PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 103(117) (2018), 181–190 DOI: https://doi.org/10.2298/PIM1817181P

PSEUDO-SYMMETRIES OF GENERALIZED WINTGEN IDEAL LAGRANGIAN SUBMANIFOLDS

Miroslava Petrović-Torgašev and Anica Pantić

Abstract. Mihai obtained the Wintgen inequality, also known as the generalized Wintgen inequality, for Lagrangian submanifolds in complex space forms and also characterized the corresponding equality case. Submanifolds M which satisfy the equality in these optimal general inequalities are called generalized Wintgen ideal submanifolds in the ambient space $\tilde{M}$. For generalized Wintgen ideal Lagrangian submanifolds M^n in complex space forms $\tilde{M}^n(4c)$, we will show some properties concerning different kinds of their pseudosymmetry in the sense of Deszcz.

1. Introduction

For surfaces M^2 in E^3, the *Euler inequality* $K \leqslant H^2$, where K is the intrinsic *Gauss curvature* of M^2 and H^2 is the extrinsic *squared mean curvature* of M^2 in E^3, at once follows from the fact that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ where k_1 and k_2 denote the *principal curvatures* of M^2 in E^3. And, obviously, $K = H^2$ everywhere on M^2 if and only if the surface M^2 is *totally umbilical* in E^3, i.e. $k_1 = k_2$ at all points of M^2, or still, by the *theorem of Meusnier*, if and only if M^2 is a part of a *plane* E^2 or of a *round sphere* S^2 in E^3. In 1979, Wintgen proved that the *Gauss curvature* K and the *squared mean curvature* H^2 and the *normal curvature* $K^\perp$ of any surface M^2 in E^4 always satisfy the inequality $K \leqslant H^2 - K^\perp$, and that actually *the equality holds if and only if the curvature ellipse of M^2 in E^4 is a circle* [**36**]. This *Wintgen inequality* between the most important intrinsic and extrinsic scalar valued curvatures of surfaces M^2 in E^4 was shown to hold more generally for all surfaces M^2 in arbitrary dimensional space forms $\tilde{M}^{2+m}(c)$, inclusive the above characterization of the equality case, by Rouxel [**31**] and by Guadalupe and Rodriguez [**22**]. After these extensions, in 1981 and in 1983 respectively, in 1999 De Smet and Dillen, Vrancken and Verstraelen formulated the conjecture on the Wintgen inequality also named DDVV conjecture for all submanifolds in all real space forms [**9**],

2010 *Mathematics Subject Classification:* 53C40; 53C25.
Key words and phrases: generalized Wintgen ideal submanifold, pseudosymmetric space, Lagrangian submanifold, complex space form.
Research supported by the project 174012 of the Serbian Ministry of Education, Science and Technological Development.

$$\rho \leqslant H^2 - \rho^\perp \tag{1.1}$$

where ρ is the *normalized scalar curvature* of M^n defined by

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} R(e_i, e_j, e_j, e_i),$$

and $\rho^\perp$ is the *normalized normal scalar curvature function* of M^n at a point p, defined by

$$\rho^\perp(p) = \frac{2}{n(n-1)} \sqrt{\sum_{1\leqslant i<j\leqslant n} \sum_{1\leqslant \alpha<\beta\leqslant n} (R^\perp(e_i, e_j, \xi_\alpha, \xi_\beta))^2},$$

$\{e_1, \dots, e_n\}$ is any orthonormal basis of the tangent space $T_p(M^n)$ ($p \in M^n$), and R is the Riemann–Christoffel curvature tensor of M^n, where $R^\perp$ is the curvature tensor of the normal space, and $\{\xi_1, \xi_2\}$ is an orthonormal basis of the normal space. They proved the above Wintgen inequality for *for all submanifolds M^n of codimension* 2 *in all real space forms* $\tilde{M}^{n+2}(c)$ of M^n, and also characterized the equality case in terms of the shape operators of M^n in $\tilde{M}^{n+2}(c)$ [**9**]. Other extensions of the Wintgen inequality were studied for submanifolds in Kaehler, nearly Kaehler and Sasakian spaces, see e.g. [**19, 20, 28, 29**], etc. Later, Choi and Lu [**4**], Lu [**27**] and Ge and Tang [**21**] proved that indeed (1.1) holds in general for all submanifolds M^n in $\tilde{M}^{n+2}(c)$ and gave a characterization of the equality situation in terms of an explicit description of the second fundamental form.

