *and a sequence* $f_n \in \mathcal{O}([-1,1]_{\mu,A,n})$ *of holomorphic functions such that* $\|f_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n$, $n \in \mathbb{N}^*$. *Then the function series* $\sum f_n$ *is uniformly convergent on* $[-1,1]$ *to a function* f *which belongs to the Carleman class* $C_{M_\mu}([-1,1])$.

4. Proof of the main result

4.1. Direct part.

PROPOSITION 4.1. *Let* $g\colon [-\pi,\pi] \longrightarrow C$ *be a restriction of a* 2π*-periodic function of class* C^∞ *on* $\mathbb{R}$. *Let us assume that* $g \in C_{M_\mu}([-\pi,\pi])$*; then there exist constants* $A > 0$, $C > 0$, $0 < \rho < 1$ *and a sequence* $(g_n)_{n\geqslant 0}$ *of rational functions defined on* $\mathbb{C}^*$ *such that*

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,A,n}} \leqslant C\rho^n, \quad n \in \mathbb{N}, \qquad g(\theta) = \sum_{n=0}^{\infty} g_n(e^{i\theta}), \quad \theta \in [-\pi,\pi]$$

where $\mathcal{K}_{\gamma_\mu,A,n} := \big\{z \in \mathbb{C},\ 1 - Ae^{-\gamma_\mu(n)} < |z| < 1 + Ae^{-\gamma_\mu(n)}\big\}$.

PROOF. The Fourier series expansion of g can be written for all $\theta \in [-\pi,\pi]$ as

$$g(\theta) = \sum_{p\in\mathbb{Z}} a_p e^{ip\theta} \quad \text{where} \quad a_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ip\theta}d\theta, \quad p \in \mathbb{Z}.$$

According to [**16**], the following estimations hold

$$|a_p| \leqslant C_0 e^{-C_1\omega_\mu(|p|)}, \quad p \in \mathbb{Z} \tag{4.1}$$

with some constants $C_0, C_1 > 0$.

Let us set for all $z \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$

$$g_0(z) := \sum_{|p|<\mathcal{N}_\mu(1)} a_p z^p, \quad g_n(z) := \sum_{\mathcal{N}_\mu(n)\leqslant|p|<\mathcal{N}_\mu(n+1)} a_p z^p.$$

Then for all $n \in \mathbb{N}^*$, g_n is a rational function defined on $\mathbb{C}^*$. Furthermore the following estimates hold

$$|g_n(z)| \leqslant C_0 \sum_{\mathcal{N}_\mu(n)\leqslant|p|<\mathcal{N}_\mu(n+1)} C_0 e^{-C_1\omega_\mu(p)}(|z|^p + |z|^{-p}), \quad z \in \mathbb{C}^*. \tag{4.2}$$

If $z \in \mathcal{K}_{\frac{C_1}{2e},n}$, then the estimates become

$$|g_n(z)| \leqslant C_0 \sum_{\mathcal{N}_\mu(n) \leqslant |p| < \mathcal{N}_\mu(n+1)} e^{-C_1\omega_\mu(p)} \left[\left(1+\frac{C_1}{2e}e^{-\gamma_\mu(n)}\right)^p + \left(1-\frac{C_1}{2e}e^{-\gamma_\mu(n)}\right)^{-p}\right].$$

We have for large values of n

$$\left(1-\frac{C_1}{2e}e^{-\gamma_\mu(n)}\right)^{-1} \leqslant 1+\frac{C_1}{e}e^{-\gamma_\mu(n)}.$$

It follows that we have for such values of n

$$\|g_n\|_{\infty\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_0(1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)) \max_{\mathcal{N}_n \leqslant p < \mathcal{N}_{n+1}} 2\exp\left[-C_1\omega_\mu(p)+C_1 p\frac{e^{-\gamma_\mu(n)}}{e}\right].$$

On the other hand we have for n sufficiently large

$$\frac{e^{-\gamma_\mu(n)}}{e} = \omega'_\mu(\mathcal{N}_\mu(n))$$

Consequently we have for such values of n

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_0(1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)) \max_{\mathcal{N}_\mu(n) \leqslant p < \mathcal{N}_\mu(n+1)} 2\exp\big[-C_1\big(\omega_\mu(p)-\omega'_\mu(\mathcal{N}_\mu(n))p\big)\big]$$

