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# A SUBDIRECTLY IRREDUCIBLE SYMMETRIC HEYTING ALGEBRA WHICH IS NOT SIMPLE

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**Abstract:** The aim of this paper is to provide a negative answer to the question of whether every subdirectly irreducible symmetric Heyting algebra (that is a Heyting algebra with a De Morgan negation) is simple or not.

## Introduction

An algebra $\langle A, \vee, \wedge, \to, ', 0, 1\rangle$ is a symmetric Heyting algebra (briefly: symmetric algebra) if $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a De Morgan algebra. If a Heyting algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is also a dual Heyting algebra (that is, if the dual of $A$ is also a Heyting algebra) then $\langle A, \vee, \wedge, \to, \leftarrow, 0, 1\rangle$ is a double Heyting algebra, $\leftarrow$ denoting the dual implication.

For each symmetric algebra $A$, $B(A)$ denotes the center of $A$ (the subalgebra of all complemented elements of $A$) and $K(A)$ the subalgebra of $B(A)$ formed by the elements $k$ such that the De Morgan negation, $k'$, coincides with the complement $\overline{k}$ of $k$. We denote $x^* = x \to 0$ (the pseudocomplement of $x$). We define inductively the operator $\nu$ by:

$$\nu^0(x) = x\,, \qquad \nu^{n+1}(x) = (\nu^n(x))'^* \ .$$

Also, it can be defined ([8])

$$t_0(x) = x\,, \qquad t_{n+1}(x) = t_n(x) \wedge \nu^{n+1}(x) \ .$$

We can express $B(A)$ and $K(A)$ in terms of $\nu$ by:

$$B(A) = \Big\{x \in A\colon\ \nu^2(x) = x\Big\}\ ,$$

$$K(A) = \Big\{x \in A\colon\ \nu(x) = x\Big\}\ .$$

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It is well known that congruences of double Heyting algebras are in 1–1-correspondence with normal filters ([5]). Analogously ([6]), congruences of symmetric algebras are in 1–1-correspondence with homomorphism kernels and this correspondence preserves order. This enables us to study properties (simplicity, reducibility, etc.) by means of kernels instead of congruences. We will use the following characterization of a kernel ([6], Ch. III, §4, 4.11): a filter $F$ is a kernel if and only if $\nu(F) \subset F$. Given a subset $X$ of $A$, the kernel generated by $X$, $N(X)$, is the set $N(X) = \{y \in A\colon\ y \geq y_1 \wedge ... \wedge y_k,\ y_i = t_{p_i}(x_i)$, with $x_i \in X\}$ ([8]). It is easy to see that a principal kernel $N(x)$ is the principal filter generated by $x$ if and only if $x \in K(A)$.

In [6], Ch. III, §7, Theorem 7.1, Monteiro shows that every finite symmetric Heyting algebra is semisimple. (This property does not hold for every variety. For instance, there are finite Heyting algebras that are not semisimple). In some varieties every algebra (finite and infinite) is semisimple. This is the case for monadic boolean algebras, which are the algebraic counterpart of first-order monadic functional calculus. Therefore, Monteiro poses the problem of whether infinite symmetric Heyting algebras are also semisimple.

The purpose of this paper is to exhibit an example of symmetric Heyting algebra that is not semisimple. Indeed, in Theorem 2.6 $P^+$ is a subdirectly irreducible symmetric algebra that is not simple.

In [5] Köhler gives a construction of a double Heyting algebra $L_1 *_\varphi L_2$ where $L_1$, $L_2$ are double Heyting algebras and $\varphi$ is a bounded $(*, 1)$-homomorphism. By taking $L_1 = L_2 = 2^N$ and some $\varphi$ he finds an example of subdirectly irreducible double Heyting algebra that is not simple. In 2.7 we present a much simpler example obtained by completely different methods of construction and proof. However, it is isomorphic to Köhler's example. In fact, it suffices to take $\mathbf{Z}$ instead of $\mathbb{N}$ in $L_1$ and $L_2$.