The submanifolds M^n in $\tilde{M}^{n+m}(c)$ satisfying equality in Wintgen inequality are called *Wintgen ideal submanifolds*; for many examples and for geometrical properties of such submanifolds, see e.g. [**1, 4–9, 16, 21, 22, 24, 27, 29–31, 36, 37**]. Recently, Mihai [**29**] established a *generalized Wintgen inequality* for Lagrangian submanifolds in complex space forms and characterized the equality case in terms of the shape operators.

A Riemannian submanifold M^n in an ambient manifold $\tilde{M}$ is called *totally geodesic* when its second fundamental form h vanishes identically, *totally ubmilical* when $h = g\vec{H}$, $\vec{H}$ is its mean curvature vector, *minimal* when $\vec{H} = 0$ and *pseudoumbilical* when the mean curvature vector field $\vec{H}$ determines an umbilical normal direction on M in $\tilde{M}$.

We recall that an n-dimensional Riemannian manifold M^n, ($n \geqslant 3$) is said to be a *pseudosymmetric space in the sense of Deszcz* or a *Deszcz symmetric space* [**10, 23, 24, 33–35**] if the (0,6) tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent, i.e., $R \cdot R = LQ(g, R)$ for some function $L : M \to R$ (on the open subset of M on which $Q(g, R) \neq 0$). Hereby, by the action of the curvature operator R as a derivation on the $(0, 4)$ curvature tensor R results the $(0, 6)$ curvature tensor $R \cdot R$,

$$\begin{aligned}(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) &= (R(X, Y) \cdot R)(X_1, X_2, X_3, X_4)\\ &= -R(R(X, Y)X_1, X_2, X_3, X_4) - R(X_1, R(X, Y)X_2, X_3, X_4)\\ &\quad - R(X_1, X_2, R(X, Y)X_3, X_4) - R(X_1, X_2, X_3, R(X, Y)X_4),\end{aligned}$$

and the *Tachibana tensor* $Q(g,R)$ (see, e.g. [**13**, **23**, **26**]) is the $(0,6)$ tensor which results from the action as a derivation on the (0,4) curvature tensor R, by the metrical endomorphism, i.e., $Q(g,R) = -\wedge_g \cdot R$, or

$$\begin{aligned} Q(g,R)(X_1,X_2,X_3,X_4;X,Y) &= -(\wedge_g \cdot R)(X_1,X_2,X_3,X_4;X,Y) \\ &= -((X \wedge_g Y)\cdot R)(X_1,X_2,X_3,X_4) \\ &= R((X\wedge_g Y)X_1,X_2,X_3,X_4) + R(X_1,(X\wedge_g Y)X_2,X_3,X_4) \\ &\quad + R(X_1,X_2,(X\wedge_g Y)X_3,X_4) + R(X_1,X_2,X_3,(X\wedge_g Y)X_4), \end{aligned}$$

where X, Y, X_1, X_2, X_3, X_4 are arbitrary tangent vector fields on M. The Riemannian manifolds M^n for which $R \cdot R = 0$ are called *Szabó symmetric spaces* or *semisymmetric spaces* (see, e.g. [**24**, **32**]). The geometrical meaning of the tensor $R\cdot R$ and the Tachibana tensor $Q(g,R)$ is given in [**23**]. And a Riemannian manifold M^n of dimension $n \geqslant 3$ is Deszcz symmetric or pseudosymmetric when its Deszcz sectional curvature or double sectional curvature function $L(p,\pi,\bar{\pi})$ is *isotropic*, i. e., at all of its points p has the same value $L(p)$ for all curvature dependent tangent planes π and $\bar{\pi}$ at a point p [**23**]