But by virtue of Proposition 2.1, ω_μ is ultimately strictly concave. It follows that the function $h_n\colon \mathbb{R}_+^* \to \mathbb{R}$, $x \mapsto -C_1[\omega(x) - \omega'_\mu(\mathcal{N}_\mu(n))x]$ is ultimately strictly concave, thence we have for large values of n that for all $x \in [\mathcal{N}_\mu(n), \mathcal{N}_\mu(n+1)]$ we have

$$h'_n(x) = -C_1[\omega'(x) - \omega'_\mu(\mathcal{N}_\mu(n))] < 0$$

Thence the function h_n is, for large values of n, strictly decreasing on the interval $[\mathcal{N}_\mu(n), \mathcal{N}_\mu(n+1)]$. It follows that the following estimates hold for large values of n

$$\begin{aligned}
&\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}}\\
&\quad \leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]\exp[-C_1(\omega(\mathcal{N}_\mu(n))-\mathcal{N}_\mu(n)\omega'(\mathcal{N}_\mu(n)))]\\
&\quad \leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]\exp[-C_1\varphi_\mu(\mathcal{N}_\mu(n))]\\
&\quad \leqslant C_0[1+\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n)]e^{-C_1 n}\\
&\quad \leqslant C_0\big[e^{\frac{C_1}{2}}(\mathcal{N}_\mu(n+1)-\mathcal{N}_\mu(n))e^{-\frac{C_1}{2}(n+1)}+1\big]e^{-\frac{C_1}{2}n}
\end{aligned}$$

Since $\mathcal{N}_\mu$ is ultimately strictly convex, we can write for large values of n

$$\begin{aligned}
\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} &\leqslant C_0\big[e^{\frac{C_1}{2}}\mathcal{N}'_\mu(n+1)e^{-\frac{C_1}{2}(n+1)}+1\big]e^{-\frac{C_1}{2}n}\\
&\leqslant C_0\left[e^{\frac{C_1}{2}}\frac{e^{-\frac{C_1}{2}\varphi_\mu(\mathcal{N}_\mu(n+1))}}{\varphi'_\mu(\mathcal{N}_\mu(n+1))}+1\right]e^{-\frac{C_1}{2}n}
\end{aligned}$$

According to (2.11) we have

$$C_0\left[e^{\frac{C_1}{2}}\frac{e^{-\frac{C_1}{2}\varphi_\mu(\mathcal{N}_\mu(n+1))}}{\varphi'_\mu(\mathcal{N}_\mu(n+1))}+1\right]e^{-\frac{C_1}{2}n} \underset{n\to+\infty}{\sim} C_0 e^{-\frac{C_1}{2}n}$$

Thence we have

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,\frac{C_1}{2e},n}} \leqslant C_2 e^{-\frac{C_1}{2}n}, \quad n \in \mathbb{N}$$

where $C_2 > 0$ is a constant. □

PROPOSITION 4.2. *Let $f \in C_{M_\mu}([-1,1])$; then there exists a function $F \in C_{M_\mu}(\mathbb{R})$ with support contained in the interval $[-2,2]$ and whose restriction to $[-1,1]$ is the function f.*

PROOF. According to Proposition 2.2, there exist $F_1 \in C_{M_\mu}([-3,-1]$ and $F_2 \in C_{M_\mu}([1,3])$ such that $F_1^{(n)}(-1) = f^{(n)}(-1)$, $F_2^{(n)}(1) = f^{(n)}(1)$, $n \in \mathbb{N}$. On the other hand, according to [**22**], there exists $\Phi \in C_{M_\mu}(\mathbb{R})$ with support contained in $[-2,2]$ such that $\Phi(x) = 1$, $x \in [-1,1]$. The function F defined by

$$\begin{aligned} &F(x) = f(x), && x \in [-1,1] \\ &F(x) = F_1(x)\Phi(x), && x \in [-3,-1] \\ &F(x) = F_2(x)\Phi(x), && x \in [1,3] \\ &F(x) = 0, && x \in \mathbb{R} \setminus [-3,3] \end{aligned}$$

satisfies the required conditions. □

END OF THE PROOF OF THE DIRECT PART OF THE MAIN THEOREM. Let $f \in C_{M_\mu}([-1,1])$. There exists, according to Proposition 4.2, a function $F \in C_{M_\mu}(\mathbb{R})$ whose support is contained in the interval $[-2,2]$ and whose restriction to $[-1,1]$ is the function f.

