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A NEW CURVATURELIKE TENSOR FIELD IN AN ALMOST CONTACT RIEMANNIAN MANIFOLD II

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Memories of Professor Mileva Prvanović

ABSTRACT. In the last paper, we introduced a new curvaturlike tensor field in an almost contact Riemannian manifold and we showed some geometrical properties of this tensor field in a Kenmotsu and a Sasakian manifold. In this paper, we define another new curvaturelike tensor field, named $(\mathrm{CHR})_3$-curvature tensor in an almost contact Riemannian manifold which is called a contact holomorphic Riemannian curvature tensor of the second type. Then, using this tensor, we mainly research $(\mathrm{CHR})_3$-curvature tensor in a Sasakian manifold. Then we define the notion of the flatness of a $(\mathrm{CHR})_3$-curvature tensor and we show that a Sasakian manifold with a flat $(\mathrm{CHR})_3$-curvature tensor is flat. Next, we introduce the notion of $(\mathrm{CHR})_3$-η-Einstein in an almost contact Riemannian manifold. In particular, we show that Sasakian $(\mathrm{CHR})_3$-η-Einstein manifold is η-Einstain. Moreover, we define the notion of $(\mathrm{CHR})_3$-space form and consider this in a Sasakian manifold. Finally, we consider a conformal transformation of an almost contact Riemannian manifold and we get new invariant tensor fields (not the conformal curvature tensor) under this transformation. Finally, we prove that a conformally $(\mathrm{CHR})_3$-flat Sasakian manifold does not exist.

1. Almost contact Riemannian manifolds

A real $(2n+1)$-dimensional differentiable Riemannian manifold (M^{2n+1}, g) is said to be an almost contact Riemannian manifold if it has a $(1,1)$-tensor φ and a 1-form η which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any $Y, X \in TM^{2n+1}$, where ξ is defined by $g(\xi, X) = \eta(X)$. From $(1.1)_3$, the vector field ξ is unit and we say this vector field the *structure vector field* of

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the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold M^{2n+1}, we define a 2-form F as $F(X,Y) = g(\varphi X, Y)$ for any $X, Y \in TM^{2n+1}$, where TM^{2n+1} denotes the tangent bundle of M^{2n+1}. Then the 2-form F is skew-symmetric and we say that this tensor field is the *fundamental* 2-*form* of this almost contact Riemannian manifold.

In an almost contact Riemannian manifold, a section which is given by X and φX for a unit vector field X is called a φ-section of X. The sectional curvature $R(X, \varphi X, \varphi X, X)$ is said to be the *φ-holomorphic sectional curvature of X*, where R denotes the Riemannian curvature tensor with respect to g.

In a $(2n+1)$-dimensional almost contact Riemannian manifold M^{2n+1}, we define $\overset{*}{R}(X,Y,Z,W)$ which is called the *second Riemannian curvature tensor* as $\overset{*}{R}(X,Y,Z,W) = R(X,Y,\varphi Z,\varphi W)$ for any $X,Y,Z,W \in TM^{2n+1}$.

An almost contact Riemannian manifold $(M^{2n+1}, \varphi, g, \xi)$ is called a normal contact Riemannian or a Sasakian manifold if the structure vector field ξ and the fundamental 2-form F satisfies

$$\nabla_X \xi = \varphi X, \quad (\nabla_X F)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$$

for any $X, Y, Z \in TM^{2n+1}$.

In a Sasakian manifold, the Riemannian curvature tensor R and the Ricci tensor ρ with respect to g satisfy

$$\begin{aligned}
R(X,Y,Z,\xi) &= \eta(X)g(Y,Z) - \eta(Y)g(X,Z),\\
R(X,Y,\varphi Z,\varphi W) &= R(\varphi X,\varphi Y,Z,W) = R(X,Y,Z,W) + g(X,Z)g(Y,W)\\
&\quad - g(Y,Z)g(X,W) + F(X,W)F(Y,Z) - F(Y,W)F(X,Z),\\
R(\varphi X,\varphi Y,\varphi Z,\varphi W) &= R(X,Y,Z,W) + \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z)\\
&\quad + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W),\\
\rho(\varphi X,\varphi Y) &= \rho(X,Y) - 2n\eta(X)\eta(Y),\\
\rho(\varphi X,Y) + \rho(X,\varphi Y) &= 0, \qquad \rho(X,\xi) = 2n\eta(X)
\end{aligned} \tag{1.3}$$

for any $X, Y, Z, W \in TM^{2n+1}$.

A Sasakian manifold is said to be a *Sasakian space form* if it has a constant φ-holomorpic sectional curvature. Then the curvature tensor field R satisfies [**2**]

$$\begin{aligned}
R(X,Y,Z,W) = {} & \frac{c+3}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}\\
&+\frac{c-1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)\\
&\quad + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\\
&\quad - F(Y,Z)F(X,W) + F(X,Z)F(Y,W)\\
&\quad - 2F(X,Y)F(Z,W)\},
\end{aligned} \tag{1.4}$$

where c is a constant holomorphic sectional curvature.

A Sasakian space form with 0 holomorphic sectional curvature is called to be *flat*.

An almost contact Riemannian manifold M^{2n+1} is said to be *η-Einstein* if the Ricci tensor ρ with respect to g satisfies $\rho(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ for certain differentiable functions a and b on M^{2n+1} which are called the *associated functions* of ρ and any $X, Y \in TM^{2n+1}$. In particular, in an η-Einstein Sasakian manifold, associated functions a and b satisfy the following relation

$$a + b = 2n, \quad \tau = 2n(a+1), \tag{1.5}$$

where τ is the scalar curvature with respect to g.

In a Sasakian manifold, the *C-Bochner curvature tensor* $CB(X,Y,Z,W)$ is defined by [**3**]

$$\begin{aligned} CB(X,Y,Z,W) &= R(X,Y,Z,W) + \frac{1}{2(n+2)}\{\rho(X,Z)g(\varphi Y,\varphi W) \\ &-\rho(Y,Z)g(\varphi X,\varphi W) + \rho(Y,W)g(\varphi X,\varphi Z) - \rho(X,W)g(\varphi Y,\varphi Z) \\ &-\tilde{\rho}(X,Z)F(Y,W) + \tilde{\rho}(Y,Z)F(X,W) - \tilde{\rho}(Y,W)F(X,Z) \\ &+\tilde{\rho}(X,W)F(Y,Z) - 2\tilde{\rho}(X,Y)F(Z,W) - 2\tilde{\rho}(Z,W)F(X,Y)\} \\ &-\frac{\tau + 2n(2n+3)}{4(n+1)(n+2)}\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\} \\ &+\frac{\tau - 3(2n+4)}{4(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+\frac{\tau + 2n}{4(n+1)(n+2)}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta Z)g(X,W) - \eta(X)\eta(W)g(Y,Z)\}, \end{aligned} \tag{1.6}$$

where we put $\tilde{\rho}(X,Y) = \rho(X,\varphi Y)$.

2. A curvaturelike tensor field in an almost contact Riemannian manifold

In this section, we define a new curvaturelike tensor field in an almost contact Riemanian manifold.

In a differentiable manifold M, a $(0,4)$-type tensor field $T(X,Y,Z,W)$ is called *curvaturelike* if it satisfies

$$T(X,Y,Z,W) = -T(Y,X,Z,W), \quad T(X,Y,Z,W) = T(Z,W,X,Y),$$
$$T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0$$

for any $X,Y,Z,W \in TM$ [**4**].