Next, we consider the $(0,4)$ tensor $R\cdot S$, obtained by the action of the curvature operator $R(X,Y)$ on the $(0,2)$ symmetric *Ricci tensor* S, given by

$$\begin{aligned} (R\cdot S)(X_1,X_2;X,Y) &= (R(X,Y)\cdot S)(X_1,X_2) \\ &= -S(R(X,Y)X_1,X_2) - S(X_1,R(X,Y)X_2) \end{aligned}$$

and the *Ricci Tachibana tensor* $Q(g,S)$ [**26**] which is given by

$$\begin{aligned} Q(g,S)(X_1,X_2;X,Y) &= ((X\wedge_g Y)\cdot S)(X_1,X_2) \\ &= -S((X\wedge_g Y)X_1,X_2) - S(X_1,(X\wedge_g Y)X_2). \end{aligned}$$

A Riemannian manifold is said to be *Ricci pseudosymmetric space*, or *Ricci Deszcz symmetric space* if $R\cdot S = L_S Q(g,S)$, for some real valued function L_S on M [**10**,**17**,**26**,**33**]. It is known that every Deszcz symmetric manifold automatically is Ricci pseudosymmetric. The converse is not true (see, e.g. [**17**]).

The property to be a Deszcz symmetric space is not invariant under conformal transformations of Riemannian spaces [**14**, **15**]. The formally related curvature condition for a Riemannian manifold M^n of dimension $n \geqslant 4$ to have *a pseudosymmetric Weyl tensor C* however is conformally invariant. We recall that the latter property means that $C \cdot C = L_C Q(g,C)$ for some function $L_C : M^n \to R$ on the open part of M where $Q(g,C) \neq 0$ [**10**–**12**, **18**]. We refer to [**11**, **12**] for recent results on manifolds satisfying that condition.

There are close connections between Wintgen ideal submanifolds and intrinsic pseudosymmetry conditions. For instance, for $n > 3$, every Wintgen ideal submanifold M^n in $\tilde{M}^{n+m}(c)$ has pseudosymmetric conformal Weyl tensor, and the minimal Wintgen ideal submanifolds are characterized by the fact that $L_C = -\frac{n-3}{(n-1)(n-2)}K_{inf}$. Moreover, the Deszcz symmetric Wintgen ideal submanifolds are either totally umbilical (with $L = 0$), (in particular, then M being a real space form), or minimal (in which case M is pseudosymmetric of constant type, namely $L = c$) [**8**].

It should be observed that for submanifolds M^n in $\tilde{M}^{n+m}(c)$ with flat normal connection, and thus in particular for hypersurfaces ($m = 1$), the Wintgen inequality actually reduces to a Chen inequality $\rho \leqslant \|H\|^2 + c$ and the corresponding ideal submanifolds M then are totally umbilical in $\tilde{M}$ and hence spaces of constant curvature (a special Deszcz symmetric spaces, with $L = 0$). As general basic reference for optimal inequalities relating various extrinsic and intrinsic characteristics of submanifolds, we refer to Chen's book [**2**].

2. Generalized Wintgen inequality for Lagrangian submanifolds

Let $\tilde{M}^m(4c)$ be a complex space form of constant holomorphic sectional curvature $4c$. Denote by J its standard complex structure.
The *Riemann–Christoffel curvature tensor* $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = c\{&g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(JX,Z)g(JY,W)\\ &-g(JX,W)g(JY,Z) + 2g(JX,Y)g(JZ,W)\},\end{aligned}$$

for any tangent vector fields X, Y, Z, W on $\tilde{M}^m(4c)$.