Let us consider the function g defined on the interval $[-\pi,\pi]$ by

$$\begin{aligned} &g(\theta) = F(\tan(\theta/2)), && \theta \in \,]-2\arctan(2), 2\arctan(2)[\\ &g(\theta) = 0, && \theta \in \mathbb{R} \setminus \,]-2\arctan(2), 2\arctan(2)[\end{aligned}$$

According to Cartan [**11**, Theorem III, pp. 24–27], the restriction of g to the interval $J := [-2\arctan(2), 2\arctan(2)]$ belongs to the Carleman class $C_{M_\mu}(J)$. But g is itself the restriction to $[-\pi,\pi]$ of a 2π-periodic, of class $\mathcal{C}^\infty$ which is vanishing on the set $[-\pi,\pi] \setminus J$. Thence $g \in C_{M_\mu}([-\pi,\pi])$.

According to Proposition 4.1 there exists constants $0 < A < 1$, $C > 0$, $0 < \rho < 1$ and a sequence $(g_n)_{n\geqslant 1}$ of rational functions defined on $\mathbb{C}^*$ such that

$$\|g_n\|_{\infty,\mathcal{K}_{\gamma_\mu,A,n}} \leqslant C\rho^n, \ \ n \in \mathbb{N}, \quad g(\theta) = \sum_{n=0}^{\infty} g_n(e^{i\theta}), \ \ \theta \in [-\pi,\pi].$$

Let $x \in [-2,2]$. There exists a unique $\theta \in [-2\arctan(2), 2\arctan(2)]$ such that $x = \tan\left(\frac{\theta}{2}\right)$, thence we have $F(x) = g(\theta) = \sum_{n=1}^{+\infty} g_n\left(\frac{i-x}{i+x}\right)$. On the other hand let $z \in \mathbb{C}$ be such that $|\operatorname{Im}(z)| < 1$ (then $z \in \mathbb{C} \setminus \{i,-i\}$). Let us set $\zeta = \frac{i-z}{i+z}$; then we have $|\operatorname{Im}(z)| \geqslant \frac{|1-|\zeta||}{1+|\zeta|}$. It follows that the following implication holds for every $A' \in \,]0,1[$

$$|\operatorname{Im}(z)| \leqslant A'e^{-\gamma_\mu(n)} \Rightarrow \frac{1 - A'e^{-\gamma_\mu(n)}}{1 + A'e^{-\gamma_\mu(n)}} \leqslant |\zeta| \leqslant \frac{1 + A'e^{-\gamma_\mu(n)}}{1 - A'e^{-\gamma_\mu(n)}}$$

If we choose $A' \in]0,1[$ sufficiently small, then we will obtain for every $n \in \mathbb{N}$

$$0 < 1 - Ae^{-\gamma_\mu(n)} < \frac{1 - A'e^{-\gamma_\mu(n)}}{1 + A'e^{-\gamma_\mu(n)}} \leqslant \frac{1 + A'e^{-\gamma_\mu(n)}}{1 - A'e^{-\gamma_\mu(n)}} < 1 + Ae^{-\gamma_\mu(n)}$$

Let us set $\mathcal{B}_n := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < A'e^{-\gamma_\mu(n)}\}$. Thence the points i and $-i$ belong to $\mathbb{C} \setminus \mathcal{B}_n$ and we have $\frac{i-z}{i+z} \in \mathcal{K}_{\gamma_n,A,n}$, $z \in \mathcal{B}_n$. For each $n \in \mathbb{N}$, the function P_n defined on $\mathbb{C}\backslash\{i,-i\}$ by $P_n(z) = g_n(\frac{i-z}{i+z})$ is a rational function satisfying

$$\|P_n\|_{\infty,\mathcal{B}_n} \leqslant C\rho^n, \quad n \in \mathbb{N}$$

We have also for all $x \in [-2,2]$ that $F(x) = \sum_{n=1}^{\infty} P_n(x)$. But $[-1,1]_{\gamma_\mu,A',n} \subset \mathcal{B}_n$ for all $n \in \mathbb{N}$; thence we have

$$f(x) = \sum_{n=1}^{\infty} P_n(x), \;\; x \in [-1,1], \qquad \|P_n\|_{\infty,[-1,1]_{\gamma_\mu,A',n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}.$$

Then, it follows

$$f(x) = \sum_{n=1}^{\infty} P_n(x), \;\; x \in [-1,1], \qquad \|P_n\|_{\infty,[-1,1]_{\mu,Ae^{-\varepsilon'},n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}. \qquad \square$$