In an almost contact Riemannian manifold (M^{2n+1},φ,ξ,g), we define a (0,4)-type tensor field $(\mathrm{CHR})_3(X,Y,Z,W)$ as

$$\begin{aligned} 16(\mathrm{CHR})_3(X,Y,Z,W) &= 3\{R(X,Y,Z,W) + R(\varphi X,\varphi Y,Z,W) \\ &+ R(\varphi Z,\varphi W,X,Y) + R(\varphi X,\varphi Y,\varphi Z,\varphi W)\} - R(\varphi W,\varphi Y,X,Z) \\ &- R(\varphi X,\varphi Z,W,Y) - R(\varphi Y,\varphi Z,X,W) - R(\varphi X,\varphi W,Y,Z) \\ &+ R(\varphi X,Z,\varphi W,Y) + R(X,\varphi Z,W,\varphi Y) + R(\varphi X,W,Y,\varphi Z) \\ &+ R(X,\varphi W,\varphi Y,Z) \\ &+ \eta(X)P(Z,W,Y) - \eta(Y)P(Z,W,X) \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&+\eta(Z)P(X,Y,W)-\eta(W)P(X,Y,Z)\\
&+\eta(X)\eta(W)Q(Y,Z)-\eta(X)\eta(Z)Q(Y,W)\\
&\qquad+\eta(Y)\eta(Z)Q(W,X)-\eta(Y)\eta(W)Q(Z,X),
\end{aligned}
$$

where we put

$$
\begin{aligned}
P(X,Y,Z)=3\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\}+R(\varphi X,\varphi Z,Y,\xi)&\\
+R(\varphi Z,\varphi Y,X,\xi)-R(X,\varphi Z,\varphi Y,\xi)-R(\varphi Z,Y,\varphi X,\xi),&
\end{aligned}\tag{2.2}
$$

$$
Q(X,Y)=3R(\xi,X,Y,\xi)-R(\xi,\varphi X,\varphi Y,\xi)\tag{2.3}
$$

for any $X,Y,Z,W\in TM^{2n+1}$. Then, we can easily check the above tensor field is curvaturelike. We call this tensor field a $(\mathrm{CHR})_3$-*curvature tensor* in an almost contact Riemannian manifold or a *contact holomorphic Riemannian curvature tensor of the second type*.

About the tensors $P(X,Y,Z)$ and $Q(X,Y)$, we can easily see

$$
\begin{aligned}
P(\varphi X,\varphi Y,Z)&=P(X,Y,Z)-\eta(X)Q(Y,Z)+\eta(Y)Q(X,Z),\\
P(X,\varphi Y,\varphi Z)&=3\{R(X,\varphi Y,\varphi Z,\xi)-R(\varphi X,Y,\varphi Z,\xi)\}\\
&\quad-R(\varphi X,Z,\varphi Y,\xi)+R(Z,Y,X,\xi)-R(X,Z,Y,\xi)\\
&\quad+R(Z,\varphi Y,\varphi X,\xi)+\eta(Y)Q(Z,X)\\
&\quad-2\{\eta(Z)S(X,Y)+\eta(Y)S(X,Z)\},\\
P(\varphi X,Y,\varphi Z)&=3\{R(\varphi X,Y,\varphi Z,\xi)-R(X,\varphi Y,\varphi Z,\xi)\}+R(X,Z,Y,\xi)\\
&\quad-R(Z,\varphi Y,\varphi X,\xi)+R(\varphi X,Z,\varphi Y,\xi)-R(Z,Y,X,\xi)\\
&\quad-\eta(X)Q(Y,Z)+2\{\eta(X)S(Y,Z)+\eta(Z)S(X,Y)\},\\
P(X,Y,\xi)&=0,\quad P(\xi,X,Y)=Q(X,Y),\quad P(X,\xi,Y)=-Q(X,Y),
\end{aligned}\tag{2.4}
$$

$$
Q(\varphi X,\varphi Y)=-Q(X,Y)+2S(X,Y),\qquad Q(\xi,X)=0,\tag{2.5}
$$

where we put $S(X,Y)=R(\xi,X,Y,\xi)+R(\xi,\varphi X,\varphi Y,\xi)$. In particular, by virtue of (1.3), the tensor fields $P(X,Y,Z)$ and $Q(X,Y)$ in a Sasakian manifold are respectively satisfied

$$
\begin{aligned}
P(X,Y,Z)&=2\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}=2R(X,Y,Z,\xi),\\
Q(X,Y)&=2\{g(X,Y)-\eta(X)\eta(Y)\}=2g(\varphi X,\varphi Y)
\end{aligned}
$$

for any $X,Y,Z\in TM^{2n+1}$.

An almost contact Riemannian manifold is said to be $(\mathrm{CHR})_3$-*flat* if the $(\mathrm{CHR})_3$-curvature tensor vanishes, identically.

In an almost contact Riemannian manifold, the $(\mathrm{CHR})_3$-curvature tensor satisfies the following equations

$$
\begin{aligned}
&(\mathrm{CHR})_3(X,Y,Z,\xi)=0,\\
&(\mathrm{CHR})_3(X,Y,\varphi Z,\varphi W)=(\mathrm{CHR})_3(\varphi X,\varphi Y,Z,W)=(\mathrm{CHR})_3(X,Y,Z,W),\\
&(\mathrm{CHR})_3(X,\varphi X,\varphi X,X)=R(X,\varphi X,\varphi X,X)-2\eta(X)R(X,\varphi X,\varphi X,\xi)\\
&\qquad\qquad+\eta(X)^2R(\xi,\varphi X\varphi X,\xi)
\end{aligned}
$$

for any $X, Y, Z, W \in TM^{2n+1}$.

3. $(\mathrm{CHR})_3$-curvature tensor in a Sasakian manifold

In this section, we consider a $(\mathrm{CHR})_3$-flat Sasakian manifold and we prove this manifold is flat Sasakian.

By virtue of (1.3) and Bianchi identity [**6**], in a Sasakian manifold, the $(\mathrm{CHR})_3$-curvature tensor satisfies

$$
\begin{aligned}
(3.1)\quad (\mathrm{CHR})_3(X,Y,Z,W) &= R(X,Y,Z,W) \\
&\quad + \frac{3}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\} \\
+ \frac{1}{4}\{g(\varphi X,W)g(\varphi Y,Z) &- g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W) \\
&+ \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z) \\
&\qquad + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W)\}.
\end{aligned}
$$

Let us consider the $(\mathrm{CHR})_3$-flat Sasakian manifold, then we have from (3.1), the curvature tensor R is written as

$$
\begin{aligned}
R(X,Y,Z,W) &= \frac{3}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\
- \frac{1}{4}\{g(\varphi X,W)g(\varphi Y,Z) &- g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W) \\
&+ \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z) \\
&\qquad + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W)\}.
\end{aligned}
$$

Comparing the above equation and (1.4), we obtain

THEOREM 3.1. *A $(\mathrm{CHR})_3$-flat Sasakian manifold is flat Sasakian.*

4. The properties of the $(\mathrm{CHR})_3$-curvature tensor.

We put $\rho(\mathrm{CHR})_3(X,Y) = \sum_{i=1}^{2n+1}(\mathrm{CHR})_3(e_i,X,Y,e_i)$ for a local orthonormal frame $(e_1, e_2, \dots, e_{2n+1})$ of an almost contact Riemannian manifold M^{2n+1}. Then we say $\rho(\mathrm{CHR})_3(X,Y)$ a $(\mathrm{CHR})_3$-*Ricci tensor*.