A submanifold M^n in complex space form is called a *totally real submanifold* if $J(T_pM^n) \subset T_p^{\perp}M^n$, at any point $p \in M^n$. The totally real submanifold M of $\tilde{M}$ is called a *Lagrangian submanifold* if $n = m$.

Let M^n be an n-dimensional totally real submanifold of a complex space form $\tilde{M}^m(4c)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on M^n and $\{\xi_1, \ldots, \xi_{2m-n}\}$ an orthonormal frame in the normal bundle $T^{\perp}M^n$. Denote by h and A the second fundamental form and the shape operator of M^n in $\tilde{M}^m(4c)$. Then the *Gauss* and *Ricci equations* are given by

$$\begin{aligned}R(X,Y,Z,W) = c\{&g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}\\ &+ g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z))\end{aligned}$$

for any tangent vector fields X, Y, Z, W on M^n,

$$R^{\perp}(X,Y,\xi,\eta) = c\{g(JX,\xi)g(JY,\eta) - g(JX,\eta)g(JY,\xi)\} - g([A_\xi, A_\eta]X,Y),$$

for any tangent vector fields $X, Y,$ on M^n and $\xi, \eta \in T^{\perp}M^n$.

In the recent paper [**29**] Mihai established a *generalized Wintgen inequality* for Lagrangian submanifolds in complex space forms.

THEOREM 2.1. [**29**] *Let M^n be an n-dimensional Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then*

$$(\rho^{\perp})^2 \leqslant (\|H\|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c)c + \frac{2c^2}{n(n-1)}, \tag{2.1}$$

and equality holds identically if and only if with respect to suitable orthonormal frames $\{e_1, e_2, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_n\}$, the shape operators of M^n in $\tilde{M}^n(4c)$ are given by

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 - \mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix},$$

$$A_{\xi_4} = \dots = A_{\xi_n} = 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n.

Lagrangian submanifolds M^n in complex space forms $\tilde{M}^n(4c)$ satisfying equality in generalized Wintgen inequality (2.1) are called *generalized Wintgen ideal Lagrangian submanifolds.* A frame $\{e_1, e_2, \dots, e_n; \xi_1, \dots, \xi_n\}$ with the corresponding shape operators from Theorem 2.1 is called a *Choi–Lu frame* on such M^n in $\tilde{M}^n(4c)$ and its distinguished tangent plane $e_1 \wedge e_2$ is called the *Choi–Lu plane* of the generalized Wintgen ideal Lagrangian submanifolds concerned [**8**, **16**].

3. Pseudosymmetry properties of generalized Wintgen ideal Lagrangian submanifolds

First consider the *pseudosymmetry condition in the sense of Deszcz* of the generalized Wintgen ideal Lagrangian submanifolds M^n in a complex space form $\tilde{M}^n(4c)$, $(n \geqslant 4)$. Their Riemann–Christoffel curvature tensors are obtained by inserting the shape operators from Theorem 2.1 in the equation of Gauss. Up to the algebraic symmetries of the (0,4) curvature tensor R of such generalized Wintgen ideal submanifolds, all components of R are zero except possibly the following ones

$$R_{1212} = 2\mu^2 - c_1, \tag{3.1}$$

$$R_{1k1k} = -\lambda_2\mu - c_1, \qquad (k \geqslant 3) \tag{3.2}$$

$$R_{1k2k} = -\lambda_1\mu, \qquad (k \geqslant 3) \tag{3.3}$$

$$R_{2k2k} = \lambda_2\mu - c_1, \qquad (k \geqslant 3) \tag{3.4}$$

$$R_{klkl} = -c_1, \qquad (k \neq l, k, l \geqslant 3), \tag{3.5}$$

where $c_1 = c + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Then expressing the condition $R \cdot R = LQ(g, R)$ for the Deszcz symmetry to be satisfied by the $(0, 6)$ tensors $R \cdot R$ and $Q(g, R)$