4.2. Converse part.

Proof. Let $A > 0$ and for each $n \in \mathbb{N}$, a function $f_n \colon [-1,1]_{\mu,A,n} \to \mathbb{C}$ which is holomorphic on $[-1,1]_{\mu,A,n}$ such that

$$f_n \in \mathcal{O}([-1,1]_{\mu,A,n}), \;\; n \in \mathbb{N}^*, \qquad \|f_n\|_{\infty,[-1,1]_{\mu,A,n}} \leqslant C\rho^n, \;\; n \in \mathbb{N}^*.$$

It follows that $\|f_n\|_{\infty,[-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}} \leqslant C\rho^n$, $n \in \mathbb{N}^*$. Thence the function series $\sum f_{n|[-1,1]}$ converges uniformly on $[-1,1]$ to a continuous function f.

We have $[-1,1] \subset [-1,1]_{\gamma_\mu,\frac{A}{2}e^{-\varepsilon},n} \subset [-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}$. Cauchy's inequalities allow us to write for all $p \in \mathbb{N}$

$$\|f_n^{(p)}\|_{\infty,[-1,1]} \leqslant Cp!\Big(\frac{2}{A}e^{\varepsilon}\Big)^p \exp\big[p\gamma_\mu(n) - \ln(\rho^{-1/2})n\big]\rho^{-n/2}. \tag{4.3}$$

On the other hand the supremum, for sufficiently large $p \in \mathbb{N}$, of the function $u \mapsto p\gamma_\mu(u) - \ln\big(1/\sqrt{\rho}\big)u$ on $[0,+\infty]$ is reached in the real $u_p > 0$ that satisfies $\gamma_\mu'(u_p) = \frac{1}{p}\ln\big(1/\sqrt{\rho}\big)$. Since for sufficiently large $p \in \mathbb{N}$, we have $\gamma_\mu'(u_p) = 1/t_0(u_p)$, it follows that $t_0(u_p) = p/\ln\big(1/\sqrt{\rho}\big)$. Consequently we can write

$$\begin{aligned} \sup_{n\in\mathbb{N}} \big[p\gamma_\mu(n) - \ln\big(1/\sqrt{\rho}\big)n\big] &\leqslant p(\gamma_\mu(u_p) - u_p\gamma_\mu'(u_p)) \\ &\leqslant p\mu(t_0(u_p)) \leqslant p\mu\big(p/\ln\big(1/\sqrt{\rho}\,\big)\big) \end{aligned} \tag{4.4}$$

Thence we have for $p \in \mathbb{N}$ sufficiently large we have for all $n \in \mathbb{N}$

$$\|f_n^{(p)}\|_{\infty,[-1,1]} \leqslant Cp!\Big(\frac{2}{A}e^{\varepsilon}\Big)^p \sqrt{\rho}^n e^{p\mu(p/\ln(1/\sqrt{\rho}))}$$

It follows that the function series $\sum f_n^{(p)}$ are for sufficiently large values of p normally convergent. Thence the function f is of class C^∞ on $[-1,1]$ and we have

$$\begin{aligned}\|f^{(p)}\|_{\infty,[-1,1]} &\leqslant \frac{2C}{A(1-\sqrt{\rho})}\Big(\frac{2}{A}\Big)^p p! \exp\big[p\big(\mu\big(p/\ln\big(1/\sqrt{\rho}\big)\big)-\mu(p)\big)\big]e^{p\mu(p)}\\ &\leqslant B^{p+1}p^p e^{p\mu(p)}\end{aligned}$$

for some constant $B>0$. Thence we have $f\in C_{M_\mu}([-1,1])$. □

5. Application: Alternative construction of Dyn'kin's pseudoanalytic extension for the Carleman class $C_{M_\mu}([-1,1])$

COROLLARY 5.1. *Let be $f\in C_{M_\mu}([-1,1])$. There exists a function $F\in C^\infty(\mathbb{C})$ with compact support such that*

$$F|_{[-1,1]}=f,\qquad |\bar\partial F(z)|\leqslant C_1 H_\mu\left[\frac{C_2}{\rho(z,[-1,1])}\right],\ \ z\in\mathbb{C}\setminus[-1,1]$$

where $C_1, C_2>0$ are constants.