By virtue of (2.1), (2.2) and (2.3), the local representations of $(\mathrm{CHR})_3$-curvature tensor, the tensor fields P and Q are respectively written by

$$
\begin{aligned}
(4.1)\quad 16(\mathrm{CHR})_{3kjih} &= 3(R_{kjih} + \overset{*}{R}_{kjih} + \overset{*}{R}_{ihkj} + \overset{*}{R}_{kjml}\varphi_i{}^m\varphi_h{}^l) \\
&\quad - \overset{*}{R}_{hjki} - \overset{*}{R}_{kihj} - \overset{*}{R}_{jikh} - \overset{*}{R}_{khji} \\
+ R_{tisj}\varphi_k{}^t\varphi_h{}^s &+ R_{kths}\varphi_i{}^t\varphi_j{}^s + R_{thjs}\varphi_k{}^t\varphi_i{}^s + R_{ktsi}\varphi_h{}^t\varphi_j{}^s \\
&+ \eta_k P_{ihj} - \eta_j P_{kji} + \eta_i P_{kjh} - \eta_h P_{kji} \\
&\qquad + \eta_k\eta_h Q_{ji} - \eta_k\eta_i Q_{jh} + \eta_j\eta_i Q_{hk} - \eta_j\eta_h Q_{ik},
\end{aligned}
$$

$$
(4.2)\qquad P_{jih} = 3(R_{jihl} + \overset{*}{R}_{jihl})\xi^l + (\overset{*}{R}_{jhil} + \overset{*}{R}_{hijl})\xi^l
$$

$$- (R_{jtsl}\varphi_h{}^t\varphi_i{}^s + R_{tisl}\varphi_h{}^t\varphi_j{}^s)\xi^l,$$

$$Q_{ih} = (3R_{mihl} - R_{mtsl}\varphi_i{}^t\varphi_h{}^s)\xi^m\xi^l, \tag{4.3}$$

where we put $\overset{*}{R}_{kjih} = R_{tsih}\varphi_k{}^t\varphi_j{}^s$ and the indices $\{k,j,...,h\}$ run over the range $\{1,2,...,2n+1\}$.

Moreover, equations (2.4) and (2.5) are respectively written as

$$\begin{aligned} P_{tsh}\varphi_j{}^t\varphi_i{}^s &= P_{jih} - \eta_j Q_{ih} + \eta_i Q_{jh}, \\ P_{jts}\varphi_i{}^t\varphi_h{}^s &= 3(R_{jtsl}\varphi_i{}^t\varphi_h{}^s\xi^l - R_{tisl}\varphi_j{}^t\varphi_h{}^s)\xi^l \\ &\quad - R_{thsl}\varphi_j{}^t\varphi_i{}^s\xi^l + R_{hijl}\xi^l - R_{jhil}\xi^l \\ &\quad + R_{htsl}\varphi_i{}^t\varphi_j{}^s\xi^l + \eta_i Q_{hj} - 2(\eta_i S_{hj} + \eta_h S_{ij}), \\ P_{tis}\varphi_j{}^t\varphi_h{}^s &= 3(R_{tisl}\varphi_j{}^t\varphi_h{}^s - R_{jtsl}\varphi_i{}^t\varphi_h{}^s)\xi^l + R_{jhil}\xi^l \\ &\quad - R_{htsl}\varphi_i{}^t\varphi_j{}^s\xi^l + R_{thsl}\varphi_j{}^t\varphi_i{}^s\xi^l - R_{hijl}\xi^l \\ &\quad - \eta_j Q_{ih} + 2(\eta_j S_{ih} + \eta_h S_{ji}), \\ P_{ihl}\xi^l = 0, \quad & P_{lih}\xi^l = Q_{ih}, \quad P_{ilh}\xi^l = -Q_{ih}, \end{aligned} \tag{4.4}$$

$$Q_{ts}\varphi_i{}^t\varphi_h{}^s = -Q_{ih} + 2S_{ih}, \quad Q_{lh}\xi^l = 0, \tag{4.5}$$

$$S_{ih} = \{R_{mihl} + R_{mtsl}\varphi_i{}^t\varphi_h{}^s\}\xi^m\xi^l. \tag{4.6}$$

Since, we know $\rho(\mathrm{CHR})_{3ji} = (\mathrm{CHR})_{3kjih}g^{kh}$, we have from (4.1), (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} 8\rho(\mathrm{CHR})_3(X,Y) &= \rho(X,Y) + \rho(\varphi X, \varphi Y) \\ &\quad - 3\{\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\} - \{\rho(X,\xi)\eta(Y) + \rho(Y,\xi)\eta(X)\} \\ &\quad + 3\{\overset{*}{\rho}(X,\xi)\eta(Y) + \overset{*}{\rho}(Y,\xi)\eta(X)\} + \rho(\xi,\xi)\eta(X)\eta(Y) \\ &\quad - \{R(\xi,X,Y,\xi) + R(\xi,\varphi X,\varphi Y,\xi)\}, \end{aligned} \tag{4.7}$$

where we put

$$\rho(X,Y) = \sum_{i=1}^{2n+1} R(e_i,X,Y,e_i), \quad \overset{*}{\rho}(X,Y) = \sum_{i=1}^{2n+1} \overset{*}{R}(e_i,X,Y,e_i).$$

In particular, the $(\mathrm{CHR})_3$-Ricci tensor in a Sasakian manifold is written by

$$\rho(\mathrm{CHR})_3(X,Y) = \rho(X,Y) - \frac{1}{2}\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\}. \tag{4.8}$$

From the above equation, we can easily have $\rho(\mathrm{CHR})_3(X,\xi) = 0$.

Next, we put $\tau(\mathrm{CHR})_3 = \sum_{i=1}^{2n+1}\rho(\mathrm{CHR})_3(e_i,e_i) = \rho(\mathrm{CHR})_{3ji}g^{ji}$ which is called the $(\mathrm{CHR})_3$-*scalar curvature* of $(\mathrm{CHR})_3$-curvature tensor. Then from (4.7), we have $4\tau(\mathrm{CHR})_3 = \tau - 2\rho(\xi,\xi) + 3\overset{*}{\tau}$, where we put $\overset{*}{\tau} = \sum_{i=1}^{2n+1}\overset{*}{\rho}(e_i,e_i)$. Then, in a Sasakian manifold, we have from (4.8)

$$\tau(\mathrm{CHR})_3 = \tau - n(3n+1). \tag{4.9}$$

Now, an almost contact Riemannian manifold M^{2n+1} is called $(\mathrm{CHR})_3$-*η-Einstein* if its $(\mathrm{CHR})_3$-Ricci tensor $\rho(\mathrm{CHR})_3(X,Y)$ has the form

$$\rho(\mathrm{CHR})_3(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{4.10}$$

for certain functions α and β which are called *associated functions* of $\rho(\mathrm{CHR})_3$. In particular, if our manifold is Sasakian, then we have from (4.8) and (4.10) the Ricci tensors is written as

$$\rho(X,Y) = \Big(\alpha + \frac{3n-1}{2}\Big)g(X,Y) + \Big(\beta + \frac{n+1}{2}\Big)\eta(X)\eta(Y),$$

that is, our manifold is η-Einstein.