for some function $L : M \to R$, by evaluating these tensors on the tangent vectors $\{e_1, e_2, \ldots, e_n\}$, one finds that this pseudosymmetry is characterized by the following system of algebraic equations

$$\begin{aligned} \lambda_1\mu(2\mu^2 - c_1 + L) &= 0 \\ \lambda_2\mu(2\mu^2 - c_1 + L) &= 0 \\ \lambda_1^2\mu^2 + (2\mu^2 + \lambda_2\mu)(\lambda_2\mu - c_1 + L) &= 0 \\ \lambda_1^2\mu^2 + (2\mu^2 - \lambda_2\mu)(-\lambda_2\mu - c_1 + L) &= 0 \\ \lambda_1^2\mu^2 - \lambda_2\mu(-\lambda_2\mu - c_1 + L) &= 0 \\ \lambda_1^2\mu^2 + \lambda_2\mu(\lambda_2\mu - c_1 + L) &= 0 \\ \lambda_1\mu(-c_1 + L) &= 0. \end{aligned}$$

This system of equations is obtained by the evaluations of tensors $R\cdot R$ and $Q(g,R)$ on the seven combinations

$$\begin{gathered} (e_1, e_3, e_1, e_3; e_1, e_2), (e_1, e_3, e_2, e_3; e_1, e_2), (e_1, e_2, e_1, e_3; e_2, e_3), \\ (e_1, e_2, e_2, e_3; e_1, e_3), (e_1, e_4, e_3, e_4; e_1, e_3), \\ (e_2, e_4, e_3, e_4; e_2, e_3), (e_2, e_4, e_3, e_4; e_1, e_3). \end{gathered}$$

All other choices of combinations of basic vectors $\{e_1, \ldots, e_n\}$ lead either to any equation of the above system or to a triviality. And this system is satisfied if and only if (I) $\mu = 0$, in which case $L = 0$, or, (II) $\mu \neq 0$ and $\lambda_1 = \lambda_2 = 0$, in which case $L = c_1 = \lambda_3^2 + c$.

Therefore, by virtue of the above calculations and the fact that every totally umbilical submanifold of a complex space form is totally geodesic [**3**], we obtained the following

THEOREM 3.1. *A generalized Wintgen ideal Lagrangian submanifold M^n in a complex space form $\tilde{M}^n(4c)$, $(n \geqslant 4)$, is a Deszcz symmetric Riemannian manifold if and only if it is totally geodesic (with $L = 0$) or a minimal or pseudoumbilical submanifold of this complex space form $\tilde{M}^n(4c)$.*

Using the nonzero components of Riemann–Christoffel curvature tensor R, the nontrivial components of the $(0,2)$ Ricci tensor S of generalized Wintgen ideal Lagrangian submanifolds M^n in the complex space form $\tilde{M}^n(4c)$ in a $\{e_1, e_2, \ldots, e_n\}$ frame are found to be

$$S_{11} = -2\mu^2 + (n-2)\lambda_2\mu + (n-1)c_1, \tag{3.6}$$

$$S_{22} = -2\mu^2 - (n-2)\lambda_2\mu + (n-1)c_1, \tag{3.7}$$

$$S_{12} = (n-2)\lambda_1\mu, \tag{3.8}$$

$$S_{kk} = (n-1)c_1, \quad (k \geqslant 3). \tag{3.9}$$

Then expressing the pseudosymmetry condition $R \cdot S = L_S Q(g,S)$ satisfied by the (0,4) tensors $R \cdot S$ and $Q(g,S)$ for some function $L_S : M \to R$, by evaluating these tensors on the tangent vectors $\{e_1, e_2, \ldots, e_n\}$, one finds that this *Ricci* *pseudosymmetry*, i. e. *Ricci Deszcz symmetry* is characterized by

$$
\begin{aligned}
(n-2)\lambda_1\mu(2\mu^2 - c_1 + L_S) &= 0\\
(n-2)\lambda_2\mu(2\mu^2 - c_1 + L_S) &= 0\\
\mu[(2\mu - (n-2)\lambda_2)(\lambda_2\mu + c_1 - L_S) - (n-2)\lambda_1^2\mu] &= 0\\
(n-3)\lambda_1\mu &= 0\\
\mu[(2\mu + (n-2)\lambda_2)(\lambda_2\mu - c_1 + L_S) + (n-2)\lambda_1^2\mu] &= 0.
\end{aligned}
$$