**1** – Let $P$ be a partially ordered set (briefly: a poset) and $P^+$ the class of all increasing or hereditary subsets of $P$ (subsets $C$ such that $x \in C$, $y \geq x$ implies $y \in C$). It is easy to see that $P^+$ is a complete sublattice of $2^P$. $P^+$ is indeed a Heyting algebra whose elements are interpretations of sentences of intuitionistic logic in the "Kripke semantics" ([4], Ch. 8, 8.4). The implication can be expressed by $A \to B = (A \cap B^c]^c$ $(A, B \in P^+)$; here $^c$ denotes set complementation and $(X] = \{y \in P\colon\ y \leq x$, for some $x \in X\}$. The dual implication can be defined by: $A \leftarrow B = [A^c \cup B)$. Thus, $P^+$ has a structure of double Heyting algebra.

Let $R(P^+) = \{C \in P^+\colon\ C^{**} = C\}$ (the set of regular elements). Then $R(P^+)$ is a Boolean algebra that is complete (see [1], VIII, 4, Theorem 4).

Consider the following condition in $P$.

**(M)** For every $q$ there is a maximal element $p$ such that $q \leq p$.

In the rest of the section we assume that (M) holds.

**1.1 Lemma.** *If $A \in R(P^+)$, $A \neq P$, then there is a maximal element $p$ such that $p \notin A$.* ∎

For a maximal $p$ we will write in the following $p^*$ instead of $\{p\}^*$ and $(p]$ instead of $(\{p\}]$.

**1.2 Proposition.** *If $p$ is a maximal element of $P$ then $p^*$ is a coatom of $R(P^+)$. Conversely, for every coatom $C$ of $R(P^+)$ there is a maximal $p$ such that $p^* = C$.*

**Proof:** If $A$ is a regular element of $P^+$ that contains $p^*$, then $A$ contains every maximal $q$ for $q \neq p$. Moreover, if some $r$ belongs to $A - p^*$, then by (M) we have that $r \leq p$. So, $A$ contains every maximal, that is, by 1.1, $A = P$.

On the other hand, let $C$ be a coatom of $R(P^+)$. Therefore, by 1.1, there exists a maximal $p$ such that $p \notin C$. Since $C$ is increasing, $(p] \subset C^c$, which implies $p^* \supset C$. Thus $p^* = C$. ∎

**1.3 Proposition.** *$R(P^+)$ is atomic.*

**Proof:** It suffices to prove ([1], III, 1) that $O$ is the infimum of the set of all coatoms. Indeed, the intersection of all $p^*$ with $p$ maximal do not contain any maximal. ∎

**1.4 Corollary.** *For every $A \in R(P^+)$: $A = \bigcap p^*$, the intersection over all maximal element $p$ such that $p \notin A$.* ∎

If there exists an anti-isomorphism $g$ of $P$ onto $P$ such that $g^2 = \mathrm{id}_P$, then $P^+$ admits a De Morgan negation given by: $A' = (g(A))^c$. In fact, $'$ (the Birula–Rasiowa operator, [7]) is a De Morgan operator in $2^P$ ([6], Ch. II, 1) that is closed in $P^+$. So, $\langle P^+, \cap, \cup, \rightarrow, ', \emptyset, P\rangle$ is a symmetric Heyting algebra. We assume the existence of $g$ in the rest of the section.

**1.5 Definition.** For $p$ and $q$ maximal elements of $P$ and $n \geq 1$ we define: $p =\!=_n\!=\!= q$ iff there exist $r_0, r_1, ..., r_n$, such that $r_1, r_3, ..., r_{2k+1}, ...$ are minimal elements and $r_0, r_2, r_4, ..., r_{2k}, ...$ maximal elements of $P$, $r_0 = q$, $r_0 \geq r_1$, $r_1 \leq r_2$, $r_2 \geq r_3$, ..., and $r_{n-1}$ comparable with $g^n(p)$.