PROOF. According to the main result there exist constants $A\in\,]0,1[$, $C>0$, $\rho\in\,]0,1[$, and a sequence of rational functions $(f_n)_{n\in\mathbb{N}}$ defined on some strip $B:=\{z\in\mathbb{C}: |\operatorname{Im}(z)|\leqslant A\}$ such that

$$\|f_n\|_{\infty,[-1,1]_{\mu,A,n}}\leqslant C\rho^n,\ \ n\in\mathbb{N}^*,\qquad \sum_{n=1}^{+\infty} f_{n|[-1,1]}=f.$$

It follows that $\|f_n\|_{\infty,[-1,1]_{\gamma_\mu,Ae^{-\varepsilon},n}}\leqslant C\rho^n$, $n\in\mathbb{N}^*$.

On the other hand, there exists, for each $n\in\mathbb{N}^*$, a function $\theta_n\colon\mathbb{C}\to[0,1]$ of class C^∞ on $\mathbb{C}$ ($\mathbb{C}$ is here identified with $\mathbb{R}^2$) and a family of positive constants $(L_\alpha)_{\alpha\in\mathbb{N}^2}$ **[22]** such that

$$\begin{aligned}\theta_n(z)&=1,\quad z\in[-1,1]_{\mu,\frac{A}{8},n}\\ \theta_n(z)&=0,\quad z\in\mathbb{C}\setminus[-1,1]_{\mu,\frac{A}{2},n}\\ |D^\alpha\theta_n(z)|&\leqslant L_\alpha e^{|\alpha|\mu(n)},\quad \alpha\in\mathbb{N}^2,\ z\in\mathbb{R}^2\end{aligned}$$

where $|\alpha|:=p+q$ and $D^\alpha:=\frac{\partial^{p+q}}{\partial x^p\partial y^q}$ for $\alpha=(p,q)$.

We denote by F_n the function defined by

$$\begin{aligned}F_n(z)&=\theta_n(z)f_n(z), z\in[-1,1]_{\gamma_\mu,A,n}\\ F_n(z)&=0, z\in\mathbb{C}\setminus[-1,1]_{\gamma_\mu,A,n}\end{aligned}$$

The function F_n is of class C^∞ on $\mathbb{C}$ and satisfies the condition

$$F_n|_{[-1,1]_{\mu,\frac{A}{8},n}}=f_n|_{[-1,1]_{\mu,\frac{A}{8},n}}.$$

Since $\|F_n\|_{\infty,\mathbb{C}}\leqslant C\rho^n$, $n\in\mathbb{N}$, it follows that the function series $\sum F_n$ is uniformly convergent on $\mathbb{C}$ to a continuous function F with compact support contained in $[-1,1]_A$. Furthermore it is clear that F is an extension to $\mathbb{C}$ of f.

Let $\alpha \in \mathbb{N}^2$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$. If $z \in \mathbb{C} \smallsetminus [-1,1]_{\mu,\frac{A}{2},n}$, then we have $D^\alpha F_n(z) = 0$. But when $z \in [-1,1]_{\mu,\frac{A}{8},n}$ we can write, in view of Cauchy's inequalities and inequality (4.4)

$$\begin{aligned}
|D^\alpha F_n(z)| &\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} |D^\beta \theta_n(z)||D^{\alpha-\beta} f_n(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\mu(n)} |D^{\alpha-\beta} f_n(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} |f_n^{(|\alpha|-|\beta|)}(z)| \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} C(4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot (|\alpha|-|\beta|)! \sqrt{\rho}^{\,n} \exp\big[(|\alpha|-|\beta|)\gamma_\mu(n) - \ln\big(1\sqrt{\rho}\big)n\big] \\
&\leqslant \sum_{\beta\leqslant\alpha} \tbinom{\alpha}{\beta} L_\beta e^{|\beta|\varepsilon} e^{|\beta|\gamma_\mu(n)} C(4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot (|\alpha|-|\beta|)! \sqrt{\rho}^{\,n} \exp\Big[\sup_{m\in\mathbb{N}} \big((|\alpha|-|\beta|)\gamma_\mu(m) - \ln\big(1/\sqrt{\rho}\big)m\big)\Big] \\
&\leqslant \sqrt{\rho}^{\,n} \sum_{\beta\leqslant\alpha} C\tbinom{\alpha}{\beta} e^{|\beta|\varepsilon} L_\beta (|\alpha|-|\beta|)! (4/A)^{|\alpha|-|\beta|} \\
&\quad \cdot \exp\Big[(|\alpha|-|\beta|)\mu\Big(\frac{(|\alpha|-|\beta|)}{\ln(1/\sqrt{\rho})}\Big)\Big]
\end{aligned}$$

It follows that the function series $\sum D^\alpha F_n(z)$ is for all $\alpha \in \mathbb{N}^2$ normally convergent on $\mathbb{C}$. Thence the function $F = \sum_{n=1}^{+\infty} F_n$ is of class C^∞ on $\mathbb{C}$.