This system of equations is obtained by the evaluations of tensors $R\cdot S$ and $Q(g,S)$ on the five combinations $(e_1,e_1;e_1,e_2)$, $(e_1,e_2;e_1,e_2)$, $(e_1,e_3;e_1,e_3)$, $(e_1,e_3;e_2,e_3)$ and $(e_2,e_3;e_2,e_3)$, all other choices of combinations of basic vectors $\{e_1,\dots,e_n\}$ lead either to any equation of the above system or to a triviality. And this system of equations is satisfied if and only if (I) $\mu = 0$, $\lambda_1,\lambda_2,\lambda_3 \in R$ in which case $L_S = 0$, or, (II) $\mu \neq 0$, $\lambda_1 = \lambda_2 = 0, \lambda_3 \in R$ and $L_S = \lambda_3^2 + c$. Based on this result on the Ricci Deszcz symmetry for Lagrangian submanifolds M^n in the complex space form $\tilde{M}^n(4c), (n \geqslant 4)$ and by virtue of Theorem 2.1, we thus obtained the following

THEOREM 3.2. *Any generalized Wintgen ideal Lagrangian submanifold M^n in a complex space form $\tilde{M}^n(4c)$, $(n \geqslant 4)$, is Deszcz symmetric if and only if it is Ricci Deszcz symmetric.*

In the sequel, we will characterize generalized Wintgen ideal Lagrangian submanifolds in a complex space form *with pseudosymmetric Weyl conformal curvature tensor C*.

From (3.6)–(3.9), the scalar curvature of generalized Wintgen ideal Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ is found to be given by

$$\tau = n(n-1)c_1 - 4\mu^2. \tag{3.10}$$

Then, combining (3.1)–(3.5) and (3.6)–(3.9) and (3.10), up to algebraic symmetries, the components of the Weyl conformal curvature tensor C of such M^n are trivially zero or are determined by

$$
\begin{aligned}
C_{1212} &= \frac{2(n-3)}{n-1}\mu^2,\\
C_{1i1i} &= -\frac{2(n-3)}{(n-1)(n-2)}\mu^2, \quad (i \geqslant 3)\\
C_{ijij} &= \frac{4}{(n-1)(n-2)}\mu^2, \quad (i \neq j, i, j \geqslant 3).
\end{aligned}
$$

The nontrivial components of the $(0,6)$ tensors $C\cdot C$ and the *Weyl–Tachibana tensor* $Q(g,C)$, [**13**, **25**] by the algebraic symmetries of them, are all determined by

$$
\begin{aligned}
(C\cdot C)(e_1,e_2,e_2,e_3;e_1,e_3) &= -\frac{2\mu^2(n-3)}{n-2}C_{1313},\\
(C\cdot C)(e_1,e_4,e_3,e_4;e_1,e_3) &= -\frac{2\mu^2}{n-2}C_{1313},
\end{aligned}
$$

$$Q(g,C)(e_1,e_2,e_2,e_3;e_1,e_3) = \frac{2\mu^2(n-3)}{n-2},$$
$$Q(g,C)(e_1,e_4,e_3,e_4;e_1,e_3) = \frac{2\mu^2}{n-2}.$$

The *pseudosymmetry condition* $C \cdot C = L_C Q(g,C)$ is then characterized by the following:

$$\frac{2\mu^2(n-3)}{n-2}(L_C + C_{1313}) = 0, \qquad \frac{2\mu^2}{n-2}(L_C + C_{1313}) = 0.$$

Thus the condition $C\cdot C = L_C Q(g,C)$ is satisfied for $L_C = -C_{1313} = \frac{2(n-3)}{(n-1)(n-2)}\mu^2$.