We note some properties of $=\!=_n\!=\!=$ whose proofs are straightforward:

**1)** $=\!=_n\!=\!=$ is symmetric for all $n$.

**2**) $p =\!=_1\!=\!= q$ iff $q \geq g(p)$.

**3**) $p =\!=_2\!=\!= p$.

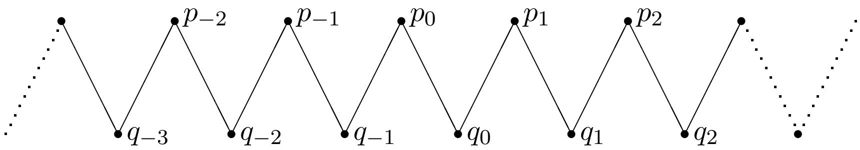
**4**) $p =\!=_n\!=\!= q$ and $q =\!=_m\!=\!= r$ imply $p =\!=_{n+m}\!=\!= r$.

**1.6 Lemma.** *Let $p$ be a maximal element of $P$, $n \geq 1$. Then: $\nu^n(p^*) = \bigcap t^*$, the intersection being over all $t =\!=_n\!=\!= p$.*

**Proof:** By induction; 1.4 implies that $\nu(p^*) = \bigcap\{t^* \colon\ t \notin \nu(p^*)\}$. By using the fact that $t$ is a maximal element and the definitions of $^*$ and $'$ we can see that $t \notin \nu(p^*)$ if and only if $t \geq g(p)$. If we suppose that the equality holds for $k = n-1$, then property 4) of 1.5 implies that it also holds for $k = n$. ∎

**2** – The aim of this section is to provide an example of subdirectly irreducible symmetric Heyting algebra that is not simple. The same algebra, considered as a double Heyting algebra, is also subdirectly irreducible but not simple.

Let $P$ be the poset whose underlying set is $\{p_k \colon\ k \in Z\} \cup \{q_i \colon\ i \in Z\}$ and whose order relation is given by the following diagram:




Then, $P$ satisfies condition (M) of §1.

Let $g \colon\ P \to P$ be defined by $g(p_0) = q_0$, $g(p_k) = q_{-k}$. We characterize the elements of $P^+$ in the following lemma.

**2.1 Lemma.** *Let $X_i = \{q_i\}^c$, for $i \in Z$. Then $C \in P^+$ if and only if $C = \bigcap_{i \in I} X_i \cap \bigcap_{k \in K} p_k^*$, where $I = \{i \in Z \colon\ q_i \notin C\}$, $K = \{k \in Z \colon\ p_k \notin C\}$.* ∎

Let $\mathbb{F}$ be the set of all $C \in P^+$ whose complement $C^c$ is finite.

**2.2 Lemma.** *$\mathbb{F}$ is a filter that contains $X_j$, $\nu(X_j)$ and $\nu^2(X_j)$ for every integer $j$.*

**Proof:** It is obvious that $\mathbb{F}$ is a filter and that $X_j \in \mathbb{F}$. Moreover: $X_j' = \{p_{-j}\}$, so $\nu(X_j) = p_{-j}^*$ which belongs to $\mathbb{F}$. By 1.6, $\nu^2(X_j) = \bigcap_{p_k \geq q_j} p_k^* = p_j^* \cap p_{j+1}^*$, which also belongs to $\mathbb{F}$. ∎

**2.3 Proposition.** *$\mathbb{F}$ is a non trivial kernel of $P^+$.*

**Proof:** Let $C \in \mathbb{F}$. By Lemmas 2.1 and 2.2, $C = \bigcap_{i\in I} X_i \cap \bigcap_{k\in K} \nu(X_{-k})$, with $I$ and $K$ finite. So $\nu(C) = \bigcap_{i\in I} \nu(X_i) \cap \bigcap_{k\in K} \nu^2(X_{-k})$ belongs to $\mathbb{F}$. ∎