Conversely, if our manifold M^{2n+1} is η-Einstein, then we can easily see that Sasakian manifolds are $(\mathrm{CHR})_3$-η-Einstein. Moreover, by virtue of (1.5), we obtain $\alpha+\beta=0$ in a $(\mathrm{CHR})_3$-η-Einstein Sasakian manifold and the scalar curvature τ and the $(\mathrm{CHR})_3$-scalar curvature $\tau(\mathrm{CHR})_3$ are written as

$$\tau = n(2\alpha+3n+1), \quad \tau(\mathrm{CHR})_3 = 2n\alpha. \tag{4.11}$$

Thus we have

THEOREM 4.1. *A $(2n+1)$-dimensional Sasakian manifold M^{2n+1} is $(\mathrm{CHR})_3$-η-Einstein if and only if M^{2n+1} is η-Einstein and the scalar curvatures τ and $\tau(\mathrm{CHR})_3$ are respectively written by (4.11) which includes only the associated function α.*

COROLLARY 4.1. *In an η-$(\mathrm{CHR})_3$-Einstein Sasakian manifold M^{2n+1}, the scalar curvatures τ and $\tau(\mathrm{CHR})_3$ are constant if and only if one of its associated functions is constant.*

5. $(\mathrm{CHR})_3$-space form

In this section, we define a notion of a $(\mathrm{CHR})_3$-space form in an almost contact Riemannian manifold. Then we consider a Sasakian $(\mathrm{CHR})_3$-space form. And in this manifold, we determine the $(\mathrm{CHR})_3$-curvature tensor by the structure tensors.

DEFINITION 5.1. An almost contact Riemannian manifold is called a $(\mathrm{CHR})_3$-space form if its $(\mathrm{CHR})_3$-curvature tensor satisfies

$$(\mathrm{CHR})_3(X,\varphi X,\varphi X,X) = c\|X\|^4$$

for a certain constant c and any $X \in TM^{2n+1} - \{\xi\}$.

By virtue of (2.1), in an almost contact Riemannian manifold, we have

$$(\mathrm{CHR})_3(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X)$$

for any $X \in TM^{2n+1} - \{\xi\}$. This means that a Sasakian $(\mathrm{CHR})_3$-space form is a Sasakian space form. Thus we have from (1.4) and (3.1)

THEOREM 5.1. *In a $(2n+1)$-dimensional Sasakian $(\mathrm{CHR})_3$-space form, the $(\mathrm{CHR})_3$-curvatuee tensor, $(\mathrm{CHR})_3$-Ricci tensor and the $(\mathrm{CHR})_3$-scalar curvature are respectively given by*

$$(\mathrm{CHR})_3(X,Y,Z,W) = \frac{c}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - F(X,W)F(Y,Z) + F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W) + \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W)\},$$

$$\rho(\mathrm{CHR})_3(X,Y) = \frac{cn}{2}\{g(X,Y) - \eta(X)\eta(Y)\}, \quad \tau(\mathrm{CHR})_3 = cn^2.$$

for any $X, Y, Z, W \in TM^{2n+1}$.

6. Conformal transformations of almost contact Riemannian manifolds

In an almost contact Riemannian manifold M^{2n+1}, we consider a following conformal transformation;

$$\bar{g} = e^{2f}g, \quad \bar{g}^{-1} = e^{-2f}g^{-1} \tag{6.1}$$

for a certain positive differentiable function f on M^{2n+1}. Then it is well known [**6**] that

$$\bar{\nabla}_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - g(X,Y)U, \tag{6.2}$$

where $\bar{\nabla}$ (resp. ∇) means the covariant derivation with respect to $\bar{g}$ (resp. g), $\theta = df$ and $g(U,X) \overset{\mathrm{def}}{=} \theta(X)$. Moreover, between the Riemannian curvature tensors $\bar{R}(X,Y,Z,W)$ with respect to $\bar{g}$ and $R(X,Y,Z,W)$ with respect to g, we have the following relation [**6**]

$$e^{-2f}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(X,W)\sigma(Y,Z) + g(Y,Z)\sigma(X,W) - g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(X,Z), \tag{6.3}$$

where we put

$$\sigma(X,Y) = (\nabla_X\theta)(Y) - \theta(X)\theta(Y) + \frac{1}{2}g(X,Y)\theta(U). \tag{6.4}$$

From (6.3) and (6.4), we obtain $\bar{\rho}(X,Y) = \rho(X,Y) + (2n-1)\sigma(X,Y) + \sigma g(X,Y)$, where $\bar{\rho}$ denotes the Ricci tensor with respect to $\bar{g}$ and

$$\sigma = \sum_{i=1}^{2n+1} \sigma(e_i,e_i) = \sigma_{ji}g^{ji}. \tag{6.5}$$

From the above two equations, we easily get $e^{2f}\bar{\tau} = \tau + 4n\sigma$.

Similarly with (2.1), $(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$ with respect to $\bar{g}$ is given by

$$16(\overline{\mathrm{CHR}})_3(X,Y,Z,W) = 3\{\bar{R}(X,Y,Z,W) + \bar{R}(\varphi X,\varphi Y,Z,W) + \bar{R}(X,Y,\varphi Z,\varphi W) + \bar{R}(\varphi X,\varphi Y,\varphi Z,\varphi W)\} - \bar{R}(X,Z,\varphi W,\varphi Y) - \bar{R}(\varphi X,\varphi Z,W,Y) - \bar{R}(X,W,\varphi Y,\varphi Z) - \bar{R}(\varphi X,\varphi W,Y,Z) + \bar{R}(\varphi X,Z,\varphi W,Y) + \bar{R}(X,\varphi Z,W,\varphi Y) + \bar{R}(\varphi X,W,Y,\varphi Z) + \bar{R}(X,\varphi W,\varphi Y,Z)$$

$$+\eta(X)\bar{P}(Z,W,Y)-\eta(Y)\bar{P}(Z,W,X)$$
$$+\eta(Z)\bar{P}(X,Y,W)-\eta(W)\bar{P}(X,Y,Z)$$
$$+\eta(X)\eta(W)\bar{Q}(Y,Z)-\eta(X)\eta(Z)\bar{Q}(Y,W)$$
$$+\eta(Y)\eta(Z)\bar{Q}(W,X)-\eta(Y)\eta(W)\bar{Q}(Z,X),$$

where we put

$$\bar{P}(X,Y,Z)=3\{\bar{R}(X,Y,Z,\xi)+\bar{R}(\varphi X,\varphi Y,Z,\xi)\}+\bar{R}(\varphi X,\varphi Z,Y,\xi)$$
$$+\bar{R}(\varphi Z,\varphi Y,X,\xi)-\bar{R}(X,\varphi Z,\varphi Y,\xi)-\bar{R}(\varphi Z,Y,\varphi X,\xi), \tag{6.6}$$

$$\bar{Q}(X,Y)=3\bar{R}(\xi,X,Y,\xi)-\bar{R}(\xi,\varphi X,\varphi Y,\xi). \tag{6.7}$$

for any $X,Y,Z,W\in TM^{2n+1}$.

Let us consider the relation between the tensor fields $(\mathrm{CHR})_3(X,Y,Z,W)$ and $(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$.

By virtue of (6.3), (6.6) and (6.7) are respectively written as

$$e^{-2f}\bar{P}(X,Y,Z)=P(X,Y,Z)+\eta(X)\{3\sigma(Y,Z)-\sigma(\varphi Y,\varphi Z)\}$$
$$-\eta(Y)\{3\sigma(X,Z)-\sigma(\varphi X,\varphi Z)\}+2\{g(Y,Z)\sigma(X,\xi)-g(X,Z)\sigma(Y,\xi)$$
$$+g(\varphi Y,Z)\sigma(\varphi X,\xi)-g(\varphi X,Z)\sigma(\varphi Y,\xi)-2g(\varphi X,Y)\sigma(\varphi Z,\xi)\}$$
$$+\eta(Z)\{\eta(Y)\sigma(X,\xi)-\eta(X)\sigma(Y,\xi)\}, \tag{6.8}$$

$$e^{-2f}\bar{Q}(X,Y)=Q(X,Y)+3\sigma(X,Y)-\sigma(\varphi X,\varphi Y)+2g(X,Y)\sigma(\xi,\xi)$$
$$+\eta(X)\eta(Y)\sigma(\xi,\xi)-3\{\eta(X)\sigma(Y,\xi)+\eta(Y)\sigma(X,\xi)\}. \tag{6.9}$$