From the specific forms of the shape operators from Theorem 2.1, we observe that, for generalized Wintgen ideal Lagrangian submanifolds M^n in a complex space form $\tilde{M}^n(4c)$, $\inf K = K_{12} = c_1 - 2\mu^2$, from which we obtain that

$$\mu^2 = \frac{1}{2(n+1)(n-2)}[\tau - n(n-1)\inf K].$$

Thus, we can state the following

Theorem 3.3. *Let M^n $(n \geqslant 4)$ be a generalized Wintgen ideal Lagrangian submanifold in complex space form $\tilde{M}^n(4c)$.*

(i) *Then M^n is conformally flat if and only if M^n is a totally geodesic submanifold in $\tilde{M}^n(4c)$.*

(ii) *If M^n is a nonconformally flat submanifold, then M^n has a pseudosymmetric Weyl conformal curvature tensor C and corresponding function of pseudosymmetry is given by*

$$L_C = \frac{2(n-3)}{(n-1)(n-2)}\mu^2 = \frac{n-3}{(n^2-1)(n-2)^2}[\tau - n(n-1)\inf K].$$

References

1. B.-Y. Chen, *Classification of Wintgen ideal surfaces in Euclidean 4-space with equal Gauss and normal curvatures*, Ann. Glob. Anal. Geom. **38(2)** (2010), 145–160.
2. ______, *Pseudo-Riemannian Geometry, δ-invariants and Applications*, World Scientific, Singapore, 2011.
3. B.-Y. Chen, K. Ogiue, *Two theorems on Kaehler manifolds*, Michigan Math. J. **21** (1974), 225–229.
4. T. Choi, Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry (I)*, Math. Z. **260** (2008), 409–429.
5. M. Dajczer, R. Tojeiro, *Submanifolds of codimension two attaining equality in an extrinsic inequality*, Math. Proc. Cambridge Philos. Soc. **146** (2009), 461–474.
6. ______, *All superconformal surfaces in R^4 in terms of minimal surfaces*, Math. Z. **261** (2009), 869–890.
7. S. Decu, M. Petrović-Torgašev, A. Šebeković, L. Verstraelen, *On the intrinsic Deszcz symmetries and the extrinsic Chen character of Wintgen ideal submanifolds*, Tamkang J. Math. **41** (2010), 109–116.
8. ______, *On the Roter type of Wintgen ideal submanifolds*, Rev. Roumaine Math. Pures Appl. **57** (2012), 75–90.
9. P. J. De Smet, F. Dillen, L. Verstraelen, L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), 115–128.