**2.4 Lemma.** *For every $j$, $k$, $i$ there is an $r > 0$ such that $p_k \notin \nu^{2r}(X_j)$, $q_i \notin \nu^{2r}(X_j)$.*

**Proof:** Let $r$ be such that $j - r + 1 < k < j + r$. By 1.6 it is easy to see that $\nu^{2r}(X_j) = \bigcap_{h=j-r+1}^{j+r} p_h$. So, $p_k \notin \nu^{2r}(X_j)$. We can choose $r$ such that we also have $p_i \notin \nu^{2r}(X_j)$. Then $q_i \notin \nu^{2r}(X_j)$ because $\nu^{2r}(X_j)$ is increasing. ∎

**2.5 Proposition.** *$\mathbb{F}$ is contained in every non-trivial kernel $\mathbb{G}$ of $P^+$.*

**Proof:** Let $\mathbb{G}$ be a non-trivial kernel. Then, there is some $j$ such that $X_j \in \mathbb{G}$. Let $C \in \mathbb{F}$, $K$ and $I$ as in 2.1. Let $k$ be the maximum of the set $\{|t|\colon t \in K\}$, $i$ the maximum of the set $\{|s|\colon s \in I\}$. Then, by 2.4, there exists an $r > 0$ such that $\nu^{2r}(X_j)$ do not contain any $p_t$ with $|t| \le k$ and any $q_s$ with $|s| \le i$. Therefore $\nu^{2r}(X_j) \subset C$ which implies that $C \in \mathbb{G}$. ∎

**2.6 Theorem.** *$P^+$ is a subdirectly irreducible symmetric algebra which is not simple.* ∎

In the following remark we denote by $D^-$ the dual pseudocomplement of $D$.

**2.7 Remark.** $P^+$ is a subdirectly irreducible double Heyting algebra which is not simple.

Indeed: Let $\mathbb{F}$ be as in 3.1. Let $C \in \mathbb{F}$. Then $C^{-*} = ([C^c)]^c$ and trivially $(C^{-*})^c$ is finite. So, $\mathbb{F}$ is a non-trivial normal filter. Let $\mathbb{G}$ be a non-trivial normal filter, $X_j \in \mathbb{G}$. It is easy to see that $(X_j)^{-*} = p_j^* \cap p_{j+1}^*$. Since $(p_i^*)^{-*} = p_{i-1}^* \cap p_i^* \cap p_{i+1}^*$, we have: $X_j^{(-*)^n} = \bigcap_{i=j-n+1}^{j+n} p_i^*$. The sequence $\{X_j^{(-*)^n}\}$ is decreasing. So, it is clear that for any $C \in \mathbb{F}$ there is an $n > 0$ such that $C \supset X_j^{(-*)^n}$. Thus, $\mathbb{F}$ is a minimum normal filter.

In this sense, it is interesting to note the following fact. Consider the construction of the double Heyting algebra $L_1 *_\varphi L_2$ given in [5], where $L_1$, $L_2$ are double Heyting algebras and $\varphi$ is a bounded $(*, 1)$-homomorphism. Let $L_1 = L_2 = 2^Z$ and $\varphi$ be defined by $\varphi(X) = X \cap X + 1$. Then $2^Z *_\varphi 2^Z$ is isomorphic to the double Heyting algebra $P^+$. The isomorphism is given by $\alpha(C) = (X, Y)$, with $X = \{k \in Z\colon p_k \in C\}$, $Y = \{k \in Z\colon q_{k-1} \in C\}$, for $C \in P^+$. It is clear that $\alpha^{-1}$ is defined by $\alpha^{-1}(X, Y) = \bigcup_{k\in Y}\{p_{k-1}, q_{k-1}, p_k\} \cup \bigcup_{r\in X}\{p_r\}$.

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