To calculate $e^{-2f}(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$, we separate the following three parts;

Part 1:

$$(\bar{\mathrm{I}})\overset{\text{put}}{=}3\{\bar{R}(X,Y,Z,W)+\bar{R}(\varphi X,\varphi Y,Z,W)+\bar{R}(X,Y,\varphi Z,\varphi W)$$
$$+\bar{R}(\varphi X,\varphi Y,\varphi Z,\varphi W)\}-\bar{R}(X,Z,\varphi W,\varphi Y)-\bar{R}(\varphi X,\varphi Z,W,Y)$$
$$-\bar{R}(X,W,\varphi Y,\varphi Z)-\bar{R}(\varphi X,\varphi W,Y,Z)+\bar{R}(\varphi X,Z,\varphi W,Y)$$
$$+\bar{R}(X,\varphi Z,W,\varphi Y)+\bar{R}(\varphi X,W,Y,\varphi Z)+\bar{R}(X,\varphi W,\varphi Y,Z). \tag{6.10}$$

Part 2:

$$(\overline{\mathrm{II}})\overset{\text{put}}{=}\eta(X)\bar{P}(Z,W,Y)-\eta(Y)\bar{P}(Z,W,X)$$
$$+\eta(Z)\bar{P}(X,Y,W)-\eta(W)\bar{P}(X,Y,Z). \tag{6.11}$$

Part 3:

$$(\overline{\mathrm{III}})\overset{\text{put}}{=}\eta(X)\eta(W)\bar{Q}(Y,Z)-\eta(X)\eta(Z)\bar{Q}(Y,W)x$$
$$+\eta(Y)\eta(Z)\bar{Q}(W,X)-\eta(Y)\eta(W)\bar{Q}(Z,X). \tag{6.12}$$

By virtue of (6.3), (6.8) and (6.9), we obtain

$$e^{-2f}(\bar{\mathrm{I}})=(\mathrm{I})+2g(X,W)\{\sigma(Y,Z)+\sigma(\varphi Y,\varphi Z)\}$$
$$+2g(Y,Z)\{\sigma(X,W)+\sigma(\varphi X,\varphi W)\}-2g(X,Z)\{\sigma(Y,W)+\sigma(\varphi Y,\varphi W)\}$$

$$- 2g(Y,W)\{\sigma(X,Z) + \sigma(\varphi X,\varphi Z)\} + 2g(\varphi X,W)\{\sigma(\varphi Y,Z) - \sigma(Y,\varphi Z)\}$$
$$+ 2g(\varphi Y,Z)\{\sigma(\varphi X,W) - \sigma(X,\varphi W)\} - 2g(\varphi X,Z)\{\sigma(\varphi Y,W) - \sigma(Y,\varphi W)\}$$
$$- 2g(\varphi Y,W)\{\sigma(\varphi X,Z) - \sigma(X,\varphi Z)\} - 4g(\varphi X,Y)\{\sigma(\varphi Z,W) - \sigma(Z,\varphi W)\}$$
$$- 4g(\varphi Z,W)\{\sigma(\varphi X,Y) - \sigma(X,\varphi Y)\} + \eta(X)\eta(W)\{\sigma(Y,Z) - 3\sigma(\varphi Y,\varphi Z)\}$$
$$- \eta(Y)\eta(W)\{\sigma(X,Z) - 3\sigma(\varphi X,\varphi Z)\} - \eta(X)\eta(Z)\{\sigma(Y,W) - 3\sigma(\varphi Y,\varphi W)\}$$
$$+ \eta(Y)\eta(Z)\{\sigma(X,W) - 3\sigma(\varphi X,\varphi W)\}.$$

$$e^{-2f}(\overline{\mathrm{II}}) = (\mathrm{II}) + 2\eta(X)\eta(Z)\{3\sigma(Y,W) - \sigma(\varphi Y,\varphi W)\}$$
$$- 2\eta(X)\eta(W)\{3\sigma(Y,Z) - \sigma(\varphi Y,\varphi Z)\} - 2\eta(Y)\eta(Z)\{3\sigma(X,W) - \sigma(\varphi X,\varphi W)\}$$
$$+ 2\eta(Y)\eta(W)\{3\sigma(X,Z) - \sigma(\varphi X,\varphi Z)\} - 2g(\varphi Y,W)\{\eta(X)\sigma(\varphi Z,\xi)$$
$$- \eta(Z)\sigma(\varphi X,\xi)\} + 2g(\varphi Y,Z)\{\eta(X)\sigma(\varphi W,\xi) - \eta(W)\sigma(\varphi X,\xi)\}$$
$$+ 2g(\varphi X,W)\{\eta(Y)\sigma(\varphi Z,\xi) - \eta(Z)\sigma(\varphi Y,\xi)\} - 2g(\varphi X,Z)\{\eta(Y)\sigma(\varphi W,\xi)$$
$$- \eta(W)\sigma(\varphi Y,\xi)\} - 4g(\varphi Z,W)\{\eta(X)\sigma(\varphi Y,\xi) - \eta(Y)\sigma(\varphi X,\xi)\}$$
$$- 4g(\varphi X,Y)\{\eta(Z)\sigma(\varphi W,\xi) - \eta(W)\sigma(\varphi X,\xi)\} + 2g(Y,W)\{\eta(X)\sigma(Z,\xi)$$
$$+ \eta(Z)\sigma(X,\xi)\} - 2g(Y,Z)\{\eta(X)\sigma(W,\xi) + \eta(W)\sigma(X,\xi)\}$$
$$-2g(X,W)\{\eta(Y)\sigma(Z,\xi)+\eta(Z)\sigma(Y,\xi)\}+2g(X,Z)\{\eta(Y)\sigma(W,\xi)+\eta(W)\sigma(Y,\xi)\},$$

$$e^{-2f}(\overline{\mathrm{III}}) = (\mathrm{III}) + \eta(X)\eta(W)\{3\sigma(Y,Z) - \sigma(\varphi Y,\varphi Z)\}$$
$$- \eta(X)\eta(Z)\{3\sigma(Y,W) - \sigma(\varphi Y,\varphi W)\} + \eta(Y)\eta(Z)\{3\sigma(X,W) - \sigma(\varphi X,\varphi W)\}$$
$$- \eta(Y)\eta(W)\{3\sigma(X,Z) - \sigma(\varphi X,\varphi Z)\} + 2\sigma(\xi,\xi)\{\eta(X)\eta(W)g(Y,Z)$$
$$- \eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(Z)g(X,W) - \eta(Y)\eta(W)g(X,Z)\},$$

where $(\bar{\mathrm{I}})$ denotes the geometric equation with respect to $\bar{g}$ of (I) with respect to g and etc.. By virtue of the above three equations, we have

$$(6.13)\quad 8e^{-2f}(\overline{\mathrm{CHR}})_3(X,Y,Z,W) = 8(\mathrm{CHR})_3(X,Y,Z,W)$$
$$+ g(\varphi X,\varphi W)D(Y,Z) - g(\varphi X,\varphi Z)D(Y,W) + g(\varphi Y,\varphi Z)D(X,W)$$
$$- g(\varphi Y,\varphi W)D(X,Z) + g(\varphi X,W)\{A(Y,Z) + B(Y,Z)\}$$
$$+ g(\varphi Y,Z)\{A(X,W) + B(X,W)\} - g(\varphi X,Z)\{A(Y,W) + B(Y,W)\}$$
$$- g(\varphi Y,W)\{A(X,Z) + B(X,Z)\} - 2g(\varphi X,Y)\{A(Z,W) + B(Z,W)\}$$
$$- 2g(\varphi Z,W)\{A(X,Y) + B(X,Y)\} + g(Y,W)C(X,Z) - g(Y,Z)C(X,W)$$
$$+ g(X,Z)C(Y,W) - g(X,W)C(Y,Z) + \sigma(\xi,\xi)\{\eta(X)\eta(W)g(Y,Z)$$
$$- \eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(Z)g(X,W) - \eta(Y)\eta(W)g(X,Z)\},$$