10. R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg., Série A, **44** (1992), 1–34.
11. R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotloś, M. Yawata, *On semi-Riemannian manifolds satisfying some conformally invatiant curvature condition*, Colloquium Math. **131** (2013), 149–170.
12. R. Deszcz, M. Głogowska, J. Jełowicki, G. Zafindratafa, *Curvature properties of some class of warped product manifolds*, Int. J. Geom. Meth. Modern Phys. **13** (2016), 1550135 (36 pages).
13. R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz, M. Scherfner, *On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type*, Kragujevac J. Math. **35** (2011), 223–247.
14. R. Deszcz, M. Hotloś, *On conformally related four dimensional pseudosymmetric metrics*, Rend. Sem. Fac. Univ. Cagliari **59** (1989), 165–175.
15. ———, *On conformally related pseudosymmetric metrics*, Tensor, N.S., **56** (1995), 55–67.
16. R. Deszcz, M. Petrović-Torgašev, Z. Şentürk, L. Verstraelen, *Characterization of the pseudo-symmetries of ideal Wintgen submanifolds of dimension* 3, Publ. Inst. Math., Nouv. Sér. **88(102)** (2010), 53–65.
17. R. Deszcz, P. Verheyen, L. Verstraelen, *On some generalized Einstein metric conditions*, Publ. Inst. Math., Nouv. Sér. **60(74)** (1996), 108–120.
18. R. Deszcz, Ş. Yaprak, *Curvature properties of certain pseudo-symmetric manifolds*, Publ. Math. Debrecen **45** (1994), 333–345.
19. F. Dillen, J. Fastenakels, J. Van der Veken, *A pinching theorem for the normal scalar curvature of invariant submanifolds*, J. Geom. Phys. **57** (2007), 833–840.
20. ———, *Remarks on an inequality involving the normal scalar curvature*, in: F. Dillen, I. Van de Woestijne (eds.), *Pure and Applied Differential Geometry PADGE 2007*, Ber. Math., Shaker Verlag, Aachen, 2007, 83–92.
21. J. Ge, Z. Tang, *A proof of the DDVV conjecture and its equality case*, Pacific J. Math. **237** (2008), 87–95.
22. I. V. Guadalupe, L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. **106** (1983), 95–103.
23. S. Haesen, L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta Math. **122** (2007), 59–72.
24. ———, *Natural Intrinsic Geometrical Symmetries*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications), **5** (2009), Special Issue "Elie Cartan and Differential Geometry", 15 pages.
25. B. Jahnara, S. Haesen, M. Petrović-Torgašev, L. Verstraelen, *On the Weyl curvature of Deszcz*, Publ. Math. Debrecen **74** (2009), 417–431.
26. B. Jahnara, S. Haesen, Z. Şentürk, L. Verstraelen, *On the parallel transport of the Ricci curvatures*, J. Geom. Phys. **57** (2007), 1771–1777.
27. Z. Lu, *Normal Scalar Curvature Conjecture and its applications*, J. Funct. Anal. **261** (2011), 1284–1308.
28. A. Mihai, *Scalar normal curvature of Lagrangian 3-dimensional submanifolds in complex space forms*, in: F. Dillen, I. Van de Woestijne (eds.), *Pure and Applied Differential Geometry PADGE 2007*, Ber. Math., Shaker Verlag, Aachen, 2007, 171–177.
29. I. Mihai, *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms*, Nonlinear Anal. **95** (2014), 714–720.
30. M. Petrović-Torgašev, L. Verstraelen, *On Deszcz symmetries of Wintgen ideal submanifolds*, Arch. Math. (Brno) **44** (2008), 57–67.
31. B. Rouxel, *Sur une famille de A-surfaces d'un espace euclidien E^4*, Proc.10 Österreichischer Mathematiker Kongress, Insbruck, 1981, 185.
32. Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R \cdot R = 0$. The local version*, J. Diff. Geom. **17** (1982), 531–582.
33. L. Verstraelen, *Comments on the pseudo-symmetry in the sense of Ryszard Deszcz*, in: F. Dillen et al. (eds.), *Geometry and Topology of Submanifolds*, Vol. VI, World Scientific, Singapore, 1994, 119–209.

34. ______, *A concise mini history of Geometry*, Kragujevac J. Math. **38** (2014), 5–21.
35. ______, *Natural extrinsic geometrical symmetries-an introduction*, in: B. Suceavă et al. (eds.), *Recent Advances in the Geometry of Submanifolds Dedicated to the Memory of Franki Dillen (1963–2013)*, Contemp. Math. **674** (2016), 5–16.
36. P. Wintgen, *Sur l'inégalité de Chen–Willmore*, C. R. Acad. Sci. Paris **288** (1979), 993–995.
37. Zhenxiao Xie, *Three special classes of Wintgen ideal submanifolds*, J. Geom. Phys. **114** (2017), 523–533.

University of Kragujevac
Faculty of Science
Institute of Mathematics and Informatics
Kragujevac
Serbia
mirapt@kg.ac.rs
anica.pantic@kg.ac.rs