Let $z \in \mathbb{C} \smallsetminus [-1,1]$. Then we have $\bar\partial F(z) = \sum_{n=1}^{+\infty} \bar\partial F_n(z)$. On the other hand, we have

$$\bar\partial F_n(z) = 0 \ \text{ if } \ \rho(z,[-1,1]) \in \big[0, \tfrac{A}{8} e^{-\varepsilon} e^{-\gamma_\mu(n)}\big[\cup \big] A e^{-\varepsilon} e^{-\gamma_\mu(n)}, +\infty\big[.$$

If $\rho(z,[-1,1]) \in \big[\frac{A}{8} e^{-\mu(n)}, A e^{-\mu(n)}\big[$, then, again by virtue of (4.4), we have

$$\begin{aligned}
|\bar\partial F_n(z)| &= |f_n(z)||\bar\partial\theta_n(z)| \\
&\leqslant \frac{C}{2}\rho^n \Big(\Big|\frac{\partial\theta_n}{\partial x}(z)\Big| + \Big|\frac{\partial\theta_n}{\partial y}(z)\Big|\Big) \\
&\leqslant \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^\varepsilon e^{\gamma_\mu(n) - \frac{1}{2}\ln(\frac{1}{\rho})n} \sqrt{\rho}^{\,n} \\
&\leqslant \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^\varepsilon e^{\mu(2/\ln(1/\sqrt{\rho}))} \sqrt{\rho}^{\,n}
\end{aligned}$$

Let us set

$$A_1 := \frac{C}{2}(L_{(1,0)} + L_{(0,1)}) e^\varepsilon e^{\mu(2/\ln(1/\sqrt{\rho}))}, \quad \lambda := -\ln\sqrt{\rho} > 0$$

Thence the following estimate holds

$$
\begin{aligned}
|\bar\partial F(z)| &\leqslant \sum_{\frac{A}{8}e^{-\mu(n)}\leqslant\rho(z,[-1,1])\leqslant Ae^{-\mu(n)}} A_1 e^{-\lambda n}\\
&\leqslant A_1 \sum_{\frac{A}{8\rho(z,[-1,1])}\leqslant e^{\mu(n)}} e^{-\lambda n}\\
&\leqslant A_1 \sum_{\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])}\leqslant e^{\gamma_\mu(n)}} e^{-\lambda n}
\end{aligned}
$$

It follows that if z is sufficiently close to $[-1,1]$, then the last estimate will become

$$
\begin{aligned}
|\bar\partial F(z)| \leqslant A_1 &\sum_{h_\mu(\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])})\leqslant n} e^{-\lambda n}\\
&\leqslant \frac{A_1}{1-e^{-\lambda}} \exp\Big[-\lambda h_\mu\Big(\frac{A}{8e^{\varepsilon}\rho(z,[-1,1])}\Big)\Big]
\end{aligned}
$$

But we know that the function h_μ is regularly varying. Thence there exists a constant $A_2 > 0$ such that we have ultimately

$$
\lambda h_\mu\Big(\frac{A}{8e^{\varepsilon}}x\Big) \geqslant h_\mu(A_2 x)
$$

Consequently we have for z sufficiently close to $[-1,1]$

$$
|\bar\partial F(z)| \leqslant \frac{A_1}{1-e^{-\frac{\lambda}{2}}} \exp\Big[-h_\mu\Big(\frac{A_2}{\rho(z,[-1,1])}\Big)\Big]
$$

Thence there exists a constant $A_3 > 0$ such that

$$
|\bar\partial F(z)| \leqslant A_3 H_\mu\Big(\frac{A_2}{\rho(z,[-1,1])}\Big), \quad z\in\mathbb{C}
$$

The proof of the corollary is complete. □

Acknowledgment. I would like to express my profound gratitude to Professor Jean Pierre Demailly, to Professor Alexander Borichev and to Professor Konstantin Mikhailovich Dyakonov for their precious assistance and their valuable helps without which my whole mathematical works would not have been realized.

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(Received 10 12 2015)