where we put

$$A(X,Y) = \sigma(\varphi X,Y) - \sigma(X,\varphi Y),$$
$$B(X,Y) = \eta(X)\sigma(\varphi Y,\xi) - \eta(Y)\sigma(\varphi X,\xi),$$
$$C(X,Y) = \eta(X)\sigma(Y,\xi) + \eta(Y)\sigma(X,\xi),$$

$$D(X,Y)=\sigma(X,Y)+\sigma(\varphi X,\varphi Y).$$

7. An invariant tensor field under a conformal transformation

The equation (6.13) is written locally by

$$\begin{aligned}(7.1)\qquad 8e^{-2f}(\overline{\mathrm{CHR}})_{3kjih} &= 8(\mathrm{CHR})_{3kjih}+(g_{kh}-\eta_k\eta_h)D_{ji}\\ &+(g_{ji}-\eta_j\eta_i)D_{kh}-(g_{ki}-\eta_k\eta_i)D_{jh}-(g_{jh}-\eta_j\eta_h)D_{ki}\\ &+\varphi_{kh}(A_{ji}+B_{ji})+\varphi_{ji}(A_{kh}+B_{kh})\\ &-\varphi_{ki}(A_{jh}+B_{jh})-\varphi_{jh}(A_{ki}+B_{ki})-2\varphi_{kj}(A_{ih}+B_{ih})\\ &-2\varphi_{ih}(A_{kj}+B_{kj})+g_{jh}C_{ki}-g_{ji}C_{kh}+g_{ki}C_{jh}-g_{kh}C_{ji}\\ &+\sigma_{ml}\xi^m\xi^l(\eta_k\eta_h g_{ji}-\eta_k\eta_i g_{jh}+\eta_j\eta_i g_{kh}-\eta_j\eta_h g_{ki}).\end{aligned}$$

Using (7.1), we get

$$(7.2)\qquad 4\rho(\overline{\mathrm{CHR}})_3(X,Y)=4\rho(\mathrm{CHR})_3(X,Y)+(n+2)D(X,Y),$$
$$-(n+2)C(X,Y)+\{\sigma-\sigma(\xi,\xi)\}g(X,Y)-\{\sigma-(n+3)\sigma(\xi,\xi)\}\eta(X)\eta(Y),$$

$$(7.3)\qquad e^{2f}\tau(\overline{\mathrm{CHR}})_3=\tau(\mathrm{CHR})_3+(n+1)\{\sigma-\sigma(\xi,\xi)\}.$$

Since, we have from (7.2) and (7.3)

$$\begin{aligned}(7.4)\qquad D(X,Y)&=\frac{1}{n+2}\Big\{4\rho(\overline{\mathrm{CHR}})_3(X,Y)-\frac{\tau(\overline{\mathrm{CHR}})_3}{n+1}\bar g(X,Y)\Big\}\\ &-\frac{1}{n+2}\Big\{4\rho(\mathrm{CHR})_3(X,Y)-\frac{\tau(\mathrm{CHR})_3}{n+1}g(X,Y)\Big\}\\ &+C(X,Y)+\frac{\sigma-(n+3)\sigma(\xi,\xi)}{n+2}\eta(X)\eta(Y).\end{aligned}$$

The above equation and (7.1) give us

$$\begin{aligned}(7.5)\quad & g(\varphi X,\varphi Y)D(Z,W)\\ &=\frac{e^{-2f}}{n+2}\Big\{4\rho(\overline{\mathrm{CHR}})_3(Z,W)-\frac{\tau(\overline{\mathrm{CHR}})_3}{n+1}\bar g(Z,W)\Big\}\bar g(\varphi X,\varphi Y)\\ &-\frac{1}{n+2}\Big\{4\rho(\mathrm{CHR})_3(Z,W)-\frac{\tau(\mathrm{CHR})_3}{n+1}g(Z,W)\Big\}g(\varphi X,\varphi Y)\\ &+g(\varphi X,\varphi Y)C(Z,W)+\frac{\sigma-(n+3)\sigma(\xi,\xi)}{n+2}g(\varphi X,\varphi Y)\eta(Z)\eta(W).\end{aligned}$$

By virtue of (7.5) and (6.13), we have

$$\begin{aligned}(7.6)\quad & e^{-2f}\bar T(X,Y,Z,W)-T(X,Y,Z,W)\\ &=g(\varphi X,W)\{A(Y,Z)+B(Y,Z)\}+g(\varphi Y,Z)\{A(X,W)+B(X,W)\}\\ &-g(\varphi X,Z)\{A(Y,W)+B(Y,W)\}-g(\varphi Y,W)\{A(X,Z)+B(X,Z)\}\\ &-2g(\varphi X,Y)\{A(Z,W)+B(Z,W)\}-2g(\varphi Z,W)\{A(X,Y)+B(X,Y)\}\\ &+\frac{\sigma-\sigma(\xi,\xi)}{n+2}\{g(X,W)\eta(Y)\eta(Z)+g(Y,Z)\eta(X)\eta(W)\end{aligned}$$

$$- g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)\},$$

where we put

$$T(X,Y,Z,W) = 8(\mathrm{CHR})_3(X,Y,Z,W) - \frac{1}{n+2}\{E(Y,Z)g(\varphi X,\varphi W) + E(X,W)g(\varphi Y,\varphi Z) - E(Y,W)g(\varphi X,\varphi Z) - E(X,Z)g(\varphi Y,\varphi W)\}, \tag{7.7}$$

$$E(X,Y) = 4\rho(\mathrm{CHR})_3(X,Y) - \frac{\tau(\mathrm{CHR})_3}{n+1}g(X,Y). \tag{7.8}$$

Equation (7.6) is locally written by

$$\begin{aligned} (7.6') \quad e^{-2f}\bar{T}_{kjih} - T_{kjih} &= \varphi_{kh}(A_{ji}+B_{ji}) + \varphi_{ji}(A_{kh}+B_{kh}) \\ &- \varphi_{ki}(A_{jh}+B_{jh}) - \varphi_{jh}(A_{ki}+B_{ki}) - 2\varphi_{kj}(A_{ih}+B_{ih}) \\ &- 2\varphi_{ih}(A_{kj}+B_{kj}) + \frac{\sigma-\sigma(\xi,\xi)}{n+2}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h \\ &- g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \end{aligned}$$

Contracting (7.6′) with φ^{kh}, we obtain

$$A_{ji} + B_{ji} = \frac{\varphi^{ts}}{2(n+1)}(e^{-2f}\bar{T}_{tjis} - T_{tjis}) - \frac{\sigma-\sigma(\xi,\xi)}{n+1}\varphi_{ji}. \tag{7.9}$$

Substituting (7.9) into (7.6′), we have

$$\begin{aligned} (7.10) \quad & e^{-2f}\Big\{\bar{T}_{kjih} - \frac{\bar{\varphi}^{ts}}{2(n+1)}(\bar{T}_{tjis}\bar{\varphi}_{kh} + \bar{T}_{tkhs}\bar{\varphi}_{ji} - \bar{T}_{tjhs}\bar{\varphi}_{ki} - \bar{T}_{tkis}\bar{\varphi}_{jh} \\ & - 2\bar{T}_{tkjs}\bar{\varphi}_{ih} - 2\bar{T}_{tihs}\bar{\varphi}_{kj}) + \frac{2\tau(\overline{\mathrm{CHR}})_3}{(n+1)^2}(\bar{\varphi}_{kh}\bar{\varphi}_{ji} - \bar{\varphi}_{ki}\bar{\varphi}_{jh} - 2\bar{\varphi}_{kj}\bar{\varphi}_{ih})\Big\} \\ & - \Big\{T_{kjih} - \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} \\ & - 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) + \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih})\Big\} \\ & = \frac{\sigma-\sigma(\xi,\xi)}{n+2}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i), \end{aligned}$$

where we put $\bar{\varphi}_{ji} = \varphi_j{}^l\bar{g}_{li}$ and $\bar{\varphi}^{ji} = \varphi_l{}^i\bar{g}^{lj}$. The right hand side of (7.10) is written by

$$\begin{aligned} (7.11) \quad & e^{-2f}\frac{\tau(\overline{\mathrm{CHR}})_3}{(n+1)(n+2)}(\bar{g}_{kh}\bar{\eta}_j\eta_i + \bar{g}_{ji}\bar{\eta}_k\eta_h - \bar{g}_{ki}\bar{\eta}_j\eta_h - \bar{g}_{jh}\bar{\eta}_k\eta_i) \\ & - \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i), \end{aligned}$$

where we put $\bar{\eta}_i = \xi^l\bar{g}_{li}$. Thus we have

Theorem 7.1. *In an almost contact Riemannian manifold, the tensor field $S(X,Y)Z$ is invariant under the conformal transformation* (6.1), *where the tensor field $S_{kji}{}^h$ is defined by*

$$S_{kji}{}^{l}g_{lh} = T_{kjih} - \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} - 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) + \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih}) - \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \tag{7.12}$$

In particular, in a Sasakian manifold, we have from (4.8) and (4.9)

$$E_{ji} = 4\rho_{ji} - \frac{1}{n+1}(\tau + 3n^2 + 3n - 2)g_{ji} - 2(n+1)\eta_j\eta_i. \tag{7.13}$$

By virtue of (3.1) and (7.13), equation (7.7) is given by

$$T_{kjih} = 8\{R_{kjih} + \frac{1}{4}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih})\} - \frac{4}{n+2}\{\rho_{ji}(g_{kh} - \eta_j\eta_h) + \rho_{kh}(g_{ji} - \eta_j\eta_i) - \rho_{jh}(g_{ki}\eta_k\eta_i) - \rho_{ki}(g_{jh} - \eta_j\eta_h)\} + \frac{2\{\tau - 2(3n+4)\}}{(n+1)(n+2)}(g_{kh}g_{ji} - g_{ki}g_{jh}) - \frac{\tau + n(3n+5)}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \tag{7.14}$$

Moreover, in a Sasakian manifold, using the Bianchi identity, we obtain

$$R_{tjis}\varphi^{ts} = -(2n-1)\varphi_{ji} - \tilde{\rho}_{ji}, \tag{7.15}$$

where we put $\tilde{\rho}_{ji} = \rho_{jl}\varphi_i{}^l$. Using this, we get

$$\varphi^{ts}T_{tjis} = \frac{2\{\tau - (6n^3 + 13n^2 + 3n - 2)\}}{(n+1)(n+2)}\varphi_{ji} - \frac{8(n+1)}{n+2}\tilde{\rho}_{ji}. \tag{7.16}$$

By virtue of (4.9), (7.14), (7.15) and (7.16), the tensor field $S(X,Y,Z,W)$ in a Sasakian manifold satisfies

$$\frac{1}{2}S(X,Y,Z,W) = 4R(X,Y,Z,W) - \frac{2}{n+2}\{\rho(Y,Z)g(\varphi X,\varphi W) + \rho(X,W)g(\varphi Z,\varphi Z) - \rho(Y,W)g(\varphi X,\varphi Z) - \rho(X,Z)g(\varphi Y,\varphi W)\} + \frac{2}{n+2}\{\tilde{\rho}(Y,Z)\varphi(X,W) + \tilde{\rho}(X,W)\varphi(Y,Z) - \tilde{\rho}(Y,W)\varphi(X,Z) - \tilde{\rho}(X,Z)\varphi(Y,W) - 2\tilde{\rho}(X,Y)\varphi(Z,W) - 2\tilde{\rho}(Z,W)\varphi(X,W)\} + \frac{\tau - 2(3n+4)}{(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} + \frac{\{\tau + 2n(2n+3)\}}{(n+1)(n+2)}\{\varphi(X,W)\varphi(Y,Z) - \varphi(X,Z)\varphi(Y,W) - 2\varphi(X,Y)\varphi(Z,W)\} - \frac{\tau + 2n}{(n+1)(n+2)}\{(g(X,W)\eta(Y)\eta(Z) + g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)\}. \tag{7.17}$$

Thus comparing the above equation and (1.6), we have

THEOREM 7.2. *In a Sasakian manifold, the tensor field* $\frac{1}{8}S(X,Y,Z,W)$ *is a C-Bochner curvature tensor.*

We call the tensor field $\frac{1}{8}S(X,Y,Z,W)$ the *conformal* $(\mathrm{CHR})_3$*-curvature tensor* or the *C-Bochner curvature tensor* in an almost contact Riemannian manifold. And an almost contact Riemannian manifold M^{2n+1} is called *conformally* $(\mathrm{CHR})_3$*-flat* if the tensor field $S(X,Y)Z$ vanishes for any $X,Y,Z \in TM^{2n+1}$, identically.

8. A conformally $(\mathrm{CHR})_3$-flat almost contact Riemannian manifold

Let an almost contact Riemannian manifold M^{2n+1} be conformally $(\mathrm{CHR})_3$-flat. Then we have from (7.12)

$$(8.1)\quad \begin{aligned} T_{kjih} = {} & \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} \\ & - 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) - \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih}) \\ & + \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \end{aligned}$$

By virtue of (7.7) and (7.8), we have

$$(8.2)\quad \begin{aligned} T_{kjih} = {} & 8(\mathrm{CHR})_{3kjih} - \frac{4}{n+2}\{(g_{kh} - \eta_k\eta_h)\rho(\mathrm{CHR})_{3ji} \\ & + (g_{ji} - \eta_j\eta_i)\rho(\mathrm{CHR})_{3kh} - (g_{ki} - \eta_k\eta_i)\rho(\mathrm{CHR})_{3jh} \\ & - (g_{jh} - \eta_j\eta_h)\rho(\mathrm{CHR})_{3ki}\} + \frac{2\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}g_{ji} - g_{ki}g_{jh}) \\ & - \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{ji}\eta_k\eta_h + g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i - g_{ki}\eta_j\eta_h). \end{aligned}$$

Contracting (8.2) with φ^{kh}, we have

$$(8.3)\quad \begin{aligned} T_{tjis}\varphi^{ts} = {} & 8(\mathrm{CHR})_{3tjis}\varphi^{ts} + \frac{4}{n+2}\{\rho(\mathrm{CHR})_{3jl}\varphi_i{}^l - \rho(\mathrm{CHR})_{3il}\varphi_j{}^l\} \\ & + \frac{2\tau(\mathrm{CHR})_3}{(n+1)(n+2)}\varphi_{ji}. \end{aligned}$$

Substituting (8.3) into (8.1), we obtain

$$(8.4)\quad \begin{aligned} 4(\mathrm{CHR})_{3kjih} = {} & \frac{2}{n+2}\{(g_{kh} - \eta_k\eta_h)\rho(\mathrm{CHR})_{3ji} + (g_{ji} - \eta_j\eta_i)\rho(\mathrm{CHR})_{3kh} \\ & - (g_{ki} - \eta_k\eta_i)\rho(\mathrm{CHR})_{3jh} - (g_{jh} - \eta_j\eta_h)\rho(\mathrm{CHR})_{3ki}\} \\ & + \frac{2\varphi^{ts}}{n+1}\{(\mathrm{CHR})_{3tjis}\varphi_{kh} + (\mathrm{CHR})_{3tkhs}\varphi_{ji} - (\mathrm{CHR})_{3tjhs}\varphi_{ki} \\ & - (\mathrm{CHR})_{3tkis}\varphi_{jh} - 2(\mathrm{CHR})_{3tkjs}\varphi_{ih} - 2(\mathrm{CHR})_{3tihs}\varphi_{kj}\} \end{aligned}$$

$$+\frac{1}{(n+1)(n+2)}\{(\rho(\mathrm{CHR})_{3jl}\varphi_i{}^l-\rho(\mathrm{CHR})_{3il}\varphi_j{}^l)\varphi_{kh}$$
$$+(\rho(\mathrm{CHR})_{3kl}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_k{}^l)\varphi_{ji}-(\rho(\mathrm{CHR})_{3jl}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_j{}^l)\varphi_{ki}$$
$$-(\rho(\mathrm{CHR})_{3kl}\varphi_i{}^l-\rho(\mathrm{CHR})_{3il}\varphi_k{}^l)\varphi_{jh}-2(\rho(\mathrm{CHR})_{3kl}\varphi_j{}^l-\rho(\mathrm{CHR})_{3jl}\varphi_k{}^l)\varphi_{ih}$$
$$-2(\rho(\mathrm{CHR})_{3il}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_i{}^l)\varphi_{kj}\}-\frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}g_{ji}-g_{ki}g_{jh}$$
$$-\eta_k\eta_h g_{ji}-\eta_j\eta_i g_{kh}+\eta_k\eta_i g_{jh}+\eta_j\eta_h g_{ki}+\varphi_{kh}\varphi_{ji}-\varphi_{ki}\varphi_{jh}-2\varphi_{kj}\varphi_{ih}).$$

Next, contraction of (8.4) with g^{kh} gives us

$$\begin{aligned}(8.5)\quad 6(2n+3)\rho(\mathrm{CHR})_{3ji}&=(2n+5)\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}\\&+6(n+2)\varphi^{ts}\{(\mathrm{CHR})_{3tjls}\varphi_i{}^l+(\mathrm{CHR})_{3tils}\varphi_j{}^l\}-6\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l.\end{aligned}$$

Contracting the above equation with $\varphi_b{}^j\varphi_a{}^i$ and changing the indices, we have

$$\begin{aligned}(8.6)\quad &(n+2)\varphi^{ts}\{(\mathrm{CHR})_{3tils}\varphi_j{}^l+(\mathrm{CHR})_{3tjls}\varphi_i{}^l\}\\&=(2n+3)\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l+\rho(\mathrm{CHR})_{3ji}+\rho(\mathrm{CHR})_3(\xi,\xi)\eta_j\eta_i\\&\quad-(n+2)\varphi^{ts}\xi^m\{\varphi_j{}^l\eta_i+\varphi_i{}^l\eta_j\}-\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}.\end{aligned}$$

Substituting (8.6) into (8.5), we get

$$\begin{aligned}&12(n+1)\{\rho(\mathrm{CHR})_{3ji}-\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l\}\\&\qquad=(2n-1)\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}\\&\qquad+6\rho(\mathrm{CHR})_3(\xi,\xi)\eta_j\eta_i-6(n+2)\varphi^{ts}(\mathrm{CHR})_{3tlms}\xi^m(\varphi_j{}^l\eta_i+\varphi_i{}^l\eta_j).\end{aligned}$$

From the above equation, we can easily get

$$\begin{aligned}(8.7)\quad &\rho(\mathrm{CHR})_3(X,Y)-\rho(\mathrm{CHR})_3(\varphi X,\varphi Y)+\rho(\mathrm{CHR})_3(\xi,\xi)\eta(X)\eta(Y)\\&\qquad-\{\rho(\mathrm{CHR})_3(\xi,X)\eta(Y)+\rho(\mathrm{CHR})_3(\xi,Y)\eta(X)\}=0.\end{aligned}$$

Next, we have from (8.7)

$$(8.8)\qquad \tau(\mathrm{CHR})_3=\frac{n+4}{3(n+2)}\rho(\mathrm{CHR})_3(\xi,\xi)+(\mathrm{CHR})_{3tmls}\varphi^{ts}\varphi^{ml}.$$

Thus we have

THEOREM 8.1. *Let M^{2n+1} be a conformally* $(\mathrm{CHR})_3$*-flat almost contact Riemannian manifold. Then the* $(\mathrm{CHR})_3$*-curvature tensor, the* $(\mathrm{CHR})_3$*-Ricci tensor* $\rho(\mathrm{CHR})_3$ *and the* $(\mathrm{CHR})_3$*-scalar curvature* $\tau(\mathrm{CHR})_3$ *satisfy* (8.4), (8.7) *and* (8.8), *respectively.*

By virtue of (7.17), in a conformally $(\mathrm{CHR})_3$-flat Sasakian manifold, the Riemannian curvature tensor satisfies

$$\begin{aligned}(8.9)\quad R(X,Y,Z,W)=\frac{1}{2(n+2)}&\{\rho(Y,Z)g(\varphi X,\varphi W)+\rho(X,W)g(\varphi Y,\varphi Z)\\&-\rho(Y,W)g(\varphi X,\varphi Z)-\rho(X,Z)g(\varphi Y,\varphi W)\}\end{aligned}$$

$$- \frac{1}{2(n+2)}\{\tilde{\rho}(Y,Z)F(X,W) + \tilde{\rho}(X,W)F(Y,Z) - \tilde{\rho}(Y,W)F(X,Z)$$
$$- \tilde{\rho}(Y,W)F(X,Z) - \tilde{\rho}(X,Z)F(X,Y) - 2\tilde{\rho}(X,Y)F(Z,W) - 2\tilde{\rho}(Z,W)F(X,Y)\}$$
$$- \frac{\tau - 2(3n+4)}{4(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}$$
$$+ \frac{\tau + 2n(2n+3)}{4(n+1)(n+2)}\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\}$$
$$+ \frac{\tau + 2n}{4(n+1)(n+2)}\{\eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W)$$
$$- \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z)\}.$$

Contracting (8.9) with g^{kh}, we get $g_{ji} = (2n+1)\eta_j\eta_i$. Thus we have

THEOREM 8.2. *There does not exist any conformally* (CHR)$_3$*-flat Sasakian manifold.*

From the above theorem, we can easily know

REMARK 8.1. There does not exist any C-Bochner flat Sasakian manifold.

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References

1. K. Matsumoto, *A new curvatuelike tensor field in an almost contact Riemannian manifold*, to appear
2. K. Matsumoto, I. Mihai, *Ricci tensor of C-totally real submanifolds in Sasakian space forms*, Nihonkai Math. J. **13** (2002), 191–198.
3. M. Matsumoto, G. Chuman, *On the C-Bochner curvature tensor*, TRU Math. **5** (1969), 21–30.
4. M. Prvanović, *Conformally invariant tensors of an almost Hermitian manifold associated with the holomorphic curvature tensor*, J. Geom. **103** (2012), 89–101.
5. S. Sasaki, *Almost Contact Manifolds*, Lecture Notes, Tôkoku University, 1965.
6. K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pargamon Press, 1965.

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